Can we enforce full compositionality in uncertainty calculi?

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Abstract

At AAAI'93, Elkan has claimed to have a result trivializing fuzzy logic. This trivialization is based on too strong a view of equivalence in fuzzy logic and relates to a fully compositional treatment of uncertainty. Such a treatment is shown to be impossible in this paper. We emphasize the distinction between i) degrees of partial truth which are allowed to be truth functional and which pertain to gradual (or fuzzy) propositions, and ii) degrees of uncertainty which cannot be compositional with respect to all the connectives when attached to classical propositions. This distinction is exemplified by the difference between fuzzy logic and possibilistic logic. We also investigate an almost compositional uncertainty calculus, but it is shown to lack expressiveness.

1. Introduction

There is a very active research trend in Artificial Intelligence concerning the management of uncertainty in knowledge-based systems. This trend is still influenced by the MYCIN experiments (Buchanan & Shortliffe, 1984), where a basic idea was to attach weights expressing uncertainty to facts and rules in a knowledge base. Then we are faced with the problem of how to propagate these weights in reasoning procedures. This problem has usually been dealt with on a rule-by-rule basis, by splitting it into three subproblems: i) computing the weight bearing on a compo-site fact from the weights bearing on the elementary parts of this fact; ii) propagating the weight bearing on the conditions of the rule to the conclusion, by integrating the weight bearing on the rule; iii) combining the weights bearing on partial conclusions pertaining to the same matter. However, investigating the validity of such a method requires a proper interpretation of the weights. Reading the literature in this area, it appears that these weights may have two interpretations: degrees of uncertainty and degrees of partial truth and that people tend to make a confusion between these two notions. One of the reasons why this confusion was made is the need for a compositionality law for computing the resulting weights in the style of many-valued logics where all the connectives are usually truth-functional. Even degrees of probability are sometimes called degrees of truth (e.g., Nilsson, 1986) although probabilistic logic excludes compositionality.

The emergence of fuzzy rule-based systems in process control problems has led AI researchers, that criticized MYCIN-like systems, to reject fuzzy logic on the same grounds of dubious compositionality assumptions. For instance, in a recent paper, Elkan (1993) has questioned its well-foundedness and cast serious doubts on the reasons of its success, arguing that "fuzzy logic collapses mathematically to two-valued logic". This claim is in fact due to the use of too strong a notion of logical equivalence which is valid in two-valued logic, but which has nothing to do with fuzzy logic. Furthermore, Elkan (1993) does not mention the important distinction between two totally different problems to which fuzzy set-based methods apply, namely the handling of *gradual* (thus non-Boolean) properties whose satisfaction by a completely known state of facts is a matter of degree on the one hand, and the handling of uncertainty pervading Boolean propositions and induced by incomplete states of knowledge (which can be represented by means of fuzzy sets) on the other hand. The first problem can be addressed by means of a truth functional fuzzy (multiple-valued) logic, while the second one is the realm of possibility theory (Zadeh, 1978; Dubois & Prade, 1988a) which is a non-fully compositional uncertainty calculus (i.e., the degree of uncertainty of a compound proposition cannot systematically be computed from the degrees of uncertainty of its components only). Elkan (1993)'s paper is thus pervaded by the wrong but alas rather common idea that truth functional fuzzy logic has something to do with uncertainty handling. Assuming a fully compositional many-valued calculus on a Boolean algebra of propositions (a structure enforced by his equivalence requirement), the logical system collapses to two-valued logic.

This paper is a presentation of the authors' view on the problem of handling uncertainty and partial truth in the framework of information systems. In Section 2 we argue in favor of a clear distinction between (un)certainty and truth and propose a practical definition of truth based on approximate matching between a proposition and the description of a state of facts. Section 3 recalls the impossibility of a fully compositional calculus for dealing with uncertainty about Boolean propositions, and illustrates this impossibility result by comparing fuzzy logic and possibilistic logic. In Section 4 we investigate the possibility of an *almost* fully compositional uncertainty calculus, but its expressiveness turns to be very limited.

