Connection Based Strategies for Deciding Propositional Temporal Logic

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Abstract

Connection methods have proven their value for efficient automated theorem proving in classical logics. However, these methods have not been extended to temporal logics due to the lack of a subformula property in existing proof procedures. We show that a slightly looser generalized subformula property exists for temporal logics. We then exploit this generalized subformula property to develop a temporal notion of polarities and connections, upon which we base an efficient proof procedure for propositional temporal logic. The proof procedure is structured around semantic tableau augmented with connections, and we propose a number of connection-based strategies. The procedure achieves many of the benefits of connection methods. The method is also sufficiently general to be extensible to other temporal logics. Experimental results indicate substantial speedup resulting from this approach.

1 Introduction

Although temporal logics (TLs) have proven to be of great value for a number of AI applications, a major drawback is the lack of efficient proof procedures. Sistla shows that the satisfiability problem for even the simple subset of linear-time propositional temporal logic (PTL) with only the eventually operator is NP-complete, and the satisfiability problems for a number of other PTL subsets are P-SPACE complete [Sistla and Clarke 1985]. One common solution to this problem is to restrict the temporal logic to an executable subset requiring lower complexity. However, these restricted logics are often inconvenient due to limited expressive power, and their use is thus limited. In particular, such logics often lack the power to perform one of the most important applications of temporal logic – the modeling of temporal properties of hardware and software systems. An alternate approach is to develop more efficient proof procedures for the TLs. While this approach may not result in worst case complexity improvements, it often achieves great proof size reductions for many formulae.

Most TL proof procedures are based on semantic tableaux, which were first presented by Beth [Beth 1959] and later refined by Smullyan [Smullyan 1968]. In the semantic tableau procedure, a tableau is generated for the negation of the proposed theorem, and the resulting tableau is checked for satisfiability. The proof succeeds iff the corresponding tableau is unsatisfiable. If the tableau is satisfiable, satisfiable paths in the tableau constitute models for the negation of the theorem, and thus counterexamples for the proposed theorem. Most PTL proof procedures are based on the tableau procedure presented by Wolper [Wolper 1985], though there are also some techniques based on Büchi automata [Vardi and Wolper 1986], clausal resolution [Cavalli and del Cerro 1984, Fisher 1991] and non-clausal resolution [Abadi 1987]. Unfortunately, these procedures are of exponential complexity, and it is thus desirable to guide the proofs intelligently.

Gough modifies Wolper's procedure to generate a PTL tableau procedure with substantial node-count reduction, due largely to the identification of nodes containing formulae which are unnecessary for the proof [Gough 1989]. Similar procedures have been applied to other TLs including a PTL with both future and past time operators [Gough 1989], and a polymodal logic with temporal and belief modalities [Wooldridge and Fisher 1994]. Although such procedures do an excellent job at reducing node-counts using algorithmic methods, they are not generally amenable to strategies that may apply when only a partial tableau is needed. In practice, in most domains where TL is useful, a single model for satisfiable formulae often suffices. For example, when formally modeling systems, a satisfiable tableau indicates a bug in the system. It is very useful to be able to generate a model corresponding to the bug; however the set of all models is unnecessary. Moreover, theorem prover performance for satisfiable formulae may be more important than for unsatisfiable formulae if we expect to use the theorem prover during design stages.

Thus, it is desirable to generate a PTL tableau procedure that can intelligently guide proofs (and non-proofs). A common method for first-order logic (FOL) is to statically identify potentially contradictory subformulae, represent these potential contradictions as *connections*, and use these connections to generate strategies to guide the proof procedure. A FOL connection graph procedure was proposed in [Kowalski 1975]. The connection method, also known as the matrix method, was proposed by Bibel (for FOL), and a good introduction to this method is given in [Bibel 1993]. The connection method can be seen as a variant of semantic tableaux in which formulae are represented clausally as matrices, one clause per row, and each path through the matrix corresponds to a tableau branch. Advantages of the connection method include a smaller proof tree due to pruning of the proof tree, and more compact and efficient data structures eliminating the need for dynamic formula construction. In [Wrightson 1987], it is shown that the semantic tableau procedure for FOL can be augmented with links between terms to provide many of the advantages of connection methods.

