A New Proof of Tractability for ORD-Horn Relations

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Abstract

This paper gives an elementary proof of the tractability of a sub-class of temporal relations in Allen's algebra and related temporal calculi, the class of preconvex relations. In Allen's case, this subclass coincides with the class of ORD-Horn relations. Nebel and Bürckert defined ORD-Horn relations and proved that path-consistency is a sufficient condition for consistency of a network for this sub-class. We prove a stronger result: for each path-consistent network in the sub-class, we give an effective method for constructing a feasible scenario without backtrack.

Introduction

In a remarkable paper, Nebel and Bürckert (Nebel & Bürckert 1994) proved two results about a subclass of Allen's algebra, the class of ORD-Horn relations:

- A constraint network with labels in the ORD-Horn class is consistent if and only if it is path-consistent.
- The class of ORD-Horn relations is the maximal subclass of Allen's algebra containing the atomic relations, closed by non empty intersection, conversion and composition which is tractable.

This paper is concerned with the first result[1]. Nebel and Bürckert results raised three questions:

- The class of ORD-Horn relations is defined as the set of relations which can be represented by Horn clauses in a suitable language involving the end points of the intervals. This is not a very convenient definition for practical purposes.
- The proof of the consistency of path-consistent networks relies on the properties of Horn theories. The question whether an independent proof of the same fact exists seemed in order, especially since analogous questions arise in higher dimensional calculi.
- Given a temporal constraint network with relations in the ORD-Horn class which is path-consistent, are there any procedures for determining a feasible scenario without backtrack?

In a 1994 paper, Ligozat (Ligozat 1994) gives an answer to the first question. Namely, ORD-Horn relations are shown to have a very simple definition: Define the dimension of an atomic relation as 2 - #(coinciding boundaries). Among the 13 atomic relations, six relations are of dimension 2: p (precedes), o (overlaps), d (during), $p^{\smile}$, $o^{\smile}$, and $d^{\smile}$; equality eq is of dimension 0, the other six relations being of dimension 1. Then ORD-Horn relations are characterized as those relations which can be made into convex relations by adding only lower-dimensional atoms.

This gives a very simple criterion for testing whether a given relation belongs to the class. (Independently, Isli (Isli 1994) gave another criterion based on different principles). Another advantage of this definition is that it allows a simple proof of the fact that the class is *closed by composition*. By contrast, Nebel and Bürckert's original proof of this fact involved a very careful and detailed analysis of the logical situation.

The question whether the tractability of ORD-Horn relations could be connected to their "geometric" characterization remained open. Another aspect of the same question arises in higher dimensions: In the case of generalized intervals in the sense of (Ligozat 1990; 1991) (Allen's ordinary intervals are 2-intervals), analogous definitions of a ORD-Horn class and of a *pre-convex* class can be given too. The definition of the first class extends Nebel and Bürckert's, whereas the definition of the pre-convex class uses the geometric characterization mentioned above. However, for higher

[1] In another paper, we give an elementary proof of the basic fact used in Nebel and Bürckert's second result, namely the fact that any subclass in Allen's algebra which contains all atomic relations and has suitable closure properties contains one of four specific relations. This fact was obtained by Nebel and Bürckert using machine enumeration of all possible cases.

dimensions (n-intervals, for $n > 2$), the class of pre-convex relations is significantly bigger, as remarked in (Ligozat 1994). Hence, another question arises: if the tractability result extends at all, does it extend to the whole pre-convex class, or only to the ORD-Horn subclass?

In this paper, we give an elementary proof of tractability for the class of ORD-Horn (or equivalently, pre-convex relations) in Allen's algebra, and sketch briefly how the proof can be extended to the higher dimensional case.

In fact, we prove a stronger result, which is connected to the effective way of how a feasible scenario can be determined, given a path-consistent network. We show that the following strategy for choosing an atomic label on the network is always successful: choose successively, among the allowed atomic relations, a relation of the highest possible dimension.

