

Analytic Solution of Qualitative Differential Equations

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Abstract

Numerical simulation, phase-space analysis, and analytic techniques are three methods used to solve quantitative differential equations. Most work in Qualitative Reasoning has dealt with analogs of the first two techniques, producing capabilities applicable to a wide range of systems. Although potentially of benefit, little has been done to provide closed-form, analytic solution techniques for qualitative differential equations (QDEs).

This paper presents one such technique for the solution of a class of ordinary linear and nonlinear differential equations. The technique is capable of deriving closed-form descriptions of the qualitative temporal behavior represented by such equations. A language QFL for describing qualitative temporal behaviors is presented, and procedures and an implementation QDIFF that solves equations in this form are demonstrated.

I. Introduction

Various techniques have been described in the literature for inferring qualitative behavior of physical systems. The first techniques were based on *simulation* [De Kleer & Brown 84, Forbus 84, Kuipers 86]. Analogous to numerical simulation, these techniques compute the progression of qualitative values over time.

More recently, qualitative phase-space approaches have been introduced [Lee&Kuipers 88, Struss 88, Sacks 87]. Augmenting simulation, these techniques explore trajectories in phase space, showing how the qualitative values in a system will change from any point in the space. Similar to the phase-space methods used in quantitative analysis [Thompson & Stewart 86], these techniques are strong at indicating convergence, stability, etc., but weaker at explicitly describing the temporal behavior of the values.

Closed-form, analytic solution of differential equations is a well-known technique in mathematics [Boyce & DiPrima 77]. Rather than using point-by-point simulation, this methodology describes entire temporal behaviors in terms of a set of *functions*. The set of these

functions includes t^n , $\exp(t)$, $\sin(t)$, $\log(t)$, etc. Manipulation of these symbols according to the laws of mathematics is used to find behaviors in closed form.

Although familiar in quantitative mathematics, closed-form analysis of differential equations has seen little attention in qualitative reasoning, although closed-form *algebraic* analysis has been described by various authors [e.g., Williams 88]. For differential equations, however, techniques such as aggregation [Weld 86] and dynamical systems theory [Struss 88] have been used to infer properties of behaviors computed in other ways. To perform qualitative, closed-form analysis, qualitative reasoning needs a set of symbolic descriptions of qualitative behavior analogous to the $\sin(t)$, $\log(t)$, etc., of quantitative mathematics, and rules to manipulate and transform these functional descriptions.

Such qualitative solutions to differential equations are desirable for several reasons. First, if an exact solution to an equation is not known, a qualitative solution can indicate the types of behavior that are possible, augmenting numerical simulation results. Also, for complex equations where an exact solution *is* known, it may be so complex as to not be comprehensible to a person examining it. A simpler, qualitative solution may be preferable for obtaining an intuitive understanding of system behavior. The advantages of qualitative descriptions of behavior are covered further in [Yip 88].

This paper discusses a preliminary set of such analytic tools. Section II presents a framework, QFL, in which to represent functions qualitatively. Section III describes how derivatives of QFL qualitative functions are computed. Section IV defines the effects of applying nonlinear functions to qualitative behaviors. Finally, Section V presents an implementation QDIFF, and some examples outlining the solution of QDEs. We close with a brief evaluation of the approach and some ideas for how it can be extended.

II. Describing Qualitative Functions

Various techniques currently exist for describing qualitative *values*. These include the $\{+,0,-\}$

representation of [DeKleer & Brown 84], values defined in terms of a *quantity space* [Forbus 84], and dynamically-defined values represented in terms of *landmarks* [Kuipers 86]. For qualitative analytic solution, a representation for behavior *over time*, similar to the quantitative functions such as $\sin(t)$, $\exp(t)$, and t^n , is needed. One way to do this is to define a generic quantitative template that describes a wide set of functions, using qualitative values for its parameters to represent particular functions. A desirable starting-point template would describe constant, increasing, and decreasing behavior, as well as a wide variety of periodic and non-periodic oscillations. One such template is:

$$f(t) = \sum_{k=1}^N A_k(t) \sin(k \cdot B(t) + \phi(k)) \quad (\text{eq. 1})$$

where $\phi(k) = \pi/2$ if k even and zero otherwise. Intuitively, the set $A_k(t)$ describes the envelope of the waveform of $f(t)$ and $B(t)$ describes the behavior of the period of oscillation of the waveform (or that there is no oscillation, if $dB(t)/dt = 0$). A great many functions can be described in this form. The variation of $A(t)$ with k allows for dynamically-varying harmonic content of the waveform, and the use of $B(t)$, rather than a constant times t allows the time scale to be varied with time. These variations from the familiar Fourier expansions [Gabel & Roberts 80], allow a wider variety of behaviors than might initially be expected.

We define a language QFL (Qualitative Function Language) in which functions are described in terms of the attributes of the *sets* of functions $A_k(t)$ and $dB_k(t)/dt$, the sets henceforth referred to as $A(t)$ and $dB(t)$. In QFL, $A(t)$ and $dB(t)$ fall into one of the following categories:

1. **inc:** $A_k(t)$ is non-negative, and for all nonzero $A_k(t)$, monotonically increases as t approaches infinity
2. **dec:** $A_k(t)$ is non-negative, and for all nonzero $A_k(t)$, monotonically decreases asymptotically as t approaches infinity.
3. **con:** - for every k , $A_k(t)$ is non-negative constant, for some k , $A_k(t)$ is nonzero.
4. **0:** - for all k , $A_k(t)$ is equal to 0.

A QFL function is represented by the expression

<label> (*<type of A(t)>*, *<type of dB(t)>*)

If $\partial B(t)$ is zero, the second argument is omitted. Figure 1 shows an example of the function $F1(\text{inc}, \text{dec})$. In addition to specifying the types of $A(t)$ and $\partial B(t)$, QFL allows functions to be specified relative to other QFL functions, by use of a set of *qualitative shape operators*.

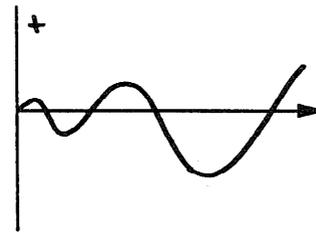


Figure 1. Example of $F1(\text{inc}, \text{dec})$

Shape operators express relationships between the amplitude envelopes of different functions. The shape operators supported by QFL include:

1. **(sharp f)** : to scale the range of function f by a positive, nonlinear scaling function which increases with distance from the origin.
2. **(flat f)** : to scale the range of function f by a positive, nonlinear scaling function which decreases with distance from the origin.
3. **(invert f)** : to nonlinearly reverse the scale of the range of function f , hence changing the type of f .

Figure 2 shows an example of the function (*sharp F1*)(*inc, dec*).

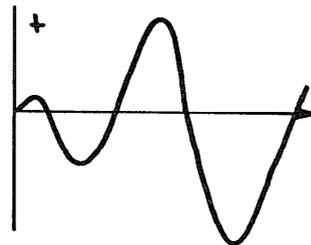


Figure 2. Example of (*sharp F1*)(*inc, dec*)

III. Derivatives of Qualitative Functions

Assume that we wish to solve a nonlinear differential equation of the form

$$\sum_{k=1}^n \{f_k(f(t)) \frac{d^k f(t)}{dt^k}\} + f_0(f(t)) = 0 \quad (\text{eq. 2})$$

for the behavior $f(t)$, where $f_k(x)$ and $f_0(x)$ are nonlinear functions of x . To process the terms of an equation in this form, we need to compute the derivatives of qualitative functions, as well as compute the results of applying nonlinear functions to qualitative behaviors.

We can elucidate the mapping between function and derivative by differentiating the template of Equation 1 and determining the implied qualitative transformations.

