Cryptographic Limitations on Learning One-Clause Logic Programs

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Abstract
An active area of research in machine learning is learning logic programs from examples. This paper investigates formally the problem of learning a single Horn clause: we focus on generalizations of the language of constant-depth determinate clauses, which is used by several practical learning systems. We show first that determinate clauses of logarithmic depth are not learnable. Next we show that learning indeterminate clauses with at most k indeterminate variables is equivalent to learning DNF. Finally, we show that recursive constant-depth determinate clauses are not learnable. Our primary technical tool is the method of prediction-preserving reducibilities introduced by Pitt and Warmuth [1990]; as a consequence our results are independent of the representations used by the learning system.

Introduction
Recently, there has been an increasing amount of research in learning restricted logic programs, or inductive logic programming (ILP) [Cohen, 1992; Muggleton and Feng, 1992; Quinlan, 1990; Muggleton, 1992a]. One advantage of using logic programs (rather than alternative first-order logic formalisms [Cohen and Hirsh, 1992]) is that its semantics and computational complexity have been well-studied; this offers some hope that learning systems based on it can also be mathematically well-understood.

Some formal results have in fact been obtained; the strongest positive result in the pac-learning model [Valiant, 1984] shows that a single constant-depth determinate clause is pac-learnable, and that a non-recursive logic program containing k such clauses is learnable against any “simple” distribution [Džeroski et al., 1992]. Some very recent work [Kietz, 1993] shows that a single clause is not pac-learnable if the constant-depth determinacy condition does not hold; specifically, it is shown that neither the language of indeterminate clauses of fixed depth nor the language of determinate clauses of arbitrary depth is pac-learnable. These negative results are of limited practical importance because they assume that the learner is required to output a single clause that covers all of the examples; however, most practical ILP learning systems do not impose this constraint. Such negative learnability results are sometimes called representation-dependent.

This paper presents representation independent negative results for three languages of Horn clauses, all obtained by generalizing the language of constant depth determinate clauses. These negative results are obtained by showing that learning is as hard as breaking a (presumably) secure cryptographic system, and thus are not dependent on assumptions about the representation used by the learning system. Specifically, we will show that determinate clauses of log depth are not learnable, and that recursive clauses of constant depth are not learnable. We will also show that indeterminate clauses with k “free” variables are exactly as hard to learn as DNF.

Due to space constraints, detailed proofs will not be given; the interested reader is referred to a longer version of the paper [Cohen, 1993a]. We will focus instead on describing the basic intuitions behind the proofs.

Formal preliminaries

Learning models
Our basic notion of learnability is the usual one introduced by Valiant [1984]. Let X be a set, called the domain. Define a concept C over X to be a representation of some subset of X, and a language L to be a set of concepts. Associated with X and L are two size complexity measures; we use the notation X_n (respectively L_n) to stand for the set of all elements of X (respectively L) of size complexity no greater than n. An example of C is a pair (x, b) where b = 1 if x ∈ C and b = 0 otherwise. If D is a probability distribution

1The prototypical example of a learning problem which is hard in a representation-dependent setting but not in a broader setting is learning k-term DNF. Pac-learning k-term DNF is NP-hard if the hypotheses of the learning system must be k-term DNF; however it is tractable if hypotheses can be expressed in the richer language of k-CNF.
function, a sample of C from X drawn according to D is a pair of multisets \( S^+, S^- \) drawn from the domain X according to D, \( S^+ \) containing only positive examples of C, and \( S^- \) containing only negative ones.

Finally, a language \( \mathcal{L} \) is \emph{pac-learnable} iff there is an algorithm \textsc{Learn} and a polynomial function \( m(\varepsilon, \delta, n_s, n_t) \) so that for every \( n_s > 0, \) every \( n_t > 0, \) every \( \varepsilon \in \mathcal{L}, \) every \( \delta : 0 < \delta < 1, \) and every probability distribution function \( D, \) for any sample \( S^+, S^- \) of \( C \) from \( X \) drawn according to \( D \) containing at least \( m(\varepsilon, \delta, n_s, n_t) \) examples

1. \textsc{Learn}, on inputs \( S^+, S^-, \varepsilon, \) and \( \delta, \) outputs a hypothesis \( H \) such that
\[
\text{Prob}(D(H - C) + D(C - H) > \varepsilon) < \delta
\]
2. \textsc{Learn} runs in time polynomial in \( \frac{1}{\varepsilon}, \frac{1}{\delta}, n_s, n_t, \) and the number of examples; and
3. The hypothesis \( H \) of the learning systems is in \( \mathcal{L}. \)

The polynomial function \( m(\varepsilon, \delta, n_s, n_t) \) is called the \emph{sample complexity} of the learning algorithm \textsc{Learn}.