Although matrix methods have been applied to modal logics ([Wallen 1990]), they are not readily adaptable to PTL since PTL tableau procedures typically do not possess the subformula property. In this paper, we present a tableau-based calculus for PTL and define a generalized subformula property possessed by this procedure. We also develop a temporal notion of polarity, from which connections can be generated. Based on the generalized subformula property and temporal connections, an efficient decision procedure for PTL is generated and presented. By using connections to guide the tableau proofs, this procedure achieves many of the benefits of connection methods.

The paper is organized as follows. Sections 2 and 3 define the PTL and a tableau procedure. Section 4 provides the definitions and theoretical basis for our technique. Section 5 presents the connection-based proof procedure. Section 6 compares our method with other approaches and summarizes results based on our PTL theorem prover. Finally, Section 7 briefly outlines potential extensions.

2 Propositional Temporal Logic

The language $\mathcal{L}$ of PTL is defined over an alphabet consisting of a countable set of atoms $\mathcal{V}$ and a set of operators. The operators include the standard $\top$, $\bot$, $\neg$, $\wedge$, and $\vee$, along with the temporal operators $\circ$ (next time), $\Box$ (always), $\Diamond$ (eventually), and $\mathcal{U}$ (strong until).

The semantics are as usual. A Kripke structure $\mathcal{K}$ for $\mathcal{L}$ is given by the triple $< S_0, S, \eta >$. S is a sequence of states, with initial state S_0. η is a set of mappings, η_i, where each η_i is a valuation function assigning Boolean values to the elements of $\mathcal{V}$ at state i.

The truth value of a formula is inductively defined as follows:

$K_i(\top) = t$		
$K_i(\bot) = f$		
$K_i(v) = \eta_i(v)$	for	$v \in \mathcal{V}$
$K_i(\neg P) = t$	iff	$K_i(P) = f$
$K_i(P \wedge Q) = t$	iff	$K_i(P) = t$ and $K_i(Q) = t$
$K_i(P \vee Q) = t$	iff	$K_i(P) = t$ or $K_i(Q) = t$
$K_i(\circ P) = t$	iff	$K_{i+1}(P) = t$
$K_i(\Box P) = t$	iff	$K_j(P) = t$ for every $j \geq i$
$K_i(\Diamond P) = t$	iff	$K_j(P) = t$ for some $j \geq i$
$K_i(P\mathcal{U}Q) = t$	iff	$K_j(Q) = t$ for some $j \geq i$ and $K_k(P) = t$ for every k such that $i \leq k < j$

Not all of these operators are needed since $\langle \wedge, \vee \rangle$ and $\langle \Box, \Diamond \rangle$ are dual pairs, and $\Box, \Diamond$ are derivable from U by the identities: $\Diamond P = \top\mathcal{U}P$ and $\Box P = \neg(\top\mathcal{U}\neg P)$. However, these operators are included for convenience. Other traditional temporal operators, such as *atnext*, *before*, *unless*, and *while* are also definable in terms of these operators.

3 A Semantic Tableau Calculus for PTL

The basic proof procedure is a variant of the one in [Wolper 1985]. There are two stages:

1. Construct the tableau as a directed graph[1].
2. Test the resulting graph for satisfiability.

Tableau construction proceeds by reducing leaf nodes according to a set of tableau rules. Tableau rules are represented in standard notation and may be linear or branching.

Example 1:

$(\langle name \rangle) \quad \dfrac{F}{F_{11},\ldots,F_{1p} \mid F_{21},\ldots,F_{2q}}$

If this rule is applied to the formula F, it produces two children: one with $F_{11}, F_{12}, \ldots$, and F_{1p}, and one with $F_{21}, F_{22}, \ldots$, and F_{2q} (the exact mechanics of node construction are discussed later). If the rule has only one child, it is said to be a linear rule.