Operator tables for *functions*, analogous to the operator transforms described for *values* [De Kleer & Brown 84, Forbus 84, Kuipers 86], can then be constructed. In the following, ∂B will be considered equivalent to $dB(t)/dt$, $\partial\partial B$ to $d^2B(t)/dt^2$, etc.

$$\frac{df(t)}{dt} = \sum_{k=1}^N \{ \partial A(t) \cdot \sin(k b(t) + \phi(k)) + k \cdot A(t) \partial b(t) \cdot \cos(k b(t) + \phi(k)) \}$$

This equation contains a component lagging $f(t)$ in phase by $\pi/2$ and a component in phase with $f(t)$. The oscillation characteristics of $f(t)$ (the argument to the \sin terms) are preserved. The results for derivatives zero through two are tabulated below:

Table I. Derivative Effect on A(t)

n	A(t) of nth derivative	
	In-phase	Out-of-phase
0	A	0
1	∂A	$A \partial B$
2	$\partial\partial A - A \partial B \partial B$	$\partial A \partial B - A \partial\partial B$

It would be desirable to express the entries in this table in algebraic terms, free of the ∂ operators, so that the solution of the differential equations could be found algebraically. This is achieved by the following process, which converts the expression $dA(t)/dt$ into a product. Let $d(t)$ be the function such that

$$\frac{dA(t)}{dt} = d(t) \cdot A(t)$$

where $d(t)$ is one of the qualitative function types. It can be shown, for the class of $A(t)$ represented in QFL, that

$$\frac{d^k A(t)}{dt^k} = \tilde{d}(t) \cdot A(t)$$

where $\tilde{d}(t)$ is of the same qualitative type as $d(t)$. The same, of course, applies to the derivatives of $\partial B(t)$. Therefore, we can rewrite the terms from Table I in terms of sums and products of $A(t)$, $\partial B(t)$, $D(t)$ (the function equivalent to the derivative of $A(t)$), and $E(t)$ (the function equivalent to the derivative of $\partial B(t)$). For example, the out-of-phase part of the second derivative from the table, $\partial A \partial B + A \partial\partial B$, would be rewritten as $D(t) A(t) \partial B(t) + A(t) E(t) \partial B(t)$, or, in the shorthand we will use from now on, $D A \partial B + A E \partial B$. By use of multiplication, such expressions can be reduced to a sum of qualitative values, given qualitative values for A , ∂B , D , and E . This is achieved with the following multiplication table:

Table II. QFL Multiplication

f(t)	g(t)	f(t) g(t)
inc	inc	inc
inc	dec	inc or dec or con
X	con	X
X	0	0

For each of the qualitative types of $A(t)$ and $\partial B(t)$, the corresponding possible types of $D(t)$ and $E(t)$ have been tabulated. Table III was computed by considering the possible behaviors and derivatives of each function type. Where ambiguous, all possible types were included:

Table III: Derivative Functions

type of f(t)	d(t) for d(t)·f(t) = ∂f(t)
dec	-inc or -con or -dec
inc	dec or inc or con
con	0
0	0

IV. Nonlinear Functions

The remaining analytic tool needed to solve differential equations in the form of Equation 2 is the mechanism for determining the qualitative effects of the nonlinear functions $f_k(t)$. As is apparent from the equation, nonlinear functions will be applied directly to the unknown $f(t)$. We take care to consider the effects of the transformation both on the characteristic $A(t)$ of $f(t)$ and on the phase of the result.

Assume that any nonlinear function $f_k(t)$ of interest can be represented as a power series in t . The following characteristics will therefore occur when applying $f_k(t)$ to qualitative behavior $f(t)$ in the form of Eq. 1:

1. The constant term in the expansion of $f_k(t)$ will lead to the appearance of terms $\sin(k B(t) + \text{phase}(k))$.
2. Quadratic terms in $f_k(t)$ will lead to contributions of the form $A_m(t) \sin(m B(t)) A_n(t) \sin(n B(t))$, when m and n are odd. Applying a trigonometric identity yields

$$A_m(t) A_n(t) \{ \cos((m - n)B(t)) + \cos((m + n)B(t)) \} = A_m(t) A_n(t) \{ \sin((m - n)B(t) + \pi/2) + \sin((m + n)B(t) + \pi/2) \}.$$

$(m - n)$ and $(m + n)$ are both even numbers, so the result will be in phase with the terms of Equation 1.
3. Quadratic terms in $f_k(t)$, when m and n are both even or for m odd and n even similarly will yield results in phase with $f(t)$.