As remarked by Bessière (personal communication) this result shows that the class of path-consistent pre-convex networks exemplifies an interesting property, which might be called *generic decomposability*, and is intermediate between minimality and decomposability.

Our proof is based on the following intuition about pre-convex relations: a pre-convex relation is basically a convex relation, i.e., it gives a convex constraint, excluding only some more particular atomic relations (that is, only the coincidency of some boundaries is disallowed). Hence, we can first relax the constraints a bit, and choose a scenario for the corresponding *convex* network, which is easily shown to be path-consistent, hence globally consistent. We cannot expect in general to get a scenario for the original network, and in fact do not, as the counter-example of van Beek (van Beek 1990) shows for the smaller class of pointizable relations. However, if the scenario is chosen to be sufficiently general, in the sense that boundaries will not be chosen to coincide unless we are compelled to do so, we might hope that it will in fact avoid the particular relations excluded by the original network.

Our proof shows that this strategy indeed works, and that dimension, which gives a measure of how many boundaries coincide, can be consistenly maximized in a path-consistent, pre-convex network.

The main result

Allen's algebra **IA** has 13 atomic relations, denoted by p (precedes), m (meets), o (overlaps), s (starts), d (during), their converses $p^\smile$, $m^\smile$, $o^\smile$, $s^\smile$, $d^\smile$, and eq (equals).

Basic operations in Allen's algebra are intersection, conversion, and composition. We denote by $(\alpha \circ \beta)$ the composition of α with β.

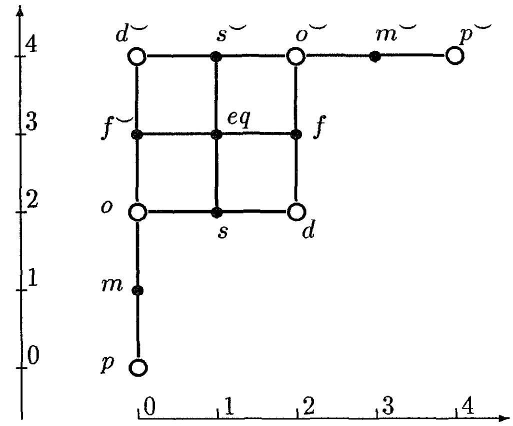

Figure 1: Atomic relations in Allen's algebra

In (Ligozat 1991), Ligozat introduced the *canonical* representation of Allen's atomic relations as pairs of integers between 0 and 4. This representation gives the set of atomic relations a partial order structure which makes it into a distributive lattice, with $p = (0,0)$ as the bottom element, and $p^\smile = (4,4)$ as the top element. Precedence in the lattice is defined componentwise: $(r_1, r_2) < (s_1, s_2)$ if and only if $r_1 < s_1$ and $r_2 < s_2$. (See Figure 1).

In that way, an arbitrary relation is represented as a subset of the lattice.

We consider two subclasses of relations in Allen's algebra: convex relations, and ORD-Horn, or pre-convex relations.

Convex relations are those relations which correspond to intervals in the lattice. For instance, $\{m, o, s, f^\smile, eq\}$ is a convex relation, because it contains all elements of the lattice between m and eq.

If r and s are two atomic relations such that $r \leq s$, we denote by $[r, s]$ the convex relation which contains all elements between r and s.

Referring to Fig. 1, notice that some atomic relations are represented by white circles (namely, p, o, d, $d^\smile$, $o^\smile$, $p^\smile$). Let us talk about white atomic relations about them, and refer to the other atomic relations as black relations.

Pre-convex relations are those relations which can be obtained from a convex relation by removing only black relations. For example, $\{o, s, f^\smile\}$ is a pre-convex relation, since it can be obtained from $[o, eq]$ by removing eq, which is a black relation. By contrast, $\{o, d, d^\smile\}$ is not pre-convex.