The set of tableau rules is as follows:

(eta)	$\dfrac{\neg\neg P}{P}$	
(alpha)	$\dfrac{P \wedge Q}{P, Q}$	$\dfrac{\neg(P \vee Q)}{\neg P, \neg Q}$
(beta)	$\dfrac{P \vee Q}{P \mid Q}$	$\dfrac{\neg(P \wedge Q)}{\neg P \mid \neg Q}$
(nu)	$\dfrac{\Box P}{P, \circ\Box P}$	$\dfrac{\neg\Diamond P}{\neg P, \neg\circ\Diamond P}$
(pi)	$\dfrac{\Diamond P}{P \mid \circ\Diamond P}$	$\dfrac{\neg\Box P}{\neg P \mid \neg\circ\Box P}$
(tau)	$\dfrac{\neg(P\mathcal{U}Q)}{\neg Q, \neg P \mid \neg Q, \neg\circ(P\mathcal{U}Q)}$	
(upsilon)	$\dfrac{P\mathcal{U}Q}{Q \mid P, \circ(P\mathcal{U}Q)}$	
(sigma)	$\dfrac{\circ P_1,\ldots,\circ P_n, B_1,\ldots,B_m}{P_1,\ldots,P_n}$	

where $B_1, \ldots, B_m$ are all either atoms or negations of atoms. Formulae to which a tableau rule is applicable are referred to as formulae of that type (e.g. a formula preceded by a $\Box$ is referred to as a nu-formula).

Before presenting the tableau procedure, the following definitions are needed:

Definition 1 *An* **eventuality formula** *is a formula of the form $\Diamond Q$, $P\mathcal{U}Q$, or $\neg\Box Q$. Q is said to be the* **eventuality** *for the first two cases, and $\neg Q$ for the third case.*

Definition 2 *An* **elementary formula** *is a formula of the form $\circ P$, $\neg \circ P$, A, or $\neg A$ where A is atomic.*

An attempted proof of a formula is performed by negating the formula and constructing the tableau for the resulting formula, F, in accordance with the following algorithm:

1. Preprocess F to eliminate any $\bot$ and $\top$ symbols, using standard tautologies.
2. Initialize the tableau, $\mathcal{T}$ to a single node containing F.
3. Repeat until there are no unexpanded nodes:
   (a) Choose an unexpanded node $\mathcal{N} \in \mathcal{T}$.
   (b) If every unmarked formula in $\mathcal{N}$ is elementary, create a node $\mathcal{N}_1$ in accordance with the sigma rule.
   (c) Otherwise,
      - Select a non-elementary formula $F \in \mathcal{N}$.

[1]While the actual proof is a directed graph, we refer to them as trees in this paper since the terminology of trees is convenient.

- If the tableau rule for F is linear (i.e. eta, alpha, nu), then create a node $\mathcal{N}_1 = \mathcal{N} \bigcup \{F_1, ..., F_p\}$ where the F_i's are the formulae generated by the appropriate tableau rule.
- Otherwise, since the tableau rule for F must have 2 branches, create nodes $\mathcal{N}_1 = \mathcal{N} \bigcup \{F_{11}, ..., F_{1p}\}$ and $\mathcal{N}_2 = \mathcal{N} \bigcup \{F_{21}, ..., F_{2q}\}$ with each branch containing the formulae generated by the corresponding branch of the appropriate tableau rule.
- In any newly created nodes, mark the formula F.

(d) For each newly created node $\mathcal{N}_i$ which is already in $\mathcal{T}$, add an arc in $\mathcal{T}$ from $\mathcal{N}$ to $\mathcal{N}_i$. Otherwise ($\mathcal{N}_i \notin \mathcal{T}$), add $\mathcal{N}_i$ to $\mathcal{T}$ along with an arc from $\mathcal{N}$ to $\mathcal{N}_i$.

(e) Mark $\mathcal{N}$ expanded.

(f) For each newly added node, if it contains both F and $\neg F$ for some formula F, mark that node expanded.

A few details in the algorithm require further discussion. First, this procedure differs from traditional FOL tableaux since tableau rules produce new nodes instead of just new formulae. Second, formulae are marked instead of being deleted after they are expanded. The reasons for this will become clear later, since these formulae will be needed when checking whether the tableau is satisfiable. Finally, the sigma rule is applicable only to nodes for which other rules do not apply - i.e. nodes for which the only unmarked formulae are elementary formulae. Such nodes are referred to as **states**. As with traditional tableau, models can be read off tableau paths, with sigma rule decompositions corresponding to a forward movement of one time unit.

Since duplicate nodes are not created, the procedure results in a directed cyclic graph. To determine satisfiability, the following elimination rules are repetitively applied:

S1: If a node contains both a formula and its negation, eliminate that node.

S2: If a node which is either the root node or the child of a state contains an unsatisfiable eventuality formula (as defined below), eliminate that node.