With condition 3 above, the definition of pac-learnability makes a relatively strong restriction on the hypotheses the learner can generate; this can lead to some counterintuitive results.\(^2\) If a learning algorithm exists that satisfies all the conditions above except condition 3, but does output a hypothesis that can be evaluated in polynomial time, we will say that \( \mathcal{L} \) is \emph{(polynomially) predictable}.\(^3\)

These learning models have been well-studied, and are quite appropriate to modeling standard inductive learners. However, the typical ILP system is used in a somewhat more complex setting, as the user will typically provide both a set of examples and a \emph{background theory} \( K \): the task of the learner is then to find a logic program \( P \) such that \( P, \) together with \( K, \) is a good model of the data. To deal with this wrinkle, we will require some additional definitions. If \( \mathcal{L} \) is some language of logic programs\(^4\) and \( K \) is a logic program, then \( \mathcal{L}[K] \) denotes the set of all pairs of the form \( (P, K) \) such that \( P \in \mathcal{L} \): each such pair represents the set of all atoms \( \varepsilon \) such that \( P \land K \models \varepsilon \). If \( K \) is a set of background theories, then the \emph{family of languages} \( \mathcal{L}[K] \) represents the set of all languages \( \mathcal{L}[K] \) where \( K \in \mathcal{K} \). We will consider \( \mathcal{L}[K] \) to be predictable (pac-learnable) only when every \( \mathcal{L}[K] \) is predictable (pac-learnable.) This requires one slight modification to the definitions of predictability pac-learnability: we must now assume a size complexity measure for background theories, and allow the sample and time complexities of the learner to also grow (polynomially) with the size complexity of the background theory provided by the user.

### Reducibilities among prediction problems

Our main analytic tool in this paper is prediction-preserving reducibility, as described by Pitt and Warmuth [1990]. This is essentially a method of showing that one language is no harder to predict than another. If \( \mathcal{L}_1 \) is a language over domain \( X_1 \) and \( \mathcal{L}_2 \) is a language over domain \( X_2, \) then we say that \( \text{predicting } \mathcal{L}_1 \) \emph{reduces to predicting } \( \mathcal{L}_2, \) written \( \mathcal{L}_1 \leq \mathcal{L}_2, \) if there is a function \( f : X_1 \rightarrow X_2, \) henceforth called the \emph{instance mapping}, and a function \( f_c : \mathcal{L}_1 \rightarrow \mathcal{L}_2, \) henceforth called the \emph{concept mapping}, so that the following all hold:

1. \( x \in C \) if and only if \( f(x) \in f_c(C) \) i.e., concept membership is preserved by the mappings;
2. the size complexity of \( f_c(C) \) is polynomial in the size complexity of \( C \) — i.e. the size of concepts is preserved within a polynomial factor; and
3. \( f_c \) can be computed in polynomial time.

Intuitively, \( f_c(C_1) \) returns a concept \( C_2 \in \mathcal{L}_2 \) that will "emulate" \( C_1 \) (i.e., make the same decisions about concept membership) on examples that have been "preprocessed" with the function \( f. \) If predicting \( \mathcal{L}_1 \) reduces to predicting \( \mathcal{L}_2 \) and a learning algorithm for \( \mathcal{L}_2 \) exists, then one possible scheme for learning a concept \( C_1 \in \mathcal{L}_1 \) would be to preprocess all examples of \( C_1 \) with \( f, \) and then use these preprocessed examples to learn some \( H \) that is a good approximation of \( C_2 = f_c(C_1). \) H can then be used to predict membership in \( C_1: \) given an example \( x \) from the original domain \( X_1 \), one can simply predict \( x \in C_1 \) to be true whenever \( f_i(x) \in H. \) Pitt and Warmuth [1990] give a rigorous argument that this approach leads to a prediction algorithm for \( \mathcal{L}_1, \) leading to the following theorem.

**Theorem 1** (Pitt and Warmuth) If \( \mathcal{L}_1 \leq \mathcal{L}_2 \) and \( \mathcal{L}_2 \) is polynomially predictable, then \( \mathcal{L}_1 \) is polynomially predictable. Conversely, if \( \mathcal{L}_1 \leq \mathcal{L}_2 \) and \( \mathcal{L}_1 \) is not polynomially predictable, then \( \mathcal{L}_2 \) is not polynomially predictable.