2. Partial Truth vs. Uncertainty

The distinction between degrees of truth and degrees of uncertainty goes back to De Finetti (1936), and seems to have been almost completely forgotten by Artificial Intelligence people. The confusion pervading the relationship between truth and uncertainty in the expert systems literature is apparently due to the lack of a dedicated paradigm for interpreting partial truth, and grades of uncertainty in a single framework. Such a paradigm can derive from a commonsense view of truth, *as the compatibility between a statement and reality*. This naïve definition of truth has been criticized by philosophers (see, e.g., Gochet in his discussion of Dubois & Prade (1988b)) but can be suitably modified by changing the debatable word "reality" into "what is known about reality" and interpreting the latter as "the description of some actual state of facts as stored in a data base". Hence computing the degree of truth of a statement S comes down to estimating its conformity with the description of what is known of the actual state of facts. As a consequence, truth evaluation comes down to a semantic matching procedure. This point of view is in accordance with Zadeh (1982) test-score semantics for natural languages. Four interesting situations can be encountered.

a) Classical two-valued logic. In order to compute truth-values, we need a precise definition of what "proposition" means. This is a matter of convention. The usual convention is that a proposition is identified with a set of "possible worlds" or "states of fact". Moreover a proposition is said to be true if and only if the actual state of facts is one of those which the proposition encompasses. By convention a proposition is true or false. If the actual state of facts is known and encoded as an item d in a database, the truth-value $\tau_d(S)$ (=1 (true) or 0 (false)) of a proposition S in a state of facts d can be computed.

b) Partial truth. This convention can be changed. Instead of defining a proposition as a binary entity that fits the actual state of fact or not, we can decide to use a more refined scale to evaluate the compatibility between a proposition S and a precisely known state of facts d. This is usual in natural language. For instance, the compatibility of "a tall man", with some individual of a given size is often graded: the man can be judged *not quite* tall, *somewhat* tall, *rather* tall, *very* tall, etc. Changing the usual true/false convention leads to a new concept of proposition whose compatibility with a given state of facts is a matter of degree, and can be measured on a scale L that is no longer {0,1}, but the unit interval for instance. It reflects linguistic levels such as "somewhat", "rather", "very", etc. This kind of convention leads to identifying a "fuzzy proposition" S with a fuzzy set of possible words; the degree of membership of a possible world to this fuzzy set evaluates the degree of fit between the proposition and the state of facts it qualifies. This degree of fit $\tau_d(S) \in L$ is called degree of truth of proposition S in the possible world d. Many-valued logics, especially truth-functional ones, provide a calculus of degrees of truth, including degrees between "true" and "false".

c) Uncertainty. On the other hand, even if we keep the convention that a proposition is either true or false, it is not always possible to determine whether it is *actually* true or false in given circumstances, because the actual state of facts is not known. In such a situation, we face uncertainty. Clearly uncertainty is a meta-level concept with respect to truth, since the uncertainty bears on whether a proposition is true or false (and nothing else). Moreover uncertainty is knowledge-dependent, i.e., refers to an agent. If uncertainty is encoded in a binary way, there can be only 3 situations: the agent is sure that S is true, he is sure that S is false, or he does not know. This last situation does *not* correspond to a third truth-value but to a suspended choice. More refined models of uncertainty will use an ordered scale U (again, the unit interval [0,1] usually) then $g(S) \in U$ will express to what extent one is sure that S is true, and $g(\neg S)$ to what extent one is sure that S is false. A typical example of degree of uncertainty is a degree of probability. Then our imperfect knowledge of the actual state of facts is modelled via a probability distribution over possible worlds, g(S) being the probability of the set of possible worlds identified with S. In other situations our knowledge of the actual state of facts is described by a set K of propositions that are believed as being true by some agent (what Gärdenfors (1988) calls a belief set). The available knowledge is then described by the set D of possible worlds where all propositions in K are true; a proposition S is surely true if $D \subseteq S$, surely false if $D \subseteq \neg S$ (the complement of S) and S is uncertain if $D \cap S \neq \emptyset$, $D \cap \neg S \neq \emptyset$. This is again the crude trichotomy mentioned above in the presence of incomplete knowledge. Between this crude model, and the sophisticated, additive probabilistic approach to uncertainty lies a third more qualitative approach. Suppose that the set D of possible states of fact is ordered in terms of plausibility, normality and the like. Then D can be viewed as a fuzzy set of possible states of facts. The overlapping between D and the ordinary set of possible worlds identified with a proposition S will be a matter of degree. This is possibility theory that handles two degrees $\Pi(S)$ and N(S) attached to S, respectively the possibility and the necessity that S is true. $\Pi(S) = 1$ means that S is true in one of the most plausible worlds in D. $N(S) = 1$ means that S is true in all possible worlds in D. Total ignorance on the truth value of S is expressed by $\Pi(S) = 1$, $N(S) = 0$. Moreover $N(S) = 1 - \Pi(\neg S)$ while $P(\neg S) = 1 - P(S)$ in the probabilistic approach. Note that the presence of uncertainty does not affect the truth-value scale which is always {0,1}: degrees of uncertainty are not truth-values.