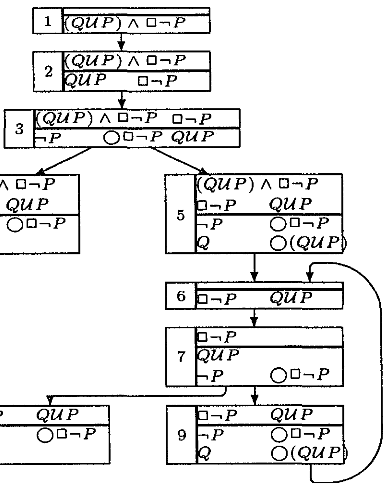
S3: If all the successors of a node have been eliminated, eliminate that node.

The second rule is needed so that an eventuality formula which is never satisfied isn't found satisfiable. Although both the eventually and (strong) until operators have semantics requiring an eventuality to be satisfied, the nature of the pi and upsilon rules allows the satisfaction of the eventualities to be infinitely delayed. Thus, the following rule is used to determine whether an eventuality is satisfiable:

An eventuality formula in a node $\mathcal{N}$ is satisfiable if there is a path in the tableau leading from $\mathcal{N}$ to to a node $\mathcal{N}_1$ containing the corresponding eventuality.

If the root node is eliminable, then the tableau is unsatisfiable and the original formula is proven; otherwise, a counterexample may be read off the tableau with the understanding that sigma rule applications correspond to a movement in time.

Elimination rule S2 differs somewhat from the other elimination rules in that it can't be applied dynamically since it requires the generation of the relevant children



1 $(Q\mathcal{U}P) \wedge \Box\neg P$
2 $(Q\mathcal{U}P) \wedge \Box\neg P$ $Q\mathcal{U}P$ $\Box\neg P$
3 $(Q\mathcal{U}P) \wedge \Box\neg P$ $\Box\neg P$ $\neg P$ $\bigcirc\Box\neg P$ $Q\mathcal{U}P$
4 $(Q\mathcal{U}P) \wedge \Box\neg P$ $\Box\neg P$ $Q\mathcal{U}P$ $\neg P$ $\bigcirc\Box\neg P$ P
5 $(Q\mathcal{U}P) \wedge \Box\neg P$ $\Box\neg P$ $Q\mathcal{U}P$ $\neg P$ $\bigcirc\Box\neg P$ Q $\bigcirc(Q\mathcal{U}P)$
6 $\Box\neg P$ $Q\mathcal{U}P$
7 $\Box\neg P$ $Q\mathcal{U}P$ $\neg P$ $\bigcirc\Box\neg P$
8 $\Box\neg P$ $Q\mathcal{U}P$ $\neg P$ $\bigcirc\Box\neg P$ P
9 $\Box\neg P$ $Q\mathcal{U}P$ $\neg P$ $\bigcirc\Box\neg P$ Q $\bigcirc(Q\mathcal{U}P)$

nodes. This distinction becomes important later in our proof procedure strategies.

Example 2:
A naive tableau for the formula $(Q\mathcal{U}P) \wedge (\Box\neg P)$ is illustrated above. Each node in the tableau is drawn with three components: an identification number on the left, a marked part on the top, and an unmarked part on the bottom. Nodes 4 and 8 are unsatisfiable by S1 and are immediately eliminated. Node 6 is the child of a state and is unsatisfiable by rule S2 since it contains $Q\mathcal{U}P$ which is unsatisfiable since no remaining descendant of node 6 contains P. Repeatedly applying rule S3, the root becomes unsatisfiable, and the proof is complete.

Theorem 1 *A formula is unsatisfiable iff the root node of its tableau is eliminable.*

Proof:
As with other proofs, we provide a brief sketch here and a complete proof in [Shankar 1997]. Soundness of non-temporal rules is as usual, while soundness of temporal rules follows from the PTL identities: $\Box P \equiv P \wedge \circ\Box P$ and $P\mathcal{U}Q \equiv Q \vee (P \wedge \circ(P\mathcal{U}Q))$. Soundness of the sigma rule follows from our interpretation of the state decompositions as movements in time in the resulting model. If the root is not eliminable, there must be a path through the tableau which contains no contradictions and satisfies all eventualities. This path constitutes a model. If the root is eliminable, a case analysis of elimination rules S1-S3 shows that the formula is unsatisfiable. $\Box$

Proof of termination relies on a generalized subformula property:

Definition 3 *Given an input formula, F, The set of (generalized)* **subformulae** *of F is the set of all subterms of F, possibly preceded with a $\neg$ and/or a $\circ$ symbol.*

The definition implicitly assumes that the $\neg$ and $\circ$ operators commute (as is the case). For the rest of this paper, the term subformula is used to refer to generalized subformulae.