### Restrictions on Logic Programs

In this paper, logic programs will always be function-free and (unless otherwise indicated) nonrecursive. A background theory \( K \) will always be a set of ground unit clauses (aka relations, a set of atomic facts, or a model) with arity bounded by the constant \( a \); the symbol \( a-K (K \) if \( a \) is an arbitrary constant) will denote the set of such background theories. The size complexity of a background theory \( K \) is its cardinality, and usually will be denoted by \( n_e. \) Examples will be represented by a single atom of arity \( n_e \) or less; thus we allow the

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\(^2\)In particular, it may be that a language is hard to learn even though an accurate hypothesis can be found, if it is hard to encode this hypothesis in the language \( \mathcal{L}. \)

\(^3\)Such learning algorithms are also sometimes called \emph{approximation algorithms}, as in the general case an approximation to the target concept may be produced.

\(^4\)We assume that the reader is familiar with the basic elements of logic programming; see Lloyd [1987] for the necessary background.
head of a Horn clause to have a large arity, although literals in the body have constant arity.5

Muggleton and Feng [1992] have introduced several additional useful restrictions on Horn clauses. If \( A \leftarrow B_1 \wedge \ldots \wedge B_r \) is an (ordered) Horn clause, then the input variables of the literal \( B_i \) are those variables appearing in \( B_i \) which also appear in the clause \( A \leftarrow B_1 \wedge \ldots \wedge B_{i-1} \); all other variables appearing in \( B_i \) are called output variables. A literal \( B_i \) is determinate (with respect to \( K \) and \( X \)) if for every possible substitution \( \sigma \) that unifies \( A \) with some \( e \in X \) such that \( K \vdash (B_1 \wedge \ldots \wedge B_{i-1})\sigma \) there is at most one substitution \( \theta \) so that \( K \vdash B_i \sigma \theta \). Less formally, a literal is determinate if its output variables have only one possible binding, given \( K \) and the binding of the input variables. A clause is determinate if all of its literals are determinate.

Next, define the depth of a variable appearing in a clause \( A \leftarrow B_1 \wedge \ldots \wedge B_r \), as follows. Variables appearing in the head of a clause have depth zero. Otherwise, let \( B_i \) be the first literal containing the variable \( V \), and let \( d \) be the maximal depth of the input variables of \( B_i \); then the depth of \( V \) is \( d + 1 \). The depth of a clause is the maximal depth of any variable in the clause.

A determinate clause of depth bounded by a constant \( i \) over a background theory \( K \in j-K \) is called \( ij \)-determinate. The learning program GOLEM, which has been applied to a number of practical problems [Muggleton and Feng, 1992; Muggleton, 1992b], learns \( ij \)-determinate programs. Closely related restrictions also have been adopted by several other inductive logic programming systems, including FOIL [Quinlan, 1991], LINUS [Lavrač and Džeroski, 1992], and GRENDEL [Cohen, 1993c].

As an example, in the determinate clause

\[
multiply(X,Y,Z) \leftarrow \ decrement(Y,W) \wedge \ multiply(X,W,V) \wedge \ plus(X,V,Z).
\]

\( W \) has depth one and \( V \) has depth two; thus the clause is 23-determinate.

Learning log-depth determinate clauses

We will first consider generalizing the definition of \( ij \)-determinacy by relaxing the restriction that clauses have constant depth. The key result of this section is an observation about the expressive power of determinate clauses: there is a background theory \( K \) such that every depth \( d \) boolean circuit can be emulated by a depth \( d \) determinate clause over \( K \). The background theory \( K \) contains these facts:

\[
\begin{align*}
\text{and}(0,0,0) & \quad \text{and}(0,1,0) & \quad \text{and}(1,0,0) & \quad \text{and}(1,1,1) \\
\text{or}(0,0,0) & \quad \text{or}(0,1,1) & \quad \text{or}(1,0,0) & \quad \text{or}(1,1,1) \\
\text{not}(0,1) & \quad \text{not}(1,0) & \quad \text{true}(1)
\end{align*}
\]

The emulation is the obvious one, illustrated by example in Figure 1. Notice an example from the circuit domain is a binary vector \( b_1, \ldots, b_n \), encoding an assignment to the \( n \) boolean inputs to the circuit, and hence must be preprocessed to the atom \( \text{circuit}(b_1, \ldots, b_n) \).