d) Uncertain partial truth. In that case truth may altogether be a matter of degree and may be ill-known. Then, all values $\alpha = \tau_d(S)$ such that d is compatible with the available information D, is a candidate truthvalue for S. When both S and D can be expressed as fuzzy sets, we can consider for each truth-value $\alpha \in L$, such that $\alpha = \tau_d(S)$ a degree of possibility $\mu_D(d)$ that α is the truth-value of S. This fuzzy set of more or less possible truth-values forms a so-called fuzzy truth-value (Zadeh, 1979) denoted $\tau_D(S)$. A fuzzy truth-value combines the ideas of partial truth and of uncertainty about truth. It is thus a more complex construct than degrees of truth and degrees of uncertainty. Changing the fuzzy set D into a probability distribution on possible worlds, $\tau_D(S)$ becomes a random truth-value over a non-binary truth set L.

A standard analogical example that points out the difference between degrees of truth and degrees of uncertainty is that of a bottle. In terms of binary truth-values, a bottle is viewed as full or empty. If one accounts for the quantity of liquid in the bottle, one may say the bottle is "half full" for instance; under this way of speaking "full" becomes a fuzzy predicate and the degrees of truth of "The bottle is full" reflects the amount of liquid in the bottle. The situation is quite different when expressing our ignorance about whether the bottle is either full or empty (given that we know only one of the two situations is the true one). To say that the probability that the bottle is full is 1/2 does not mean that the bottle is half full. Degrees of uncertainty are clearly a higher level notion than degrees of truth.

3. Fuzzy Logic vs. Possibilistic Logic

3.1. The Compositionality Problem

An important consequence of the above distinction between degrees of truth and degrees of uncertainty is that degrees of uncertainty bearing on classical propositions cannot be compositional for all connectives. Namely there cannot exist operations $\oplus$ and $*$ on [0,1], nor negation functions f such that $g(S) \neq 0,1$ for some S, $g(\top) = 1$, $g(\bot) = 0$ and the following identities simultaneously hold for all classical propositions S_1, S_2, S

$$g(\text{not } S) = f(g(S)) \quad (1) \quad ; \quad g(S_1 \wedge S_2) = g(S_1) * g(S_2) \quad (2)$$
$$g(S_1 \vee S_2) = g(S_1) \oplus g(S_2) \quad (3)$$

This result is proved independently in (Dubois & Prade, 1988b) and (Weston, 1987). A family of propositions represented by a classical language form a Boolean algebra. The lack of compositionality is then a direct consequence of the well-known fact in mathematics that a non-trivial Boolean algebra that is linearly ordered has only two elements. However weak forms of compositionality make sense; for instance $\Pi(S_1 \vee S_2) = \max(\Pi(S_1), \Pi(S_2))$ in possibility theory, but generally, $\Pi(S_1 \wedge S_2) < \min(\Pi(S_1), \Pi(S_2))$; $\Pi(S_1 \wedge S_2) = \min(\Pi(S_1),\Pi(S_2))$ holds only for logically independent propositions; see Sec. 3.3. Similarly with grades of probability where $P(S) = 1 - P(\text{not } S)$ but $P(S_1 \wedge S_2) = P(S_1) \cdot P(S_2)$ only in situations of stochastic independence. The above impossibility result is a new way of stating a well known fact, i.e., that the unit-interval is *not* a Boolean algebra. It rejects many usual uncertainty handling compositional techniques currently used in expert systems into ad-hocery.

