Abstract

We work in a calculus of intervals, formulated by James Allen for convex intervals, and by ourselves for unions of convex intervals \([All2, Lad2]\). We investigate the primitive relations and operations needed for implementing such calculi in a system which includes some set theory, and which allows the existential definition of operators in Horn clause fashion. We indicate how standard temporal logic may be rephrased in the interval calculus, and present a formalisation of a system of time units in the interval framework. We are implementing the primitives in the \textit{REFINE}^{TM} system\(^1\).

Introduction

James Allen introduced an interval calculus for reasoning about time, and we have proposed an extension of this calculus to enable the representation of time by non-convex intervals. Recent work on the convex interval calculus is by Allen, Pat Hayes, and Henry Kautz \([All2, AllHay, AllKau]\). Recent work on the non-convex calculus is by ourselves and Roger Maddux \([Lad2, Lad5, LadMad]\).

Convex intervals are those intervals considered by Allen and Humberstone \([All2, Hum]\), which span a period of time, without gaps of any sort. In a formalism based on points, these are \(1\)-dimensional convex sets of points. We are concerned with intervals that are arbitrary unions of these, which we need for expressing temporal properties of intermittent events \([Lad1, Lad2]\). We consider the interval formulation to be an abstraction of time periods, and in this view, sets of time points would be just one way of modelling intervals.

Mathematically, Allen’s calculus of convex relations is a particular relation algebra in the sense of Tarski \([JoTa1, JoTa2]\). Allen’s algebra has thirteen atoms, and thus generates a relation algebra of size \(2^{13}\). By contrast, Ladkin’s algebra is infinite, as well as having only infinite representations. We argue in \([Lad2]\) that the relations, which are a strict subset of all relations between unions of convex intervals, are not only convenient but necessary for expressive power. Mathematical results concerning Allen’s and our own calculi are contained in \([LadMad]\).

We present a specification of primitives which can be used to implement time intervals represented as unions of convex intervals. It is important to us to allow only relation structure in the interval calculus, so that we are able to maintain the structure of a relation algebra, and to restrict the proliferation of intervals denoted by basic terms in our model \([LadMad]\).

Lad\(^2\). We obtain the effect of operators on intervals by using a correspondence between an interval and a certain set of convex subintervals of that interval. Sets of convex intervals are needed in any case for our model of time units.

We show how to express standard temporal logic primitives in the interval calculus, and finally we develop a general model of time units in the interval framework.

We assume throughout that time is linearly ordered, with respect to the relation \textit{precedes}, although this work is equally applicable to branching time models. Some modification would be needed to the measure functions, and other modifications would be of a minor nature only.

Other references to time representation by intervals are \([oBen, Dow]\). Another representation of time for AI purposes using a points-based model rather than an interval model is described in \([McDer1, McDer2]\).

Notation

We assume the reader is familiar with standard logical notation, in particular the connectives and quantifiers

- \(\land \lor \neg \Rightarrow \forall \exists\)

We use certain terminology from \([All2, Lad2]\), in particular

- We refer to the relations in \([All2]\) as convex relations. All convex relations are reflexive and antisymmetric, except for equality. Non-convex intervals or relations are those for unions of convex intervals in \([Lad2]\). A picture of such a beast will look like a sequence of lines with gaps, when drawn in one dimension. The lines are the \textit{maximal convex subintervals}, or \textit{maxconsubs}.

- \(\mid\) is the convex relation \textit{meets}.
  \((i \mid j)\text{ iff }i\text{ is before }j\text{ with no interval occurring between them}\)

- \(\ll\) is the convex relation \textit{contains-in}.
  \((i \ll j)\text{ iff }i\text{ is a strict subinterval of }j\text{, i.e. }i\text{ starts }j\lor i\text{ during }j\lor i\text{ ends }j\text{ in the terminology of }[All2]\)

- \(\prec\) is the convex relation \textit{precedes}.
  \((i \prec j)\text{ iff }i\text{ is before }j\text{, with some other interval occurring between }\)