The learnability of circuits has been well-studied. In particular, the language of log-depth circuits (also familiar as the complexity class \( \text{NC}^1 \)) is known to be hard to learn, under cryptographic assumptions [Kearns and Valiant, 1989]. Thus we have the following theorem; here \( C_{ij-\text{DET}} \) denotes the language of logic programs containing a single non-recursive \( ij \)-determinate clause.

**Theorem 2** There is a small background theory \( K \in 3-K \) such that \( \text{NC}^1 \subseteq C_{(\log \, n)^3-\text{DET}}[K] \). Thus, for \( j \geq 3 \), the family of languages \( C_{(\log \, n)^j-\text{DET}}[j-K] \) is not polynomially predictable, and hence not pac-learnable, under cryptographic assumptions.

It has recently been shown that \( \text{NC}^1 \) is not predictable even against the uniform distribution [Kharitonov, 1992], suggesting that distributional restrictions will not make log-depth determinate clauses predictable.

Learning indeterminate clauses

We will now consider relaxing the definition of \( ij \)-determinacy by allowing indeterminacy. Let the free variables of a Horn clause be those variables that appear in the body of the clause but not in the head; we will consider the learnability of the language \( C_{k, \text{FREE}} \), defined as all nonrecursive clauses with at most \( k \) free variables. Clauses in \( C_{k, \text{FREE}} \) are necessarily of depth at most \( k \); also restricting the number of free variables is required to ensure that clauses can be evaluated in polynomial time.

Notice that \( C_{1, \text{FREE}} \) is the most restricted language possible that contains indeterminate clauses. We begin with an observation about the expressive power of

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5It should be noted that the parameters \( n_e, n_b, n_t \) (the complexity of the target concept) all measure different aspects of the complexity of a learning problem; as the requirement on the learner is that it is polynomial in all of these parameters, the reader can simply view them as different names for a single complexity parameter \( n = \max(n_e, n_b, n_t) \).
Background theory:

\[ \text{for } i = 1, \ldots, k \]
\[ \text{true}_i(b, y) \text{ for all } b, y : b = 1 \text{ or } y \in 1, \ldots, r \text{ but } y \neq i \]
\[ \text{false}_i(b, y) \text{ for all } b, y : b = 0 \text{ or } y \in 1, \ldots, r \text{ but } y \neq i \]

DNF formula:

\[ (v_1 \lor \overline{v_3} \lor v_4) \lor (v_2 \land \overline{v_3}) \lor (v_1 \land \overline{v_4}) \]

Equivalent clause:

\[ \text{dnf}(X_1, X_2, X_3, X_4) \leftarrow \]
\[ \text{true}_1(X_1, Y) \lor \text{false}_1(X_2, Y) \lor \text{true}_1(X_4, Y) \lor \]
\[ \text{false}_2(X_2, Y) \lor \text{false}_2(X_3, Y) \lor \]
\[ \text{true}_3(X_1, Y) \lor \text{false}_3(X_4, Y). \]

Figure 2: Reducing a DNF formula to a clause

This language: for every \( r \), there is a background theory \( K_r \) such that every DNF formula with \( r \) or fewer terms can be emulated by a clause in \( C_{\text{FREE}}[K_r] \).

The emulation is a bit more indirect than the emulation for circuits, but the intuition is simple: it is based on the observation that a clause \( p(X) + q(X, Y) \) classifies an example \( p(a) \) as true exactly when \( 0 \) and \( Y \) can be bound to some constant other than 1.

The answer to this question is no: for every \( k \) and every background theory \( K \in a-K \), every clause in \( C_{\text{FREE}}[K] \) can be emulated by a DNF formula. Let \( CI = A \leftarrow B_{c_1} \land \ldots \land B_{c_l} \) be a clause in \( C_{\text{FREE}}[K] \). As we assume clauses are nonrecursive, \( CI \) covers an example \( e \) iff

\[ \exists \sigma : K \vdash (B_{c_1} \land \ldots \land B_{c_l}) \theta \text{e} \tag{1} \]

where \( \theta_e \) is the most general unifier of \( A \) and \( e \). However, since the background theory \( K \) is of size \( n_b \) and all predicates are of arity \( a \) or less, there are at most \( n^a_b \) constants in \( K \), and hence only \( (a_n b)^k \) possible substitutions \( \sigma_1, \ldots, \sigma (a_n b)^k \) to the \( k \) free variables. Also (as we assume clauses are function-free) if \( K \) defines \( l \) different predicates, there are at most \( l \cdot (n_e + k)^a \) possible literals \( B_1, \ldots, B_{n_b} + (n_e + k)^a \) that can appear in the body of a \( C_{\text{FREE}} \) clause.