This result is based on the assumption that the propositions to evaluate are not fuzzy ones. By contrast, truth values of fuzzy propositions can be compositional when they can be precisely evaluated (i.e., under complete information). This is because closed sets of fuzzy propositions are no longer Boolean algebras but form weaker structures compatible with the unit interval. For instance, using max, min, $1 - (\cdot)$ for expressing disjunction, conjunction and negation of fuzzy propositions equips sets of such propositions with a distributive lattice structure that is compatible with the unit interval; this structure is the only one where all laws of Boolean algebra hold except the laws of non-contradiction and of excluded middle (Bellman & Giertz, 1973). Sometimes, arguments against fuzzy set theory rely on the impossibility of compositionality (e.g., Weston, 1987; Elkan, 1993). Usually these arguments are based on the wrong assumption that the algebra of propositions to be evaluated is Boolean. Note that fuzzy truth values (case d above) are not truth- functional, generally.

3.2. Fuzzy Logic Equivalence is not Classical

Elkan (1993) claims that in fuzzy logic the four following requirements hold for any propositions S_1 and S_2, τ being a truth assignment function such that $\forall S, \tau(S) \in [0,1]$

$$\tau(S_1 \wedge S_2) = \min(\tau(S_1),\tau(S_2)) \quad (4)$$
$$\tau(S_1 \vee S_2) = \max(\tau(S_1),\tau(S_2)) \quad (5)$$
$$\tau(\neg S) = 1 - \tau(S) \quad (6)$$
$$\tau(S_1) = \tau(S_2) \text{ if } S_1 \text{ and } S_2 \text{ are logically equivalent.} \quad (7)$$

While (4)-(5)-(6) are indeed the basic relations governing degrees of truth in fuzzy logic (as well as fuzzy set membership degrees) as proposed by Zadeh (1965), requirement (7) where "logically equivalent" is understood in a stronger sense than the equivalences induced by (4)-(5)-(6) has never been seriously considered by any author in the fuzzy set literature (up to a few erroneous papers which may always exist in a large corpus of publications). Indeed assuming that degrees of truth can be intermediary between 0 and 1, the propositions under consideration are no longer classical ones. Hence logical equivalence must be redefined from scratch. (7) should be understood the other way around: "S_1 is equivalent to S_2" means $\tau(S_1) = \tau(S_2)$ in all possible worlds. Obviously some classical logic equivalences still hold with fuzzy propositions obeying (4)-(5)-(6), namely the ones allowed by the De Morgan's structure induced by (4)-(5)-(6), as for instance

$S \wedge S \equiv S$; $S \vee S \equiv S$ (idempotency)
$S_1 \wedge (S_2 \vee S_3) \equiv (S_1 \wedge S_2) \vee (S_1 \wedge S_3)$;
$S_1 \vee (S_2 \wedge S_3) \equiv (S_1 \vee S_2) \wedge (S_1 \vee S_3)$ (distributivity).

But other Boolean equivalences *do not* hold, for instance

$S \wedge \neg S \not\equiv \bot$ since (4) and (6) *only* entail
$$\tau(S \wedge \neg S) = \min(\tau(S), 1 - \tau(S)) \leq 1/2$$

$S \vee \neg S \not\equiv T$ since (5) and (6) *only* entail

$$\tau(S \vee \neg S) = \max(\tau(S), 1 - \tau(S)) \geq 1/2$$

where $\tau(\bot) = 0$ and $\tau(T) = 1$. Indeed the failure of contradiction and excluded-middle laws is typical of fuzzy logic as emphasized by many authors. This is natural with gradual properties like 'tall'. For instance, in a given context, somebody who is 1.75 meter tall, may be considered neither as completely tall (i.e., tall with degree 1) nor as completely not tall (i.e., tall with degree 0); in this case we may have, for example, $\mu_{tall}(1.75) = 0.5 = \mu_{\neg tall}(1.75)$.