In (Ligozat 1994) Ligozat proved that pre-convex relations are exactly the same relations as ORD-Horn relations, in the sense of Nebel and Bürckert (Nebel &

Bürckert 1994).

We consider binary constraint networks on Allen's algebra.

Notations

A *binary constraint network* N on the interval algebra **IA** is defined as:

1. A set of nodes $(X_i), 1 \leq i \leq m$.

2. For each pair $(i,j), 1 \leq i,j \leq m$ a relation $\alpha_{i,j}$ in **IA**.

We assume that $\alpha_{j,i}$ is the converse of $\alpha_{i,j}$, and that $\alpha_{i,i}$ is equality, for all i.

A network is *path-consistent* if each label $\alpha_{i,j}$ contains at least an atomic relation, and $(\alpha_{i,j} \circ \alpha_{j,k}) \supseteq \alpha_{i,k}$ for all i, j, k, $1 \leq i,j,k \leq m$.

An *instantiation* of a variable X_i is a pair of real numbers (a_i, b_i), with $a_i < b_i$. If instantiations are given for X_i and X_j, there is exactly one atomic relation r such that the interval (a_i, b_i) is in relation r with respect to (a_j, b_j). We say that the two instantiations of X_i, X_j instantiate r.

A network is *globally consistent* if instantiations can be found for each X_i such that for each pair (i,j) an element in $\alpha_{i,j}$ is instantiated on the edge (i,j). Such a global instantiation is also called a *feasible scenario* for the network. Hence global consistency means that such a feasible scenario exists.

Path consistency amounts to the fact that, for each set of three nodes, and for any instantiation of two variables X_i and X_j instantiating an element in $\alpha_{i,j}$, there exists an instantiation of X_k which yields instantiations of $\alpha_{i,k}$ and $\alpha_{k,j}$ for the other two pairs of nodes.

One of the main results of Nebel and Bürckert's paper is the following:

Theorem 1 *Any path-consistent network labeled by ORD-Horn relations is consistent.*

Nebel and Bürckert's proof of this fact uses properties of Horn theories. By contrast, the main result of this paper is an elementary proof of the stronger statement:

Theorem 2 *Let* $N = ((X_i), (1 \leq i \leq m), (\alpha_{i,j}), (1 \leq i,j \leq m))$ *be a path-consistent network labeled by pre-convex relations. Then a feasible scenario can be obtained in the following way: Successively choose instantiations of the* X_i*s for* $1 \leq i \leq m$ *in such a way that for each* i, X_i *has as the maximal number of points distinct from* $\{a_k, b_k\}$, $k = 1, \ldots, i-1$ *allowed by the labels on* (i,k).

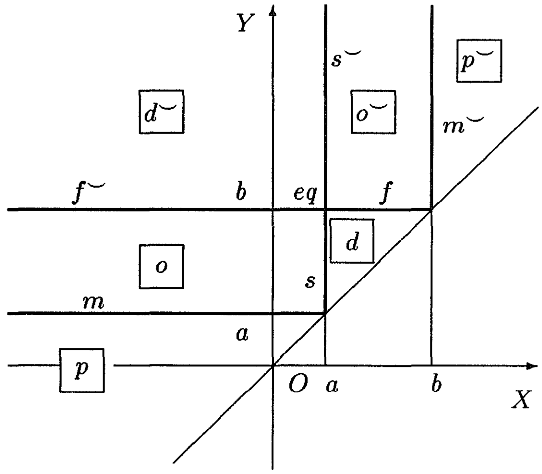

Figure 2: Regions associated to atomic relations in Allen's algebra

Subclasses of Allen's algebra

Allen's relations as regions in the plane

We already discussed the canonical representation of Allen's relations. Another representation of Allen's relations is in terms of regions in the Euclidean plane. Namely, an interval in Allen's framework is defined by a pair of real numbers (X, Y) such that $X < Y$. Hence the set of all intervals in that sense can be identified with the half plane **H** defined by the inequation $X < Y$ in the (X, Y)-plane. Let (a, b) be a fixed interval. Then, for each atomic relation r, (x, y) is in relation r with respect to (a, b) if and only if (x, y) belongs to a well defined region in the half plane which we denote by $reg(r, (a, b))$. For instance, the region associated to o is defined by $X < Y, X < a, a < Y < b$. (See Figure 2).