4. Higher-order terms in will also result in terms in phase with the original terms in Equation 1. This can be shown inductively, using the results of 2) and 3).

Using this property, nonlinear functions can be adequately defined in terms of the qualitative shape operator they impose on $A(t)$. For example, let $f_k(x)$ be $\sin(x)$, for $-\pi/2 < x < \pi/2$. Suppose that we wish to find $f_k(F(A, \partial B))$, where $A(t)$ is of type *inc*. In this case, $\sin(A(t))$ will be "flattened" more and more as $A(t)$ gets larger. Therefore, we conclude

$$f_k(F(\text{inc}, \partial B)) = (\text{flat } F)(\text{inc}, \partial B).$$

Consider a somewhat more complex nonlinear function, $f_k(x) = (1 - x^2)$. Differing values of $A(t)$ will lead to differing qualitative effects: when $|A(t)| < 1$, $f_k(A)$ will be positive, and negative when $|A(t)| > 1$. Therefore, the behavior is divided into distinct regions. In all regions, the behavior of this equation is given by *con - |sharp A|*. However, when $A(t) > 1$, we can infer the qualitative relationship $|con| < |sharp A|$, and where $A(t) < 1$, we know that $|con| > |sharp A|$.

The final step in supporting the differential equation representation of Equation 2. is the multiplication of the derivatives of $f(t)$ by the nonlinear functions $f_k(t)$. Recall that in Table I, the in-phase and out-of-phase portions of the derivatives are separated. Therefore, we wish to maintain the separation of in- and out-of phase components when multiplying these expressions by the nonlinear functions. A derivation nearly identical to that carried out above yields the following conclusion:

When multiplying $f_k(F(A, \partial B)) = g(t)$ by a derivative of $F(A, \partial B)$, the in-phase part of the product will be $g(t)$ times the in-phase part of the derivative. Similarly, the out-of phase part of the product will be $g(t)$ times the out-of phase part of the derivative.

For example, consider the term $\sin(f(t)) df(t)/dt$. From Table I, we see that the in-phase part of $df(t)/dt$ is $\partial A(t)$, and the out-of phase part is $A(t) \partial B(t)$. Recall from the preceding discussion that $\sin(F(A, \partial B)) = (\text{flat } F)(A, \partial B)$. Therefore, the in-phase part of the italicized term is $(\text{flat } F)(\partial A, \partial B)$, and the out of phase part of the term is $(\text{flat } F)(A \partial B, \partial B)$.

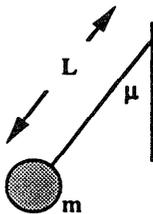


Figure 3. The Nonlinear Pendulum

V. Solving QDEs

The results outlined above lead to a technique for solving qualitative differential equations. A program called QDIFF has been implemented for just this purpose. In this section, we describe the solution method used by QDIFF and show examples of various equations and their solution.

QDIFF solves differential equations by finding values for $A(t)$ and $\partial B(t)$ that allow the in-phase and out-of phase contributions of each term in the equation to add to zero. The problem can be broken down in this way because the in-phase and out-of phase parts are linearly independent (although not necessarily orthogonal). The solution is achieved with the following procedure:

1. Gather the in-phase and out-of phase expressions for each derivative of $f(t)$ that appears in the qualitative differential equation.
2. For terms multiplied by a nonlinear function, obtain the expression, in terms of $A(t)$, that describes that function, and multiply the corresponding in-phase and out-of phase expressions from step 1) by that function.
3. Replace ∂ operators in the resulting in-phase and out-of phase sums with $D(t)$ and $E(t)$ terms, according to the translation process of Section II.
4. Constrain the values of $d(t)$ and $e(t)$ according to potential values for $A(t)$ and $\partial B(t)$ from Table II. Using multiplication via Table III, find all combinations of $A(t)$ and $\partial B(t)$ within these constraints that allow both sums to be zero.