Idempotency is thus preserved by using min and max for intersection and union respectively but not the excluded middle and contradiction laws. If we change the truth-functions in (4, 5, 6), we change the structure of the set of propositions (hence the underlying conventions). For instance using $\max(0, a + b - 1)$ in (4) and $\min(a + b, 1)$ we recover the laws of excluded middle and of non-contradiction but we lose idempotency of conjunction and disjunction. Indeed, the laws of excluded middle and non-contradiction are not consistent with idempotency of conjunction and disjunction, when truth is no longer a binary notion (Dubois & Prade, 1980). Note that Elkan (1993) finds it natural to require that propositions $\neg(A \wedge \neg B)$ and $B \vee (\neg A \wedge \neg B)$ be equivalent, and shows that this requirement is incompatible with the convention of propositions having more than 2 truth-levels in the presence of (4, 5, 6). Many-valued logics are trivialized by this result only insofar as the proposed equivalence is so intuitively compelling that any fuzzy logic system should adopt it. The intuitive appeal of this equivalence is far from striking since in the presence of (4, 5, 6), $\neg(A \wedge \neg B) \equiv \neg A \vee B$ and $B \vee (\neg A \wedge \neg B) \equiv (\neg A \vee B) \wedge (B \vee \neg B)$. Elkan's suggested equivalence is clearly related to the acceptance of the excluded middle law (for B), a unusual requirement in fuzzy logic.

3.3. Possibility and Qualitative Uncertainty

The presence or absence of compositional rules is thus a criterion to distinguish between logics of graded truth (that handle vague propositions under complete information) and logics of uncertainty (that handle usual propositions under incomplete information). This is well exemplified by the distinction between fuzzy logic and possibilistic logic.

Fuzzy sets can be used not only for modelling the gradual nature of properties but can also be used for representing incomplete states of knowledge. In this second use, the fuzzy set plays the role of a possibility distribution which provides a complete ordering of mutually exclusive states of the world according to their respective levels of possibility or plausibility. For instance, if we *only know* that "John is tall" (but not his precise height), where the meaning of 'tall' is described in the context by the mem-bership function of a fuzzy set, i.e., μ_{tall}, then the greater $\mu_{tall}(x)$ is, the greater the possibility that height(John) = x and the smaller $\mu_{tall}(x)$, the smaller this possibility.

Given a [0,1]-valued possibility distribution π describing an incomplete state of knowledge, Zadeh (1978) defines a so-called possibility measure Π such that

$$\Pi(S) = \sup\{\pi(x), x \text{ makes } S \text{ true}\} \tag{8}$$

where S is a *Boolean* proposition, i.e., a proposition which can be true or false only. It can be easily checked that for Boolean propositions S_1 and S_2, we have

$$\Pi(S_1 \vee S_2) = \max(\Pi(S_1), \Pi(S_2)) \tag{9}$$

but *only* $$\Pi(S_1 \wedge S_2) \leq \min(\Pi(S_1), \Pi(S_2)) \tag{10}$$

in the general case (equality holds when S_1 and S_2 are *logically independent*). Indeed if $S_2 \equiv \neg S_1$, $\Pi(S_1 \wedge S_2) = \Pi(\bot) = 0$, while $\min(\Pi(S), \Pi(\neg S)) = 0$ only if the information is sufficiently complete for having either $\Pi(\neg S) = 0$ (S is true) or $\Pi(S) = 0$ (S is false). If nothing is known about S, we have $\Pi(S) = \Pi(\neg S) = 1$. By duality, a necessity measure N is associated to Π according to the relation (which can be viewed as a graded version of the relation between what is necessary and what is possible in modal logic)

$$N(S) = 1 - \Pi(\neg S) \tag{11}$$

which states that S is all the more necessarily true as $\neg S$ has a low possibility to be true. It entails

$$N(S_1 \wedge S_2) = \min(N(S_1), N(S_2)) \tag{12}$$

and $$N(S_1 \vee S_2) \geq \max(N(S_1), N(S_2)). \tag{13}$$

Observe also that neither Π, nor N, are fully compositional with respect to $\wedge$, $\vee$ and $\neg$. Possibilities are only compositional with respect to disjunction, necessities with respect to conjunction. The equalities (9), (11) and (12) should not be confused with (5), (6) and (4) respectively. In (9), (11), (12) we deal with Boolean propositions pervaded with uncertainty due to incomplete information, while (4)-(5)-(6) pertain to non-Boolean propositions whose truth is a matter of degree (the information being assumed to be complete). This distinction is a crucial prerequisite in any discussion about fuzzy sets and possibility theory and their use in automated reasoning.