So, let us introduce the boolean variables \( v_{ij} \) where \( i \) ranges from one to \( n_b \cdot (n_e + k)^a \) and \( j \) ranges from one to \( (a_n b)^k \). We will preprocess an example \( e \) by constructing an assignment \( \eta_e \) to these variables: \( v_{ij} \) will be true in \( \eta_e \) if and only if \( K \vdash B_{c_i} \sigma_j \theta_e \). This means that Equation 1 is true exactly when the DNF formula

\[ (a_n b)^a \land \bigwedge_{j=1}^{l} \bigwedge_{i=1}^{n_b} v_{c_{ij}} \]

is true; hence the clause \( CI \) can be emulated by DNF over the \( v_{ij} \).

We can thus strengthen Theorem 3 as follows:

**Theorem 4** The family of languages \( C_{\text{FREE}} \) is predictable if and only if DNF is predictable.

This does not actually settle the question of whether indeterminate clauses are predictable, but does show that answering the question will require substantial advances in computational learning theory; the learnability and predictability of DNF has been an open problem in computational learning theory for several years.

**Learning recursive clauses**

Finally, we will consider learning a single recursive \( ij \)-determinate clause. A realistic analysis of recursive clauses requires a slight extension of our formalism: rather than representing an example as a single atom of arity \( n_e \), we will (in this section of the Complexity in Machine Learning 83
paper only) represent an example as a ground goal atom, \( e \), plus a set of up to \( n_e \) ground description atoms \( D = \{ d_1, \ldots, d_n \} \) of arity bounded by \( a \). A program \( P \) now classifies an example \(( e, D) \) as true iff \( P \land K \land D \mid e \). This also allows structured objects like lists to be used in examples; for example, an example of the predicate \( \text{member}(X, Y) \) might be represented as the goal atom \( e = \text{member}(b, \text{list\_ab}) \) together with the description

\[
D = \{ \text{head} \left( \text{list\_ab}, a \right), \text{tail} \left( \text{list\_ab}, \text{list\_b} \right), \text{head} \left( \text{list\_b}, b \right), \text{tail} \left( \text{list\_b}, \text{nil} \right) \}
\]

This formalism follows the actual use of learning systems like FOIL. Finally, in order to talk sensibly about one-clause recursive programs, we will also assume that the non-recursive “base case” of the target program is part of \( D \) or \( K \).

Again, the key result of this section is an observation about expressive power: there is a background theory \( K_n \) such that every log-space deterministic (DLOG) Turing machine \( M \) can be emulated, on inputs of size \( n \) or less, by a single recursive \( ij \)-determinate clause. Since DLOG Turing machines are cryptographically hard to predict, this will lead to a negative predictability result for recursive clauses.

Before describing the emulation, we will begin with some basic facts about DLOG machines. First, we can assume, without loss of generality, that the tape alphabet is \( \{0, 1\} \). Then a DLOG machine \( M \) accepting only inputs of size \( n \) can be encoded by a series of transitions of the following form:

If \( x_i = b \) and \( \text{CONFIG} = c_j \) then let \( \text{CONFIG} = c'_j \)

where \( x_i \) denotes the value of the \( i \)-th square of the input tape, \( b \in \{0, 1\} \), and \( c_j \) and \( c'_j \) are constants from a polynomial-sized alphabet \( \text{CON} = \{c_1, \ldots, c_{p(n)}\} \) of constant symbols denoting internal configurations of \( M \). One can also assume without loss of generality that there is a unique starting configuration \( c_0 \), a unique accepting configuration \( c_{\text{acc}} \), and a unique “failing” configuration \( c_{\text{fail}} \), and that there is exactly one transition of the form given above for every combination of \( i : 1 \leq i \leq n \), \( b \in \{0, 1\} \), and \( c_j \in \text{CON} - \{c_{\text{acc}}, c_{\text{fail}}\} \). On input \( X = x_1 \ldots x_n \) the machine \( M \) starts with \( \text{CONFIG} = c_0 \), then executes transitions until it reaches \( \text{CONFIG} = c_{\text{acc}} \) or \( \text{CONFIG} = c_{\text{fail}} \), at which point \( X \) is accepted or rejected (respectively).