A successive-refinement strategy is used to find values for $A(t)$ and $\partial B(t)$ in step 5. QDIFF chooses a value for one of the functions, and narrows down the space of other functions to consider by use of the specified constraints. This technique will be clarified with some examples. First, consider the pendulum shown in Figure 3. This is a nonlinear system described by the equation

$$m l^2 \partial \partial \mu + c l \mu + m g l \sin \mu = 0$$

where the damping constant $c > 0$. No reasonable exact solution to this equation is known. An approximation that is often made, for the case where μ is near 0, is

$$m l^2 \partial \partial \mu + c l \mu + m g l \mu = 0.$$

Let us first solve the linearized equation using the QDIFF algorithm. Using Table I and Table III, equivalent representations of the in-phase sum for the equation are found. The in-phase part of the differential equation terms is:

$$A + \partial A + \partial \partial A - \partial B \partial B A = 0 \text{ or } \\ con + D + |D| - |\partial B| = 0,$$

where common factors are removed. The out-of phase sum is:

$$A \partial B + \partial A \partial B + A \partial \partial B = 0 \text{ or } \text{con} + D + E = 0.$$

The term A, factored out of both equations, immediately indicates that F(0) is a solution. The term ∂B , factored out of the second sum, also easily leads to a solution when $D = -\text{con}$ (and, hence, $\partial A = \text{dec}$). This indicates that F(dec) is also a solution. Another solution occurs when $D = -\text{con}$ and $E = 0$. In this case, $\text{con} + D + E$ can equal zero. For $D = -\text{con}$, Table III shows that ∂B can equal con , which allows the in-phase sum to also be zero, indicating the solution F(dec,con), depicted in Figure 4. No other values of A and ∂B simultaneously solve both sums. The complete set of solutions is found by QDIFF is:

$$F(0), F(\text{dec}), -F(\text{dec}), F(\text{dec},\text{con}),$$

consistent with textbook solutions to the problem [Boyce & DiPrima 77].

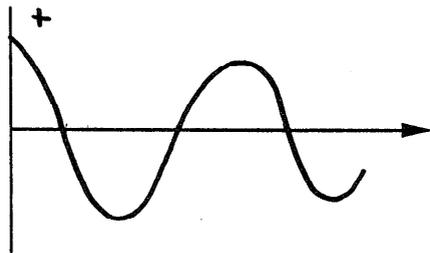


Figure 4. Linear pendulum solution F(dec,con).

An example demonstrating more powerful capabilities of the analytic approach, is the nonlinear pendulum. Assume that $-\pi/2 < \mu < \pi/2$. $\text{Sin}(F(A,\partial B))$ is represented qualitatively as (flat F)(A, ∂B), as demonstrated in Section IV. The sums for this differential equation are, in-phase:

$$\begin{aligned} |\text{flat } A| + \partial A + \partial \partial A - \partial B \partial B A = 0 \text{ or} \\ (\text{invert } A) + D + |D| - |\partial B| = 0; \end{aligned}$$

and the out-of phase sum is the same as the linear case:

$$A \partial B + \partial A \partial B + A \partial \partial B = 0 \text{ or } \text{con} + D + E = 0.$$

For this equation, the solutions F(0) and F(dec) are found in the same manner as before. It is more interesting to note, however, what happens to the "linear" solution F(dec,con). The out-of phase sum will be zero for these values of A and ∂B . However, because the constant in the linearized system has been replaced by an (invert A) in the nonlinear system, the constant-period value for ∂B no longer holds. For the case where $A = \text{dec}$, (invert A) = inc . As a result, QDIFF finds that ∂B must be of type inc for a solution to exist. The complete solution set is:

$$F(0), F(\text{dec}), -F(\text{dec}), F(\text{dec},\text{inc}).$$

This solution is consistent with the solutions demonstrated numerically in [Thompson & Stewart 86]. The oscillating result $F(\text{dec},\text{inc})$ is shown in Figure 5. This example shows that the analytic techniques described here are sufficiently powerful to identify certain qualitative differences between a linearized differential equation and the more accurate nonlinear equation from which it was derived. Identifying temporal behavior of this nature is a feature not found in most other qualitative reasoning approaches.