Example 3:
Consider the formula $\Box P$. The set of subformulae is $\{P, \neg P, \circ P, \neg \circ P, \Box P, \neg\Box P, \circ\Box P, \neg \circ \Box P\}$.

The motivation for this definition is captured in the following theorems:

Theorem 2 *Any formula generated during a proof is a subformula of the input formula.*

Theorem 3 *The above proof procedure terminates.*

Proof:
Let n be number of subterms of the input. Then, the number of subformulae of the input is at most 4n. Since the tableau procedure disallows identical nodes, there are at most 2^{4n} nodes in the proof. Thus, the tableau construction terminates. □

The above is a very loose bound on the number of nodes, and is of no practical interest.

4 Polarities and Connections

The generalized subformula property can be exploited to provide representation schemes that eliminate dynamic formula construction and reduce overhead in determining contradictions. In addition, it is desirable to also eliminate common subformulae in the formula representation. Since all formulae generated during the proof are generalized subformulae, a form of structure sharing is used to represent formulae.

A notion of connections is also needed to support contradiction determination and the use of strategies to control the proof search. First, we define the (temporal) polarity of an occurrence of an atom in a formula.

Definition 4 *A (temporal)* **polarity** *is a pair of integers* $\langle n_\neg, n_\circ \rangle$. *The set of polarities of an atom occurrence, a, in a formula, F, is denoted pol(a,F) and is defined inductively as follows:*

$$\begin{array}{lcl}
pol(a,a) & = & \{ \langle 0,0 \rangle \} \\
pol(a,F) & = & \{\} \text{ if } a \text{ does not occur in } F \\
pol(a,\neg F) & = & \{\langle m+1 (mod 2), n\rangle : \langle m,n\rangle \in pol(a,F)\} \\
pol(a,\circ F) & = & \{\langle m, n+1\rangle : \langle m,n\rangle \in pol(a,F)\} \\
pol(a,F \odot G) & = & pol(a,F) \text{ if } a\text{'s occurence is in } F \text{ and} \\
 & & \odot \in \{\vee, \wedge\} \\
pol(a,F \odot G) & = & pol(a,G) \text{ if } a\text{'s occurence is in } G \text{ and} \\
 & & \odot \in \{\vee, \wedge\} \\
pol(a,\odot F) & = & \{\langle m,n\rangle : \langle m,n_2\rangle \in pol(a,F) \\
 & & \text{for some } n_2 \leq n\} \text{ and } \odot \in \{\Box, \Diamond\} \\
pol(a,F\,\mathcal{U}\,G) & = & \{\langle m,n\rangle : \langle m,n_2\rangle \in pol(a,F) \text{ or} \\
 & & \langle m,n_2\rangle \in pol(a,G) \text{ for some } n_2 \leq n\}
\end{array}$$

Note that polarities are associated with occurrences rather than the atom itself. Intuitively, a has polarity ⟨m,n⟩ if it appears in (a normal form of) f preceded by m (mod 2) ¬ symbols and n ∘ symbols. The set of polarities of an atom occurrence may be countably infinite, though the corresponding data structure size is fixed.

Example 4:
Denoting the n'th occurrence of an atom, a, as a^n, pol(a^1,$\circ\circ a \wedge \neg a$) = { ⟨0,2⟩ }, pol($a^2$,$\circ\circ a \wedge \neg a$) = {⟨1,0⟩}, and pol(a,$\Box \circ a$) = {⟨0,1⟩,⟨0,2⟩,⟨0,3⟩,...}

The temporal notion of polarity serves the same purpose as in classical logics – namely, to statically determine whether the atom can occur positively or negatively in the proof, and using this information to guide the proof. However, a major distinction in our notion of polarities is the addition of a temporal context, which allows us to extend many classical techniques to PTL. The temporal context allows for representing not only whether the atom occurrence is positive or negative, but also at which time it is positive or negative, thus allowing for a much finer predetermination of potentially contradictory formulae. However, this generalization requires polarities to be sets instead of a 0 or 1 since the language of PTL allows for a single formula to represent an atom which can occur at multiple times, as seen in Example 4.