More generally, each relation α is represented by a region $reg(\alpha, (a, b))$ which is the union of $reg(r, (a, b))$ for $r \in \alpha$.

We define the *dimension* of a relation as the dimension of the associated region.

Clearly, among the atomic relations, p, o, d and their converses are of dimension 2; m, s, f and their converses are of dimension 1; eq is the only atom (hence the only relation) of dimension 0.

Topological closures

Because the regions associated to the relations in Allen's algebra are subsets in the plane, it makes sense to consider their topological closures in the half plane **H**. Clearly too, the closure of each region associated to an atom is obtained by adding all neighboring regions. For instance, the closure of (the region associated to) o

(see Figure 2) contains (the regions associated to) m, s, $f^{\smile}$ and eq.

We define the topological closure of a relation α as the set of relations associated to the closure of its associated region.

We denote by $C(\alpha)$ the topological closure of α. For instance, the preceding example shows that $C(o) = \{m, o, s, f^{\smile}, eq\}$.

The topological closure operator does not commute with composition. However, the continuity of composition is expressed by the following fact:

Proposition 1 *For all relations α, β, we have* $C(\alpha) \circ C(\beta) \subseteq C(\alpha \circ \beta)$.

Convex and pre-convex relations

We already defined convex relations as those relations which are represented by intervals in the canonical representation.

In terms of regions, a relation α is convex if and only if its associated region $reg(\alpha, (a, b))$ has convex projections, and coincides with

$$(pr_1(reg(\alpha, (a, b))) \times pr_2(reg(\alpha, (a, b)))) \cap \mathbf{H}$$

For any relation α, there is a smallest convex relation $I(\alpha)$ containing it. We call this relation the *convex closure* of α.

Convex closure obviously commutes with conversion. It also commutes with composition:

Proposition 2 *For any pair of relations α and β, we have* $I(\alpha \circ \beta) = I(\alpha) \circ I(\beta)$.

Corollary 1 *If N is a path-consistent network, then $I(N)$ is path-consistent.*

We can now give a alternative definition of preconvex relations in terms of convex and topological closures:

Definition 1 *A relation α is* pre-convex *if it satisfies one of the following equivalent conditions:*

1. *The topological closure of α is convex.*
2. $I(\alpha) \subseteq C(\alpha)$.
3. *$I(\alpha) \setminus \alpha$ is a union of relations of dimension smaller than $dim(\alpha)$.*

The following results are proved in (Ligozat 1994):

Proposition 3 *The class of pre-convex relations is closed by non empty intersection, conversion, and composition.*

Proposition 4 *The class of pre-convex relations in Allen's algebra coincides with the class of ORD-Horn relations.*

Regions associated to pre-convex relations

In the representation of relations as regions, pre-convex relations are "almost" convex, in the sense that their regions are obtained from those of convex relations by removing only lower dimensional pieces. Consider for instance the region associated to $\{o, s, f^{\smile}\}$ (Fig. 2). It is obtained from the region defined by $X < Y, X \leq a, a < Y \leq b$ by removing the point $X = a, Y = b$ corresponding to eq.