Possibility measures have been shown (Dubois, 1986; Dubois & Prade, 1991) to be the numerical counterpart of so-called qualitative possibility relations $\geq$ (where $S_1 \geq S_2$ reads "S_1 is at least as possible as S_2"), in the sense that $\forall \Pi$, $\exists$ an ordering $\geq$ such that $\forall S_1, \forall S_2, \Pi(S_1) \geq \Pi(S_2) \Leftrightarrow S_1 \geq S_2$. The ordering $\geq$ is supposed to be reflexive, complete ($S_1 \geq S_2$ or $S_2 \geq S_1$), transitive, non-trivial ($T > \bot$), such that $\forall S, T \geq S$ (certainty of tautology) and to satisfy the characteristic axiom

$$\forall S_1, \text{ if } S_2 \geq S_3 \text{ then } S_1 \cup S_2 \geq S_1 \cup S_3.$$

For qualitative necessity the above axiom is changed, for the corresponding ordering, by substituting $\cap$ to $\cup$. This shows the qualitative nature of possibility and necessity measures.

The case d of Section 2 which combines the case of fuzzy statements and of incomplete information can be also handled in the possibilistic framework. Let μ_D be the membership function of the fuzzy set representing the available information and μ_S be the one representing the fuzzy statement. The degree of truth $\tau_D(S)$ is then itself a fuzzy set of [0,1], which can be interpreted as a fuzzy truth-value, whose membership function is defined by

$$\mu_{\tau_D(S)}(v)=\sup_d\{\mu_D(d) \mid \mu_S(d)=v\}; \mu_{\tau_D(S)}(v)=0 \text{ if } \mu_S^{-1}(v)=0$$

i.e., $\mu_{\tau_D(S)}(v)$ is the grade of possibility that the degree of truth of S is equal to v knowing that the state of facts is restricted by D. The fuzzy truth value $\mu_{\tau_D(S)}$ can be approximated by means of two numbers $\Pi(S)$ and N(S), which extend (8) and its dual to the case of a fuzzy statement S (Dubois & Prade, 1985), namely with $\pi = \mu_D$

$$\Pi(S) = \sup_d \min(\mu_S(d), \pi(d))$$
$$N(S) = 1 - \Pi(\neg S) = \inf_d \max(\mu_S(d), 1 - \pi(d)).$$

Indeed $\Pi(S)$ and N(S) can be viewed as the degrees of possibility and necessity that S is "true", if we interpret "true" by extending its definition from {0,1} (i.e., $\mu_{true}(1) = 1$, $\mu_{true}(0) = 0$) to [0,1] by letting $\mu_{true}(v) = v$, $\forall$ v $\in$ [0,1]. We then have in any case

$$\Pi(S) = \sup_v \min(\mu_{\tau_D(S)}(v), v)$$
$$N(S) = \inf_v \max(1 - \mu_{\tau_D(S)}(v), v).$$

4. Almost Preserving Compositionality

As said above a measure of uncertainty defined on a Boolean algebra and taking its values in the interval [0,1] cannot be fully compositional with respect to all the logical connectives, just because we cannot equip [0,1] with a structure of Boolean algebra. However we may try to preserve compositionality *as far as possible*. Recently Schwartz (1992) has proposed a logic of likelihood governed by the following laws, for all S, S_1, S_2

$$g(\neg S) = 1 - g(S);\ g(S_1 \vee S_2) = \begin{cases} 1 \text{ if } S_1 \vee S_2 = T \\ \max(g(S_1), g(S_2)) \text{ if not}; \end{cases}$$
$$g(S_1 \wedge S_2) = \begin{cases} 0 \text{ if } S_1 \wedge S_2 = \bot \\ \min(g(S_1), g(S_2)) \text{ if not.} \end{cases}$$

Such a measure of likelihood g is as compositional as possible. Note that these likelihood set-functions are self-dual. Moreover only operations with a qualitative flavor are used to combine the likelihood degrees. Only a totally ordered set equipped with an order reversing involution is required as a likelihood scale. In the following we investigate what is the power of expressiveness of these measures of likelihood, in the finite case.