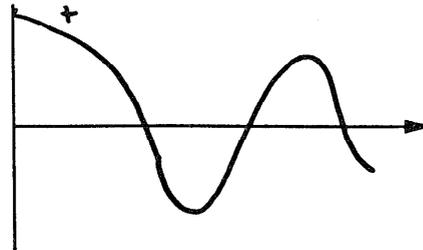


Figure 5. Nonlinear pendulum solution F1(dec,inc).

As a final example, consider the more complex system described by the differential equation

$$\partial \partial X - \mu(1 - x^2) \partial X + X = 0.$$

This is known as the *van der Pol* equation, a relation of significance in engineering as well as medical modeling. It is an interesting problem from a phase-plane perspective in that it exhibits a *limit cycle*. This example has been studied from that perspective in the piecewise-linear approach of [Sacks 87]. Here, we find that the QDIFF qualitative function perspective is also able to identify this unique behavior.

The nonlinear function $1-x^2$ leads QDIFF to divide consideration of the system behavior into distinct regions, where differing qualitative relations between the con term and the (*sharp a*) term are known (see Section IV). First, consider the behavior in the region where $|\text{sharp } a|$ is small. The sums are, in-phase:

$$\begin{aligned} A + |\text{sharp } a| \partial A - \text{con} \partial A + \partial \partial A - A \partial B \partial B = 0 \text{ or} \\ \text{con} + AD - D + |D| - |\partial B| = 0 \end{aligned}$$

where $|\text{AD}| < |-D|$ and, out-of-phase:

$$\begin{aligned} |\text{sharp } A| A \partial B - \text{con } A \partial B + \partial A \partial B + A \partial \partial B = 0 \text{ or} \\ |A| - \text{con} + D + E = 0 \end{aligned}$$

where $|A| < |\text{con}|$. Consider the case where A is of type dec . In this case, QDIFF finds that consistent values for D and E cannot be found to make the out-of phase sum be equal to zero. Likewise, QDIFF fails to find a consistent solution for A of type con . When A is of type dec , however, solutions are found. QDIFF finds solutions for ∂B of types inc , dec , and con .

When QDIFF considers the region where (*sharp A*) is large, the in-phase and out-of phase equations are unchanged, but the qualitative ordering between the *con* and *sharp a* terms is reversed. This leads to a different set of solution values for A and ∂B . The complete solution set is:

For Region I, (small A(t)):
 F(0), F(inc,dec), F(inc,con), F(inc,dec)

For Region II, (large A(t)):
 F(dec,con), F(dec,inc), F(dec,dec)

For Region III, boundary:
 F(con,con).

QDIFF found the correct solutions to the equation, with regard to the increasing and decreasing oscillations and convergence to a stable amplitude, although it did not determine whether the convergence would occur via increasing or decreasing period of oscillation. Interestingly, this convergence to a stable oscillation is equivalent to the detection of the limit cycle by phase-plane methods, but was achieved through functional, temporal techniques.

Conclusions and Further Work

The analytical technique described in this paper provides a method to augment the existing techniques of qualitative simulation and phase-space analysis. It shares several of the characteristics of its quantitative analog, including conceptually simple solution mechanisms, but the drawback that solutions outside the representational scope of QFL will not be found. It is interesting to note that simple explicit reasoning about qualitative behaviors avoids some of the problems of severe ambiguity that are found with simple simulation-only qualitative reasoning systems.

Potentially interesting extensions will briefly be mentioned here. First, a richer set of qualitative shape operators and function types would allow more expressive qualitative solutions to be found. An interesting extension would be a coupling between the analytic approach presented here and other qualitative reasoning techniques. Possibilities include the use of a QDIFF-like system to solve for waveform characteristics in the various regions found by phase-space analysis, and a QDIFF filter for use with qualitative simulation systems.

Acknowledgements

I would like to thank J. Dan Layne and Corrina Perrone for their expertise and support toward the success of this project.

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