More precisely, we have the following properties:

Let $R = reg(\alpha, (a, b))$ be the region associated to a pre-convex relation α. Let $R' = reg(I(\alpha), (a, b)))$, where $I(\alpha)$ is the convex closure of α. Recall that $\mathbf{H}$ denotes the half plane defined by $X < Y$ in the (X, Y)-plane. Then:

1. $R' = (pr_1(R') \times pr_2(R')) \cap \mathbf{H}$.
2. R' is the convex closure of R.
3. $reg(I(\alpha), (a, b)) \setminus reg(\alpha, (a, b))$ is contained in the union of vertical and horizontal lines defined by $X = a, X = b, Y = a, Y = b$.
4. If (x, y) is such that $x \in pr_1(R')$, $y \in pr_2(R')$, and $\{x, y\}$ is disjoint from $\{a, b\}$, then (x, y) instantiates a relation of maximal dimension in R.

Fixed points of pre-convex relations

The following result is a crucial component of the proof:

Proposition 5 *Let α be a pre-convex relation of dimension d. If r is an atom of maximal dimension d in α, and r' any atomic relation in α, then all $2-d$ pairs of endpoints which coincide in r also coincide in r'.*

Proof

The conclusion is vacuous if $d = 2$. If $d = 0$, then $\alpha = eq$, hence $r = r' = eq$. Finally, a 1-dimensional pre-convex relation of dimension 1 is either m, $m^{\smile}$, or a subset of $[s, s^{\smile}]$, or a subset of $[f^{\smile}, f]$. Hence the conclusion holds in all cases (see alsoProposition 7).

Proof of the main result

Convex regions on the real line: Helly's theorem in dimension 1

The following result is a slightly more precise version of Helly's theorem (Chvátal 1983) in dimension 1:

Lemma 1 *Let $I_i, 1 \leq i \leq n$ be a finite family of convex subsets of the real line $\mathbf{R}$. If the subsets are pairwise intersecting, then the intersection I of all I_i is non empty.*

Moreover, either I is of dimension 1, or there exists i_0 such that $I = I_{i_0}$ (and I_{i_0} is a point), or there exist

i_1 and i_2 such that both I_{i_1} and I_{i_2} are intervals, and I is at the same time the right endpoint of I_{i_1} and the left endpoint of I_{i_2}.

Proof

Non empty convex subsets in the real line are either points, or intervals, which can then be finite or infinite, open, closed, or semi-closed. For each I_i, consider its left limit point and right limit point (using $-\infty$ and $+\infty$ if necessary).

Consider among the left limit points of all I_is the rightmost one u (by convention, we put $u = -\infty$ if all I_is begin at $-\infty$). Let v, with the same conventions, be the leftmost of all right limit points. Since we are dealing with a finite family, u is the left limit point of one subset in the family, say of I_{i_1}, and v is the right limit point of I_{i_2}. Now the intersection of I_{i_1} and I_{i_2} must be non empty. This is only possible if $u \leq v$.

If $u < v$, any I_i in the family has its left limit point less than or equal to u, and its right limit point greater than or equal to v. Hence any point in $]u, v[$ belongs to the intersection, which is of dimension 1.

If $u = v$, then u must belong to the subsets I_{i_1} and I_{i_2}. If one at least of them is a point, we can choose (one of) the corresponding index(es) as i_0. If none is, both are of dimension 1, and the lemma is proved.

The proof

We are now in a position to prove the main result:

Proposition 6 *Let N be a path-consistent network on the interval algebra with labels in the class of pre-convex relations. Let m be the number of nodes of the network. Then, for $1 \leq n \leq m-1$, each instantiation of n variables such that the resulting instantiations on the edges are of maximal dimension extends to an instantiation of $n+1$ variables satisfying the same condition.*

Assume that N is a path-consistent network labeled by pre-convex relations. Let $I(N)$ be the associated convex network.

Assume we have been able to choose instantiations $(a_1, b_1), \ldots, (a_n, b_n)$ of $X_1, \ldots, X_n$ in a way such that (a_i, b_i) is in relation $r_{i,j}$ with respect to (a_j, b_j), for $1 \leq i, j \leq n$, where $r_{i,j}$ is a relation in $\alpha_{i,j}$ of maximal dimension. We denote by E (the excluded set) the finite set $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$.