- \(\bowtie\) is the convex relation \textit{overlaps}.
  \((i \bowtie j)\text{ iff, intuitively, }i\text{ starts before }j\text{, and finishes before }j\).
resenting the periods of time over which an action takes place,

We attach intervals to actions, tasks, events and assertions, rep-

In mathematics, duration is usually called

We note here that certain supposed problems with the defini-

The operators

Operators for Intervals

We attach intervals to actions, tasks, events and assertions, rep-

The operators

We use the word type to indicate a domain of objects of the

Interval-of(T): P is of type task / action / event / assertion

Since we reason assertionally about time intervals, this
gives us a way of passing between the task domain and
the time domain.

dissect(): the set of maximal convex subintervals of i. Dis-

combine(S): S is a set of intervals

makes an interval out of S, rather like a union operator.

In general, this interval will be non-convex. All the inter-
vals in S are subintervals of combine(S). Combine is the
constructor for the data domain of non-convex intervals.

duration(): type real, the measure of i

convexify(startint,endint): type interval, the smallest con-

alltime: type interval, the global time interval that includes all
time.

Note that we have omitted from the list of primitives the func-
tion
diameter(): type real, the largest distance between two subin-
tervals of i (including the duration of the subintervals)

Diameter may be defined in terms of duration by the equation

We do not include diameter as a primitive, since it is definable,
but it is a basic part of the constraint-expression language.

Similarly, duration need only be defined for convex intervals,
since its extension to non-convex intervals follows from the
additive property below.

Axioms

In this section, we give the axioms that specify the operators
described above. We note that the only unbounded quantifiers
appearing in the axioms are universal, and that the bounded
existential quantifiers that appear are restricted to range over
sets or intervals that are parameters in the formula. We envis-
age that these objects will be finitely bounded in most applica-
tions, in such a way that the quantifiers may be realised by an
enumeration. This is indeed the case for RENETM.

Note (i convex) and (convex i) are both shorthand for i bars i
[Lad8].

- (Vj)(i <= alltime v i = alltime) characterises alltime as
  the global time interval. All intervals are contained-in or
  equal-to alltime.

The next three axioms characterise dissect():

- (V convex j <= i)(3k E dissect(i))(j <= k v j = k)
- (V j E dissect(i))(j convex A (j <= i v j = i))
- (V j)(V k E dissect(i))(j <= k => ~ j E dissect(i))

The first axiom states that all convex subintervals of i are con-
tained in some interval k in dissect(i), or are equal to such an
interval. The second ensures that dissect(i) contains only con-

cvex subintervals of i, which in the presence of the first states that
dissect(i) contains at least the maxconvsubintervals of i. The
third axiom is not in positive Horn clause form, since the con-
sequent is negated. It may easily be turned into the right form
by observing that the negation of the antecedent is equivalent
to a positive disjunction of the other twelve interval relations
enumerated by Allen, who gave the exhaustive list of possible
relations between convex intervals. This observation allows us
to take the contrapositive statement for our axiom. This has the
correct form, even though its intent is more obscure. Our
relations include those that are the disjunction of convex inter-
val relations, so this disjunction reduces to a single predication in
the consequent.

- (V j)(V k E dissect(i))(j C dissect(i) => (j I V O .... k))

In the presence of the first two axioms for dissect, the third dis-
sect axiom ensures that dissect(i) contains only maxconvsubint-
subintervals of i.

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Finally, the axioms for convexify specify that convexify(i,j)

\[ \text{is the smallest convex interval containing } i \text{ and } j. \]

Converses is a total, and thus a commutative operation. It is also associative,

and we prefer to include both these properties explicitly, even

even though they follow from the minimal property.

\[ i \ (\text{begins-at} \land \text{ends-at}) \text{ convexify}(i,j) \land \\
\text{is convexify}(i,j) \text{ convex} \]

\[ (k \text{ convexify}(i,j)) \Rightarrow (i \leq k \land j \leq k) \Rightarrow \\
\text{convexify}(i,j) < k \lor \text{convexify}(i,j) - k) \]

\[ \text{convexify}(i,j) = \text{convexify}(j,i) \]

\[ \text{convexify}(i, \text{convexify}(j,k)) = \text{convexify}(\text{convexify}(i,j), k) \]

Note that the associative and commutative properties of convexify actually follow from its minimal property. However, any reasonable theorem prover would probably prefer to know this explicitly (as ours does).