To emulate \( M \), we will preprocess an example \( S = b_1 \ldots b_n \) into the goal atom \( e \) and the description \( D \) defined below:

\[
D = \{ \text{bit}_i(\text{news}_S, b_i) \}_{i=1}^n \cup \{ \text{tm}(\text{news}_S, c_{\text{acc}}) \}
\]

Now we will define the following predicates for the background theory \( K_n \). First, for every possible \( b \in \{0, 1\} \) and \( j : 1 \leq j \leq p(n) \), the predicate \( \text{stat}_{ij}(B, Y) \) will be defined so that given bindings for variables \( B \) and \( Y \), \( \text{stat}_{ij}(B, Y) \) will fail if \( Y = c_{\text{fail}} \); otherwise it will succeed, binding \( Y \) to \( \text{active} \) if \( B = b \) and \( Y = c_j \), and binding \( Y \) to \( \text{inactive} \) otherwise. Second, for \( j : 1 \leq j \leq p(n) \), the predicate \( \text{next}_{j}(Y, C) \) will be true if \( Y = \text{active} \) and \( C = c_j \), or if \( Y = \text{inactive} \) and \( C = c_{\text{acc}} \). It is easy to show that these definitions require only polynomially many facts to be in \( K_n \).

\[
\text{TRANS}_{ij} \equiv \text{bit}_i(S, B_{ij}) \land \text{stat}_{ij}(C, B_{ij}, Y_{ij}) \land \text{next}_{j}(Y_{ij}, C_{ij}) \land \text{tm}(S, C_{ij})
\]

Given \( K_n \) and \( D \), and assuming that \( S \) is bound to \( \text{news}_S \) and \( C \) is bound to some configuration \( c \), this conjunction will fail if \( c = c_{\text{fail}} \); otherwise, it will succeed trivially if \( x_i \neq b \) or \( c \neq c_j \); finally, if \( x_i = b \) and \( c = c_j \), \( \text{TRANS}_{ij} \) will succeed only if the atom \( \text{tm}(\text{news}_S, c_{ij'}) \) is provable. From this it is clear that the one-clause logic program

\[
\text{tm}(S, C) \leftarrow \bigwedge_{i \in \{1, \ldots, n\}} \bigwedge_{b \in \{0, 1\}} \bigwedge_{j \in \{1, \ldots, p(n)\}} \text{TRANS}_{ij}
\]

will correctly emulate the machine \( M \). Thus, letting \( R_{ij, \text{DET}} \) denote the language of one-clause \( ij \)-determinate recursive logic programs, and letting DLOG\(_n\) represent the class of problems on inputs of size \( n \) computable in deterministic space \( \log n \), we have the following theorem.

**Theorem 5** For all \( n \) there is a background theory \( K_n \) such that \( \text{DLOG}_n \subseteq R_{ij, \text{DET}}[K_n] \). Thus the family of languages \( R_{ij, \text{DET}}[X] \) is not polynomially predictable, and hence not pac-learnable, under cryptographic assumptions.

Although this result is negative, it should be noted that (unlike the case in the previous theorems) the preprocessing step used here distorts the distribution of examples: thus this result does not preclude the possibility of distribution-specific learning algorithms for recursive clauses.

**Concluding remarks**

This paper presented three negative results on learning one-clause logic programs. These negative results are stronger than earlier results [Kietz, 1993] in a number of ways; most importantly, they are not based on restricting the learner to produce hypotheses in some
designated language. Instead, they are obtained by showing that learning is as hard as breaking a secure cryptographic system.

In particular, we have shown that several extensions to the language of constant-depth determinate clauses lead to hard learning problems. First, allowing depth to grow as the log of the problem size makes learning a single determinate clause as hard as learning a log-depth circuit, which is hard under cryptographic assumptions; this shows that all learning algorithms for determinate clauses will require time worse than exponential in the depth of the target concept. Second, indeterminate clauses with $k$ “free” variables (a restriction of the language of constant-depth indeterminate clauses) are exactly as hard to learn as DNF; the learnability of DNF is a long-standing open problem in computational learning theory.

Finally, adding recursion to the language of indeterminate clauses makes them as hard to learn as a log-space Turing machine; again, this learning problem is cryptographically hard. There are however, learnable classes of one-clause linearly recursive clauses; a discussion of this is given in a companion paper [Cohen, 1993b].

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References