For each i, $1 \leq i \leq n$, consider the regions $R_i = reg(\alpha_{n+1,i}, (a_i, b_i))$ and $R'_i = reg(I(\alpha_{n+1,i}), (a_i, b_i))$.

Because of the path-consistency of N, for each pair (i, j), $1 \leq i, j \leq n$, R_i and R_j have a non empty intersection.

Consider first $pr_1(R'_i)$ and $pr_2(R'_i)$. For each coordinate, this set of projections is a finite family of convex subsets in $\mathbf{R}$. which are pairwise intersecting. Hence they have a non empty intersection, which is either of dimension 1, hence an interval, or a point. In any case, the intersection R' of all R'_is is a non empty convex region in the plane, and $R' = (pr_1(R') \times pr_2(R')) \cap \mathbf{H}$.

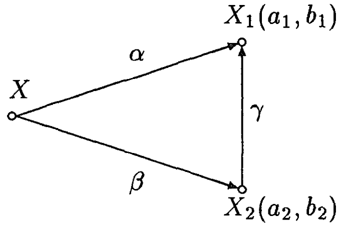

Figure 3: The triangle lemmas

Assume now that one of the projections of R' is a point a. Suppose for instance that it is the first projection.

Consider all the non empty convex regions $pr_1(R'_i)$. By lemma 1, either there exists i_0 such that $pr_1(R'_{i_0}) = \{a\}$, or there exists a pair of indexes i_1, i_2 such that $pr_1(R'_{i_1}) = (-, a]$ and $pr_1(R'_{i_2}) = [a, -)$.

In the following two lemmas, X, X_1, X_2 is a path-consistent network with three nodes. We call α, β, γ respectively the labels on (X, X_1), (X, X_2) and (X_2, X_1). We consider a given instantiation of γ by $X_1 = (a_1, b_1)$ and $X_2 = (a_2, b_2)$ (Fig 3).

Lemma 2 (First triangle lemma) *Assume that the labels are convex relations, and that the instantiation induced on (X_2, X_1) is of maximal dimension in γ.*

Then, if $R'_1 = reg(\alpha, (a_1, b_1))$ and $R'_2 = reg(\beta, (a_2, b_2))$, it cannot be the case that $pr_1(R'_1) = (-, a]$ and $pr_1(R'_2) = [a, -)$ (i.e. that both regions project onto intervals meeting at a).

Proof of the first triangle lemma

Suppose we are dealing with the first projection. Because of path-consistency, we can choose an instance (x, y) of X such that all constraints are satisfied. Hence, in particular, $x = a$. This implies that a is one of the endpoints of (a_1, b_1). By the same token, it is one of the endpoints of (a_2, b_2). Hence, for some p, q, $1 \leq p, q \leq 2$, the p-th endpoint of (a_1, b_1) is the same as the q-th endpoint of (a_2, b_2). By proposition 5, we know that this last condition remains true for all instantiations of γ.

Leaving (a_1, b_1) fixed, choose an instantiation (x, y) of X such that $x < a$, and such that the resulting relation is in α, which is possible because of the assumption on $pr_1(reg(\alpha, (a_1, b_1)))$. Path consistency implies that we should be able to choose (a'_2, b'_2) in such a way that the resulting relations on X, X_2 and X_2, X_1

belong to β and γ respectively. However, for any relation in γ, the p-th endpoint of (a_1, b_1) is the same as the q-th endpoint of (a'_2, b'_2). Now x is strictly smaller than this common point. But x should also belong to $pr_1(reg(\beta, (a'_2, b'_2)))$, whose left endpoint is the q-th endpoint of (a'_2, b'_2). Hence we get a contradiction. This proves the lemma.