**Additional Noteworthy Properties**

The following properties are all consequences of the axioms:

\[ i \text{ convex } \Rightarrow \text{duration}(i) = \text{diameter}(i) \]

this follows from the definition of diameter and the property:

\[ i \text{ convex } \Rightarrow \text{convexify}(i,i) = i \]

\[ i \leq j \Rightarrow \text{duration}(i) < \text{duration}(j) \]

which follows from the additive property of duration, given enough subintervals in the universe.

**Predicates**

The predicates we wish to use on intervals are specified in [Lad8, 
Lad8]. They form a relation algebra in the sense of Tarski 
[JoTa, JoTa2, Madl]. We mention them here only for com-
pleteness, since it is not the purpose of this paper to explain the 
interval calculus.

**Interval relations:** We include all the relations between in-
tervals defined in [Lad8].

**Axioms**

- We include the relation product table, and the other ax-
ioms needed for specifying the algebra of relations in [Lad8].

See [LadMad, Madl, JoTa2].

**Additional Defined Entities**

We need the basic (but not primitive) function diameter to ade-
quately express properties of, and constraints on, non-convex
intervals. We repeat here the definition and properties of di-
tefure.

\[ \text{diameter}(i) = \text{diameter} \]

\[ \text{type real, the largest distance between two subintervals} \]

\[ of \ i \ (including \ the \ duration \ of \ the \ subintervals) \]

\[ \text{diameter}(i) = \text{duration} \]

\[ (\text{convexify}(i,i)) \]
We also need, for purposes of specification, the predicates

\begin{align*}
\text{occupies}(P,i) & : P \text{ is type task / action / event / assertion} \\
& \text{is type interval} \\
& i \text{ is the exact interval over which } P \text{ takes place / holds / occurs / is true} \\
\text{occurs-in}(P,i) & : \text{true of all } i \text{ such that } i < \text{interval-of}(P) \\
& \text{which have the properties} \\
& \quad \bullet \text{ occupies}(P,\text{interval-of}(P)) \\
& \quad \bullet \text{ occupies}(P,i) \implies \text{occurs-in}(P,i) \\
& \quad \bullet \text{occurs-in}(P,i) \implies (\exists j < i)(\text{occupies}(P,j)) \\
\end{align*}

These predicates are provided by, and their properties follow from, the definitions

\begin{align*}
\bullet \text{ occupies}(P,i) & \iff i = \text{interval-of}(P) \\
\bullet \text{occurs-in}(P,i) & \iff \text{interval-of}(P) \ll i \\
\end{align*}

**Temporal Logic**

**Interval Constants**

We introduce three interval constants, which correspond to McTaggart's A-series notion of time [McT, Lad], and the standard syntax of temporal logic. The A-series notion conceives of time as consisting of the moving, changing present, the past and the future. This corresponds to the interpretation of the temporal operators in classical tense logic, except that 'present' is implicit. The semantics of tense logic, however, is similar to McTaggart's B-series, which consists of immutable points of time, like timestamps, at which there is no change. Change is represented in the B-series by moving from one point to another.

The evaluation of a tense-logical formula relative to a point, which is the standard semantical definition, is similar to connecting the A-series and the B-series notions. We show how to capture the A-series notion in interval calculus. The standard time-of-day clock functions as an A-series to B-series converter.

We refer the reader to [A1121 for the terminology and calculus of interval relations.