We are now ready to define connections between atom occurrences in two formulae.

Definition 5 *There is a* **connection** *between an atom occurrence a in formula F and formula G if there exists n such that pol(a,F)=⟨0,n⟩ and pol(a,G)=⟨1,n⟩.*

Connections within formulae are allowed (when F=G). The set of all connections between subformulae is statically determinable, though we choose to encode some information into our tableau rules in our implementation.

5 Connection Based Strategies for PTL

The concepts discussed in the previous section can be exploited to solve many of the problems that propositional logic matrix methods are meant to address including (though not limited to) notational redundancy, subformula classification, contradiction determination, and control redundancy.

Notational redundancy occurs due to the multiple representation (and construction) of repeated subformulae during the construction of the tableau. The subformula property allows the use of structure sharing to represent formulas, thus eliminating this problem. Subformula classification overhead results from the need to repeatedly determine the type of a subformula and thus the tableau rule that applies to it, particularly when preferentially selecting certain formula types for expansion (for example, linear over branching formulae). The subformula property clearly eliminates this problem. One of the most time consuming components of our theorem prover is in determining contradictory formulae, and the notion of connections reduces this problem since only connected formulae need be checked.

Control redundancy results from the generation of inefficient proof trees. Heuristics can greatly affect the size and shape of the proof tree, and the use of connections to direct the proof can alter the proof tree dramatically. However, care must be taken in these heuristics, since a naive extension of propositional and first-order strategies may be unsound. In particular, paths on which eventuality satisfiability must be checked can not be pruned.

The first strategy prevents time from being wasted on expanding satisfiable formulae:

Strategy 1 *If a formula F in a node $\mathcal{N}$ contains no connections (either within F or to other formulae in $\mathcal{N}$), mark $\mathcal{N}$ expanded and create a child node $\mathcal{N}' = \mathcal{N}$-{F}.*

Proof (of correctness):
Let $\mathcal{T}$ and $\mathcal{T}'$ be the original and pruned tableaux respectively, and denote their roots $\mathcal{R}$ and $\mathcal{R}'$. First, suppose $\mathcal{R}$ is eliminable. If $\mathcal{N}$ is eliminable in $\mathcal{T}$, it must be unsatisfiable. Since F has no connections, $\mathcal{N}'$ must also be unsatisfiable and thus eliminable in $\mathcal{T}'$. Since the rest of the tableau is unchanged, $\mathcal{R}'$ is eliminable. If $\mathcal{N}$ is not eliminable in $\mathcal{T}$, there must be some ancestor which has an unsatisfiable eventuality. Since F has no connections, this eventuality remains unsatisfiable in $\mathcal{T}'$, and $\mathcal{R}'$ is thus eliminable.

Now, suppose $\mathcal{R}$ is not eliminable. If every path from $\mathcal{R}$ to $\mathcal{N}$ is eliminable, then $\mathcal{R}$ is not eliminable due to some other part of the tableau and $\mathcal{R}'$ is also not eliminable for the same reason. If there is some path from $\mathcal{R}$ to $\mathcal{N}$ which is not eliminable in $\mathcal{T}$, then $\mathcal{N}'$ must also not be eliminable in $\mathcal{T}'$. Moreover, any eventuality on that path in $\mathcal{N}'$ remains satisfiable in $\mathcal{T}'$ since F has no connections. Thus, $\mathcal{R}'$ is also not eliminable.

Thus, $\mathcal{R}$ is eliminable iff $\mathcal{R}'$ is eliminable. □

Of course, F must be kept if a model is desired for satisfiable tableaux. In our theorem prover, we keep F as a deleted formula, and output it as part of the model. We also avoid the creation of node $\mathcal{N}'$ by simply deleting F after declaring any eventuality formulae it satisfies as satisfiable.