Lemma 3 (Second triangle lemma) *Assume that the labels are pre-convex relations, and that the instantiation induced on (X_2, X_1) is of maximal dimension in γ.*

If for some projection k $pr_k(reg(\alpha, (a_1, b_1)))$ is a point a, with $a \in \{a_2, b_2\}$, then $pr_k(reg(\beta, (a_2, b_2))) = \{a\}$.

Proof of the second triangle lemma

We may assume that we are dealing with the first projection pr_1. By path-consistency, we can find an instantiation (x, y) of X such that a relation in α holds on (X, X_1), and a relation in β holds on (X, X_2). Hence $x = a$ must be one of a_1 and b_1: it is the p-th element of (a_1, b_1), for $1 \leq p \leq 2$. Assume that a is the q-th element in (a_2, b_2), $1 \leq q \leq 2$. Then the instantiation of γ we are considering makes the p-th element of (a_1, b_1) coincide with the q-th element of (a_2, b_2). By proposition 5, this must also be true for any other atomic relation in γ.

Now suppose that $pr_1(reg(\beta, (a_2, b_2)))$ contains a point $x' \neq a$. Keeping (a_2, b_2) as an instantiation of X_2, choose a point (x', y') in $reg(\beta, (a_2, b_2)))$ as an instantiation of X.

By path-consistency, we can instantiate X_1 by (a'_1, b'_1) in such a way that all constraints are met. Since some atomic relation in α holds, x' is the p-th element in (a'_1, b'_1). Because of γ, the p-th element of (a'_1, b'_1) coincides with the q-th element of (a_2, b_2), which is a. Hence we get a contradiction.

We can now give the proof of the main result. We reason on the dimension of R'.

R' is of dimension 2

In that case, both projections of R' are convex intervals. Hence each R'_i is also of dimension 2, which means that each $\alpha_{n+1,i}$ is of dimension 2. Choose an instantiation (a, b) in R' for X_{n+1} such that neither a nor b belongs to E. Then the resulting instantiation of each $I(\alpha_{n+1,i})$ is an atomic relation of dimension 2, because neither a nor b coincides with a_i or b_i. Moreover, because it is of maximal dimension, this atomic relation is in fact in $\alpha_{n+1,i}$.

R' is of dimension 1

Assume now that one of the projections of R' is a point a. Suppose for instance that it is the first projection.

Choose b in the second projection of R' such that b is not in E. Instantiate X_{n+1} as (a, b). We claim that this instantiation meets all requirements.

Firstly, by the first triangle lemma, there exists i_0 such that a is the first projection of R'_{i_0}.

Now, for each i, consider R'_i. Since R' is of dimension 1, each R'_i must be of dimension 1 or 2.

If it is of dimension 2, then both its projections are intervals. Using the second triangle lemma for the triangle (X_{n+1}, X_i, X_{i_0}), we conclude that a is neither a_i nor b_i. Since neither a nor b is in $\{a_i, b_i\}$, (a, b) instantiates a relation of dimension 2 which belongs to $\alpha_{n+1,i}$.

If R'_i is of dimension 1, its first projection must be a point (if its second projection were a point, the second projection of R' would be a point too). Hence it is a, which coincides with a_i or b_i. Since b is not in E, (a, b) instantiates a relation of dimension 1 in $I(\alpha_{n+1,i})$, which is itself of dimension 1, which implies that the instantiated relation is in fact in $\alpha_{n+1,i}$.

R' is of dimension 0

Hence R' is (a, b), for some a and b.

If R'_i is of dimension 2, the second triangle lemma implies that neither a nor b is one of a_i or b_i. Hence (a, b) instantiates a relation of dimension 2, which consequently in in $\alpha_{n+1,i}$ itself.

If R'_i is of dimension 1, one of its projections is a_i or b_i, while the other is of dimension 1. Using the second triangle lemma again, we conclude that this other projection is neither a_i nor b_i.

Finally, if R'_i is of dimension 0, we are instantiating equality, as we should.

This concludes the proof.