The constants are:

\begin{itemize}
\item now: intuitively, an interval of smallest granularity. In any practical domain of application, intervals will not be infinitely divisible. If they are, there is still no logical contradiction in the axioms presented, as can be shown by a compactness argument from model theory. In this case, now would function like an interval of measure 0.
\item future: intuitively, for those who like their intervals to contain points, the interval \((\infty, \text{now})\); all future time
\item past: intuitively, the interval \((-\infty, \text{now})\); all past time
\end{itemize}

**Axioms**

\begin{itemize}
\item now convex
\item future convex
\item past convex
\item past || now || future
\item \((\forall \text{ convex } i)(\sim \text{ now } \ll i \implies ((i \ll \text{past}) \lor (i \ll \text{future})))
\item \((\exists i) (\text{now } \ll i \implies i \ll \text{past})
\item \((\exists i) \text{(now } \ll i \implies i \ll \text{future})
\item \((\forall \text{ convex } i)((\text{now } \ll i) \lor (\text{now } \ll i) \lor (i \ll \text{now}) \lor (\text{now } \ll i) \lor (i = \text{now})
\end{itemize}

These axioms characterise the constants. They state that the three convex intervals meet in the right ways, that they include all time, and that time is linearly ordered with respect to the now interval.

**The Temporal Operators**

We don't intend to present a full development of temporal logic in the interval framework in this paper. That work is still in progress. We provide here definitions of the two temporal operators \(\boxdot\) and \(\triangledown\).

The formula \(\square P\) is true if and only if \(P\) is true everywhere in the future. Similarly, \(\Diamond P\) is true if and only if \(P\) is true at some evaluation object in the future. We deliberately avoid the word "point" since in our formulations, the evaluation objects are intervals, which may or may not have point-like properties depending on the application. [Hum] shows how truth may be defined for tense-logical formulae evaluated on convex intervals.

The definitions of \(\square P\) and \(\Diamond P\) can be

\begin{itemize}
\item \(\square P \equiv (\text{int}(P) = \text{future } \lor \text{future } \ll \text{int}(P))
\item \(\Diamond P \equiv (\text{int}(P) \text{ sometimes-(overlaps } \lor \text{ contained-in ) future})
\end{itemize}

By observing that \(\diamond P \equiv \neg \neg P\), we can obtain alternative definitions of either within the interval framework.

**Conversion Functions**

Standard temporal logic has a syntax that corresponds to McTaggart's A-series time, and a semantics that corresponds to his B-series notion of time (roughly, timestamps).

We indicate that conversion by noting that it is provided already in the standard facilities available on most AI workstations, as a real-time clock, which converts now into a timestamp. We also need to construct past and future from the timestamp. We assume that the clock runs to a certain granularity, say microseconds, and point out that the clock does, in fact, specify a time interval, whose duration is one microsecond in this case. Essentially, the clock tells you which interval the query interrupt is contained in. In this context, there are next and previous operators, which return the next and the previous timestamps. They may be implemented by increment and decrement respectively. The representation is therefore just

\text{timenow}(): the call to the clock implements the conversion of now to its B-series timestamp.

If there are infinity intervals at both ends of the time structure, \((\infty \text{ and } -\infty \text{ for want of better names})\) then past and future may both be represented by

\begin{itemize}
\item past: \text{convexify}(\infty, \text{previous(now)})
\item future: \text{convexify}(\text{next(now)}, \infty)
\end{itemize}
It is consistent to add such infinity intervals. If you don't want them, the properties of the past and future B-series intervals may then be inferred from the axioms alone.

**Defining Time Units**

We need to reason about years, months, days, minutes and microseconds. We introduce a standard form for an interval which represents an instance of one of these units. All the units will be convex intervals, and we then show how to develop the types of units from these standard forms.

**Standard Time Units**

We use sequences of integers to represent our standard units. We use a linear hierarchy of standard units, year, month, day, hour, minute, second, arranged as a sequence. We illustrate its use down to seconds, hence our sequences will have lengths of up to six elements. It should be clear that the hierarchy is easily extendable to smaller units such as microseconds. We illustrate the meanings of sequences of integers, rather than giving an obvious definition.

- \([1986]\) represents the year 1986
- \([1986,3]\) represents the month of March, 1986
- \([1986,3,21]\) represents the day of 21st March, 1986
- \([1986,3,21,7]\) represents the hour starting at 7am on 21st March, 1986
- \([1986,3,21,7,30]\) represents the minute starting at 7:30am on 21st March, 1986
- \([1986,3,21,7,30,32]\) represents the 33rd second of 7:30am on 21st March, 1986 (the first second starts at 0)

We conceive of these intervals as being closed at the left end and open at the right, since this corresponds with normal usage. Notice, as we mentioned, that the standard clock times returned by a time-of-day clock in fact returns the interval in which the interrupt occurred. (It's not really possible to determine from this which end of an interval should be open and which closed, since the interrupts are serialised).