The next strategy prunes trees once they can no longer be found unsatisfiable:

Strategy 2 *If a node contains no connections, declare the node satisfiable and terminate the tree at that node.*

Proof (of correctness):
This strategy is equal to a finite number of applications of Strategy 1, and is thus sound. □

Strategies 1 and 2 are related to the PURE principle in connection graph procedures. In connection graphs, the PURE principle allows for deletion of clauses which contain literals with no complementary literals (i.e. *pure* literals). Our strategies are, however, somewhat more general even in the non-temporal case since they apply to arbitrary formulae instead of just literals.

The next strategy is one of the most powerful strategies afforded by our procedure, and does not seem to be achievable in many other PTL decision procedures. First, note that we are not interested in the generation of all possible models for formulae which are found to be satisfiable; one model suffices.

Under the above assumption, the following strategy eliminates the expansion of many nodes whose satisfiability can be deduced from the satisfiability of other nodes:

Strategy 3 *Suppose a node $\mathcal{N}_1$ is a subnode of node $\mathcal{N}_2$, and $\mathcal{N}_2$ is not reachable from $\mathcal{N}_1$. Then, expand only node $\mathcal{N}_1$ and declare $\mathcal{N}_2$ eliminable if $\mathcal{N}_1$ is eliminable.*

Note that if $\mathcal{N}_1$ is not eliminable, $\mathcal{N}_2$ may or may not be eliminable. However, this is not of concern since we are not interested in finding all possible models for satisfiable formulae.

Proof (of correctness):
Consider the pruned tableau.

First, suppose that $\mathcal{N}_1$ is eliminable. Then, the set of formulae in $\mathcal{N}_1$ is unsatisfiable. But, since $\mathcal{N}_1 \subset \mathcal{N}_2$, this implies that the set of formulae in $\mathcal{N}_2$ is also unsatisfiable and $\mathcal{N}_2$ is thus eliminable in the original tableau. Thus the root in the pruned tableau is eliminable iff the root of the original tableau is.

Now, suppose that neither $\mathcal{N}_1$ nor the root is eliminable. Then, any path from the root going through $\mathcal{N}_1$ constitutes a model for the tableau.

Finally, suppose that $\mathcal{N}_1$ is not eliminable and the root is eliminable. Then, there must be some ancestor of $\mathcal{N}_1$, $\mathcal{M}$ which contains an unsatisfiable eventuality formula F. Then, by the nature of the tableau rules, $\mathcal{N}_1$ must contain a formula corresponding to the next-time branch of F along with formulas that can be used to contradict F's eventuality at every future timepoint. But since $\mathcal{N}_1 \subset \mathcal{N}_2$, the root in the originial tableau is also eliminable. □

This strategy is a temporal generalization of the SUBS rule in FOL theorem proving. The SUBS rule allows for deletion of clauses which are supersets of other clauses, since they are subsumed by the smaller clauses. If our tableau procedure is applied only to sets of clauses where the clauses are conjunctive and connected by disjunctions, Strategy 3 is identical. Of course, the strategy is more general, since it applies to arbitrary non-clausal formulae.

Branching can exponentially increase a tree's size, and it is desirable to reduce the number of branches. To reduce the number of branches, the following strategy is used:

Strategy 4 *If node $\mathcal{N}$ contains multiple unmarked branching formulae, and expansion of one of these formulae, F, would result in no connections to the newly generated formula/s in one branch, select F for expansion.*

Proof (of correctness):
Since the correctness of the procedure is independent of the order in which rules are applied, the proof is trivial. □

In the implementation, this strategy is actually used in conjunction with Strategy 1 and possibly with Strategy 3 to eliminate branches completely.

Example 5:
Using set notation, let $\mathcal{N} = \Gamma \bigcup \{\Diamond P, \circ\Box\neg P\}$ where Γ is a set of branching formulae with no connections to P. Then Strategy 4 selects $\Diamond P$ for expansion, generating the two children node $\mathcal{N}_1 = \Gamma \bigcup \{*\Diamond P, P, \circ\Box\neg P\}$ and $\mathcal{N}_2 = \Gamma \bigcup \{*\Diamond P, \circ\Diamond P, \circ\Box\neg P\}$ (using $*$ to denote marked formulae). Strategy 1 is then applied to delete P from $\mathcal{N}_1$, and Strategy 3 can then be applied to prevent expansion of $\mathcal{N}_2$. Thus, we only need to expand $\Gamma \bigcup \{*\Diamond P, \circ\Box\neg P\}$.