Higher dimensions

In this section, we briefly describe how the main result of this paper extends to the wider context of generalized intervals, as defined and studied in (Ligozat 1990; 1991). We recall some basic definitions.

(p, q)-relations and their representations

If n is a positive integer, a n-interval $X = (X_1, \ldots, X_n)$ in a total order T is a strictly increasing sequence $X_1 < \ldots < X_n$ of elements of T. Let p and q be two positive integers. We consider the set $\mathbf{\Pi_{p,q}}$ of atomic relations between a p-interval $X = (X_1, \ldots, X_p)$ and a q-interval $Y = (Y_1, \ldots, Y_q)$. Numbering $0, \ldots, 2q$ the zones defined by Y, we can encode the set of atomic relations we get in this way as the set of non-decreasing sequences of p integers between 0 and $2q$, which do not contain more than one occurrence of the same odd integer.

This encoding gives an embedding of $\mathbf{\Pi_{p,q}}$ in the Euclidean space $\mathbf{R^p}$. The product order on this set makes it into a distributive lattice (Ligozat 1991).

On the other hand, each relation in $\mathbf{\Pi_{p,q}}$ is associated with a region in $\mathbf{H}$, where $\mathbf{H}$ is the p-dimensional cone in $\mathbf{R^p}$ defined by $X_1 < X_2 < \cdots < X_p$.

In particular, the regions associated to pre-convex relations verify the same conditions as in Allen's case.

Pre-convex relations in $\mathbf{\Pi_{p,q}}$

The notions of convex relations ((Ligozat 1990; 1991)), topological closure and interval closure can be defined exactly in the same way as in the case of Allen's relations; hence we can define a *pre-convex* relation as a relation whose topological closure is convex.

The equivalent characterizations described by 1 are still valid.

Finding feasible scenarios for pre-convex relations

We can prove in this more general setting that finding a feasible scenario can be done by using the same strategy as in Allen's case, namely, by choosing on each edge of a pre-convex constraint network a relation of maximal dimension. A close examination of the proofs of the triangle lemmas shows that they are valid in this generalized setting. Both lemmas need the following fact, which generalizes Proposition 5 (we use the canonical representation of elements of $\mathbf{\Pi_{p,q}}$ as sequences of p integers between 0 and $2q$):

Proposition 7 *Let α be a pre-convex relation of dimension d in $\mathbf{\Pi_{p,q}}$, $r = (r_1, \ldots, r_p)$ an atomic relation in α of maximal dimension d. Then, for any atomic relation $r' = (r'_1, \ldots, r'_p)$ in α, $r_i = r'_i$ for each i such that r_i is an odd integer (which is the case for $p - d$ values of i).*

Corollary 2 *Under the same conditions, every pair of endpoints which is identified by r is also identified by r'.*

Conclusions

We considered the class of pre-convex relations in Allen's algebra. This class coincides with the class of ORD-Horn relations defined by Nebel and Bürckert. We showed that, for any path-consistent network with pre-convex labels, a feasible scenario can be obtained without backtrack by using a simple strategy. The proof of this fact, which implies Nebel and Bürckert's result about the tractability of ORD-Horn relations, uses the geometric properties of pre-convex relations. The basic fact is that pre-convex relations are almost as good as convex ones, in the sense that, if they are not entirely characterized by their projections, the difference is only in lower dimensions. Because of this property, most of the reasoning can be done componentwise, and ultimately reduces to Helly's theorem in dimension one, which is an elementary result.

We also sketched how this result extends to pre-convex relations between generalized intervals.

These results show that pre-convex, rather than ORD-Horn relations, are the appropriate object of study in higher dimension.

The techniques we used for temporal calculi can also be used in other frameworks, including some qualitative spatial calculi. For instance, an analogous (but simpler) proof shows that the class of pre-convex relations in the North/ South/ East/West calculus described in (Ligozat 1993) has similar properties.

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