### Axioms for Units

Certain relations hold between these intervals. All of these intervals are convex, and hence the vocabulary of relations is Allen's [All2].

We give examples only of the axioms, since the nature of the rest may easily be inferred from the examples.

It is obvious that not all integer sequences of appropriate lengths are going to name units in our formalism. We shall not bother with checking bounds on elements of a sequence, since this is a detail of no theoretical interest. We shall assume bounds are checked somehow.

We shall use \(x, y, z, \ldots\) for integer variables, and \(a, b, \ldots\) for sequence variables. Concatenation of sequences is denoted by \(-\). All the axioms are quantifier-free statements.

- \([x,1] \text{ starts } [x,y]\)  
  January is the first month and December is the last month of the year

We also need to be able to coalesce representations, to gather months into years, and seconds into minutes. The axiom for this is

- \((\alpha \leq \delta \text{ starts } \alpha) \land (\alpha \leq \gamma \text{ ends } \alpha) \Rightarrow \text{convexify}(\alpha - \delta, \alpha - \gamma) = \alpha\)

### Interval Types Definable From Unit Types

We can now define classes of intervals, based upon the units.

- \(\text{YEARS} = \{ \alpha \mid \text{length}(\alpha) = 1 \}\)
- \(\text{MONTHS} = \{ \alpha \mid \text{length}(\alpha) = 2 \}\)
- \(\text{DAYS} = \{ \alpha \mid \text{length}(\alpha) = 3 \}\)
- Similarly for HOURS, MINUTES, SECONDS,.....

We may also define units which are not in the list of basic units. Firstly, let us assume that all variables and sequences range over the set DAYS. This will simplify notation for our examples. We define

- \(\phi_0(\alpha, \beta) \equiv (\alpha \parallel \beta)\)
- \(\phi_{i+1}(\alpha, \beta) \equiv (\exists \gamma)(\phi_i(\alpha, \gamma) \land (\gamma \parallel \beta))\) for \(0 \leq i\)
- \(\phi^*\) is the symmetric, transitive closure of \(\phi\), for any binary relation \(\phi\)

The \(\phi_i\) are the iterated meet relations for DAYS. Note that, as we have defined them, a given \(\alpha\) in DAYS meets exactly one \(\beta\) in DAYS, and is-met-by exactly one \(\gamma\) in DAYS.

- \(\text{WEEKS} =\)
  \(\{ \text{convexify}(\alpha, \beta) \mid \phi^*(\alpha, \beta)\}\)
  defines all 7 day intervals as weeks
- \(\text{MONDAYS} =\)
  \(\{ \alpha \mid (\phi^*)^*((1986,3,31), \alpha)\}\)
  Needless to say, \([1986,3,31]\) is, in fact, a Monday
We may define other days of the week in a similar way to MONDAYS, or we may choose to use an implicit definition, such as

\[(\text{SUNDAYS} \subseteq \text{DAYS}) \land (\text{combine}(\text{SUNDAYS})\text{ always-meets } \text{combine}(\text{MONDAYS}))\]

Our definitions of the interval types show that we need to maintain the distinction between a non-convex interval \(I\) and the set of its maximal convex subintervals \(\text{dissect}(I)\). All of the unit classes \(\text{YEARS}, \text{MONTHS}, \ldots\), as well as some of the defined classes such as \(\text{WEEKS}\), satisfy the condition

\[\text{combine}(S) = \text{alltime}\]

and cannot thus be distinguished purely as interval objects. One of the major reasons for introducing sets into the time structure must be to distinguish between the different classes of time units. Since the set theory is needed, we see no reason not to make cautious use of it, and we may then avoid the need for a proliferation of interval operations, since we may use \(\text{dissect}\) on an interval, perform set theoretic operations, and then use \(\text{combine}\) to create the new interval.

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**Bibliography**