Let $\Omega = \{\omega_1, \ldots, \omega_n\}$ be the finite set of atoms of the Boolean algebra 2^Ω. Let $g(\{\omega_i\}) = g_i \in [0,1]$. We have

$$\forall i, g_i = 1 - g(\Omega - \{\omega_i\}) = 1 - \max_{j \neq i} g_j = \min_{j \neq i} (1 - g_j).$$

If $\exists i$, $g_i = 1$ then $g(\Omega - \{\omega_i\}) = 0$ and then $\forall j \neq i$, $g_j = 0$. Thus it corresponds to the *deterministic* case.

Let us suppose that $\exists i$, $g_i = \alpha \in (0,1)$. Then

$$\forall j \neq i, g_j \leq \max_{k \neq i} g_k = g(\Omega - \{\omega_i\}) = 1 - g_i = 1 - \alpha.$$

Let us suppose that $\alpha = g_1 \geq g_2 \geq \ldots \geq g_n$. Then

$$g_2 = 1 - g(\Omega - \{\omega_2\}) = 1 - \max_{j \neq 2} g_j = 1 - g_1 = 1 - \alpha.$$

Since g_1 is the maximal level, it follows that $\alpha \geq 1/2$. Similarly we have: $g_3 = 1 - \max(g_1, g_2, g_4, \ldots, g_n) = 1 - \alpha, \ldots, g_n = 1 - \alpha$. Thus if $g_1 < 1$, we can only have $1 > g_1 \geq g_2 = g_3 = \ldots = g_n = 1 - g_1 > 0$. So we can only describe a *pseudo-deterministic* situation where $\exists i$, $g_i = \alpha \geq 1/2$, and $\forall j \neq i$, $g_j = 1 - \alpha \leq 1/2$. In particular, total uncertainty is described by $\forall i$, $g_i = \alpha = 1 - \alpha = 1/2$.

In this calculus, we only have at most four certainty levels corresponding respectively to the complete certainty of truth (1), the likelihood of truth (L), the unlikelihood of truth (UL = 1 – L), and the complete certainty of falsity. Especially this representation of uncertainty does not really need the unit interval since only a 4-element totally ordered set {0, UL, L, 1} is needed.

Thus this proposal corresponds to the most elementary logic of likelihood which can be imagined: there exists *one* alternative ω_0 which, without being necessarily completely certain, appears to be more likely than the others which are considered as having a smaller, undifferentiated level of likelihood. This seems to coincide with the "simplified English probabilistic logic" considered by Aleliunas (1990); this logic also distinguishes between the four levels: 0 (certainly false), unlikely, likely, 1 (certainly true).

It is interesting to see whether likelihood measures induce a comparative probability ordering on events. Namely a comparative probability ordering $\geq$ is such that $\geq$ is complete and transitive, $S \geq \emptyset$, $\forall S \subseteq \Omega$, and $\geq$ satisfies the additivity axiom (Fine, 1973): $\forall S_1$,

$$\text{if } S_1 \cap (S_2 \cup S_3) = \emptyset, \text{ then } S_2 > S_3 \Leftrightarrow S_1 \cup S_2 > S_1 \cup S_3 \quad (14)$$

where $S_1 > S_2$ means $S_1 \geq S_2$ and not ($S_2 \geq S_1$). Any non-degenerate function g classifies the events in Ω into 4 classes of level 1, L, UL and 0 respectively. Namely $\exists \omega_0$ such that the class of level L is $\{S \neq \Omega, \omega_0 \in S\}$, the class of level UL is $\{S \neq \emptyset, \omega_0 \notin S\}$. The class of level 1 is $\{\Omega\}$ and the one of level 0 is $\{\emptyset\}$. Particularly we have, for $S_1 \neq S_2$, $S_1 > S_2$ if and only if $S_1 = \Omega$ or $S_2 = \emptyset$ or ($\omega_0 \in S_1$ and $\omega_0 \notin S_2$). Let us consider whether (14) holds:

- if $S_2 = \Omega$ then $S_1 = \emptyset$ and (14) is trivial. From now on $S_1 \neq \emptyset$;
- if $S_2 \neq \Omega$, $S_3 \neq \emptyset$ then assume $S_2 > S_3$, i.e., $\omega_0 \in S_2$, $\omega_0 \notin S_3$. Since $S_1 \cap S_2 = \emptyset$, $\omega_0 \notin S_1$. Hence $\omega_0 \notin S_1 \cup S_3$ and $S_1 \cup S_2 > S_1 \cup S_3$.
  Conversely assume $\Omega \neq S_1 \cup S_2 > S_1 \cup S_3$. Clearly $S_1 \cup S_3 \neq \emptyset$; we have $\omega_0 \in S_1 \cup S_2$, $\omega_0 \notin S_1 \cup S_3$.

Hence $\omega_0 \notin S_1$, and $\omega_0 \in S_2 - S_3$. Hence $S_2 > S_3$.

Assume now $S_1 \cup S_2 = \Omega > S_1 \cup S_3$ then since $S_1 \cap (S_2 \cup S_3) = \emptyset$, it follows that $S_3 \subseteq S_2$. If $\omega_0 \in S_2 - S_3$ then $g(S_1 \cup S_2) = 1 > g(S_1 \cup S_3) =$ UL and $g(S_2) = L > g(S_3) =$ UL. If $\omega_0 \in S_3$ we have $g(S_2) = g(S_3) = L$ and $g(S_1 \cup S_2) = 1 > g(S_1 \cup S_3) =$ L. Hence (14) fails when $S_1 \cup S_2 = \Omega$.

- when $S_3 = \emptyset$ then (14) fails too, if $\omega_0 \in S_1$ since then $g(S_2) > g(S_3)$ but $g(S_1 \cup S_2) = g(S_1 \cup S_3) = L$, generally.

As a consequence the likelihood measure *almost* satisfies the axioms of a comparative probability relation. It satisfies the following reasonable relaxation of additivity: $\forall S_1, S_2, S_3$ such that $S_1 \cup S_2 \neq \Omega$, $S_1 \cap (S_2 \cup S_3) = \emptyset$, $S_3 \neq \emptyset$: $S_2 > S_3 \Leftrightarrow S_1 \cup S_2 > S_1 \cup S_3$. This section gives an answer to the following question: how far can we go with a representation of uncertainty that tries to take advantage of truth-functionality as far as possible. It is shown here that, not only full truth-functionality is not possible, but retaining this property as much as mathematical consistency allows, leads to a very crude, almost deterministic model of uncertainty.

5. Concluding Remarks

The intended purpose of this paper is to emphasize the distinction between the treatment of gradual (or vague) predicates in presence of complete information which can be handled in a fully truth functional multiple-valued way (this is for instance the case for most of the applications in fuzzy control), and the handling of uncertainty for propositions which are either true or false (and which more generally may also have intermediary degrees of truth). In this second case, possibility theory offers a qualitative way for handling uncertainty which can be cast in a logical formalism (see, e.g., Dubois, Lang and Prade, 1991). Possibility theory, as probability theory and any uncertainty calculus is not fully compositional with respect to all connectives. It is still possible to enforce an almost fully compositional calculus for uncertainty, only at the high price of an important loss of expressiveness. On the whole we agree with Elkan (1993) on the point that the truth-functional fuzzy logic is not adapted to a proper handling of uncertainty in knowledge-based system. But our agreement is not based on an alleged self-inconsistency of fuzzy logic leading to a collapse. It is based on the fact that fuzzy logic offers a calculus of truth-values not of degrees of uncertainty. Especially there is no treatment of uncertainty in fuzzy controllers. Elkan's trivialization result kills truth-functional uncertainty handling systems, and certainly does not harm fuzzy logic nor the interpolation device at work in fuzzy controllers.

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