When used this way, this strategy is actually a technique used to allow for earlier application of Strategy 1. If it is applied to a beta formula, it serves as a generalization of the disjunctive subformula deletion and UNIT rules used in connection methods. Otherwise, it is applied to a pi, tau, or upsilon formulae, and it becomes a temporal generalization of the disjunctive subformula deletion rule. Note that a similar strategy can be added for conjunctive subformula deletion; however, we treat that case through other means.

The above strategies apply to traditional as well as temporal logics; however our notion of temporal polarities and the resulting connections allows the rules to apply to PTL. In practice, many PTL proofs repeat proofs of properties for future timepoints as well as the current one, though the same proof might apply. The following strategy alleviates this problem in many cases.

Strategy 5 *Suppose a node, $\mathcal{N}$ contains an eventuality formula, F, with corresponding eventuality E. Moreover, suppose every connection from E is to a term which is in the scope of only nu or alpha formulae, and in the scope of at least one nu formula. Then, expand F without branching, by ignoring the next-time branch.*

Proof (of correctness):
Expansion of F generates E in the current-time branch. By the nature of the nu, alpha, and sigma rules, the conditions on the connection ensure that every contradiction to E at the current time point also occurs at all future time points and vice versa. Thus, if the current-time branch is eliminable due to a contradiction to E, then the next-time branch is also eliminable. The next-time branch may thus be ignored. □

The conditions for Strategy 5 need to be checked before node expansion, since expanding other formulae may make the strategy inapplicable.

Strategy 5 is similar to standard theorem proving rules which eliminate tautologies from further consideration. Traditional propositional tautologies all apply, and we present several other similar temporal strategies [Shankar 1997].

Example 6:
Consider the tableau of Example 2. This strategy applies to node 2, from which the QUP is expanded on to create a new node 3. Expanding on $\Box\neg P$ in node 3 results in node 4 which is unsatisfiable. Thus, the original 9 node tableau is pruned to 4 nodes.

6 Results

In this paper we have described a PTL proof procedure which achieves many of the benefits of connection methods within the framework of semantic tableaux. Our strategies are surprisingly simple; however, they achieve most of the benefits of the major connection method reduction rules, including the PURE, SUBS, TAUT, and UNIT; moreover, we generalize these rules substantially in some cases. There are several other connection method rules that apply only to clausal methods.

Gough [Gough 1989] provides another PTL tableau procedure which accomplishes substantial node-count reduction. The reduction occurs mostly by identifying states which contain unnecessary formulae. Strategy 3 performs a similar function, though it is somewhat less general. While Gough's procedure does an excellent job when the complete tableau needs to be constructed, it may not be needed on satisfiable tableaux, since partial tableaux often suffice. For example, [Gough 1989] presents a tableau for $\Diamond P \wedge \Diamond Q \wedge (PUQ)$ which achieves a reduction from 26 to 8 states. However, since there are no connections in this formula, Strategy 2 trivially determines that the formula is satisfiable with no expansion.

We have implemented a theorem prover based on the techniques presented here. Our results, based on a collection of 38 textbook theorems and 66 small but 'hard' problems are promising. The pruning strategies have shown an average reduction in node count of 20-50% for unsatisfiable tableaux. For satisfiable tableaux, the prover achieves arbitrarily high node reductions, though the average reduction is closer to 50%. We expect that the actual time saved will be greater than node count reductions due to the non-linear complexity of certain critical graph algorithms.

7 Extensions

One of the most important advantages of our approach is its generality. Since our notion of polarities (and thus, connections) relies on the presence of a temporal context, it is applicable to any discrete linear-time temporal logic with a subformula property, though strategies may differ. Moreover, since it does not change the basic nature of the tableau, strategies are easily generatable, though all traditional reduction rules may not apply and many require generalization. In many cases, the strategies we have presented here apply unchanged.

Some useful logics that our procedure may be applied to include temporal logic with both past and future time operators and polymodal logics with time as a modality. Wolper [Wolper 1983] provides a tableau procedure for an extended TL which augments PTL with temporal operators definable by right-linear grammars, and our procedure is also applicable to this. Our procedures may be extended to the first-order versions of the above logics, using the techniques presented in [Wrightson 1987] for first order logic.

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