



4.  $x||y$  iff neither  $x \leq y$  nor  $y \leq x$

The relations in a point algebra are always disjunctions of basic relations and they are represented as sets of basic relations. Since we have 4 different basic relations we get  $2^4 = 16$  possible disjunctive relations. The set of basic relations is denoted  $\mathcal{B}$  and the set of all 16 relations is denoted by  $\mathcal{PA}$ . Sometimes we use a short-hand notation for certain relations, for example,  $\{<, =\}$  is sometimes written as  $\leq$  and  $\{=, ||\}$  as  $\perp$ . The empty relations is denoted by  $\perp$  and it is always unsatisfied. We will implicitly assume that the converse of relations are available. For instance, if  $(<)$  is in a set of relations  $\Gamma$ , then the converse  $(>)$  is also in  $\Gamma$ .

The basic computational problem of the point algebra is the satisfiability problem where we have a set of variables and a set of constraints over the variables and the question is whether there exists a mapping from the variables to some partial order such that all constraints are satisfied.

**Definition 1** Let  $\mathcal{R} \subseteq \mathcal{PA}$  be a set of point relations and  $P$  a class of partial orders. A problem instance of  $\text{SAT}_p(\mathcal{R})$  is a set of variables  $V$  and a set of binary constraints of the form  $xRy$  where  $x, y \in V$  and  $R \in \mathcal{R}$ . A tuple  $\langle f, \langle T, \leq \rangle \rangle$  where  $f : V \rightarrow T$  is a total function and  $\langle T, \leq \rangle \in P$  is called an interpretation of  $\Pi$ .

A problem instance  $\Pi$  is satisfiable iff there exists an interpretation  $M = \langle f, \langle T, \leq \rangle \rangle$  such that  $f(x) R f(y)$  holds for every constraint  $xRy$ . Such an  $M$  is called a *model* of  $\Pi$ .

Given an instance  $\Pi$  of  $\text{SAT}_p(\mathcal{R})$  and two variables  $x, y$  we write  $x \leq^+ y$  to say that there exists zero or more variables  $z_1, \dots, z_n$  such that

$$x \leq z_1 \wedge z_1 \leq z_2 \wedge \dots \wedge z_{n-1} \leq z_n \wedge z_n \leq y$$

and we write  $x \leq^* y$  to say  $x \leq^+ y$  or  $x = y$ .

We will consider two classes of partial orders: *po* which is the class of all partial orders and *to*  $\subset$  *po* which is the class of all total orders.

## Disjunctions

We will now extend the point algebra with disjunctions.

**Definition 2** Let  $R_1, R_2$  be relations of arity  $i, j$  and define the disjunction  $R_1 \vee R_2$  of arity  $i + j$  as follows:

$$R_1 \vee R_2 = \{(x_1, \dots, x_{i+j}) \in D^{i+j} \mid (x_1, \dots, x_i) \in R_1 \vee (x_{i+1}, \dots, x_{i+j}) \in R_2\}$$

Thus, the disjunction of two relations with arity  $i, j$  is the relation with arity  $i + j$  satisfying either of the two relations.

Let  $\Gamma_1, \Gamma_2$  be sets of relations and define the disjunction  $\Gamma_1 \check{\vee} \Gamma_2$  as follows:

$$\Gamma_1 \check{\vee} \Gamma_2 = \Gamma_1 \cup \Gamma_2 \cup \{R_1 \vee R_2 \mid R_1 \in \Gamma_1, R_2 \in \Gamma_2\}$$

The disjunction of two sets of relation  $\Gamma_1 \check{\vee} \Gamma_2$ , first introduced by Cohen *et al.* (Cohen, Jeavons, & Koubarakis 1997), is the set of disjunctions of each pair of relations in  $\Gamma_1, \Gamma_2$  plus the sets  $\Gamma_1, \Gamma_2$ . It is a natural to include  $\Gamma_1$  and  $\Gamma_2$  since one wants to have the choice of using the disjunction or not. The fact that if  $R_1 \vee R_2$  is included in a set of relations, then both  $R_1$  and  $R_2$  are in the set is an property which we refer

to as the  $\check{\vee}$ -closure property. In the sequel, we will tacitly assume that all sets of relations that we consider can be constructed from  $\mathcal{PA}$  and  $\check{\vee}$  and that they, consequently, satisfy the  $\check{\vee}$ -closure property. In many cases we shall be concerned with constraints that are specified by disjunctions of an arbitrary number of relations. Thus, we make the following definition: for any set of relations,  $\Delta$ , define  $\Delta^* = \bigcup_{i=0}^{\infty} \Delta^{\vee i}$  where  $\Delta^{\vee 0} = \{\perp\}$  and  $\Delta^{\vee i+1} = \Delta^{\vee i} \check{\vee} \Delta$ .

The previously defined concepts of problem instances, interpretations, models and so on can obviously be extended to disjunctions in a natural way. It is worth noting that the problem  $\text{SAT}_{po}$  is in NP even if we allow the use of disjunctions.

We say that a set of constraints  $\Gamma$  is *maximal tractable* iff  $\text{SAT}_p(\Gamma)$  is tractable and for every set  $X \not\subseteq \Gamma$  of relations which can be constructed by the relations in  $\mathcal{PA}$  and the  $\check{\vee}$  operator,  $\text{SAT}_p(\Gamma \cup X)$  is not tractable.

Finally, we introduce the independence property as defined by Cohen *et al.* (1997). This concept will be used extensively for showing tractability results.

**Definition 3** For any sets of relations  $\Gamma$  and  $\Delta$ , we say that  $\Delta$  is independent with respect to  $\Gamma$  if for any set of constraints  $C$  in  $\text{SAT}_{po}(\Gamma \cup \Delta)$ ,  $C$  has a solution whenever every  $C' \subseteq C$ , which contains at most one constraint whose constraint relation belongs to  $\Delta$ , has a solution.

**Theorem 4** For any sets of relations  $\Gamma$  and  $\Delta$ , if  $\text{SAT}_p(\Gamma \cup \Delta)$  is tractable and  $\Delta$  is independent with respect to  $\Gamma$ , then  $\text{SAT}_p(\Gamma \check{\vee} \Delta^*)$  is tractable.

## Gadgets

Gadgets allow us to concentrate on small sets of relations while proving tractability for large sets.

**Definition 5** Let  $\Psi$  be a satisfiable instance of  $\text{SAT}_{po}(\Gamma)$  containing the variables  $x, y, z_1, \dots, z_n$  and assume  $R$  to be a relation not in  $\Gamma$ . We say that  $\Gamma$  *implements*  $R$  iff the following holds: in every problem instance  $\Theta$  containing a constraint  $xRy$  the following holds:  $\Theta$  is satisfiable iff  $\Theta$  with  $xRy$  replaced by a fresh copy of the gadget  $\Psi$  is satisfiable.

We assume that the variables  $z_1, \dots, z_n$  does not appear in the instance  $\Theta$ . It is then easy to prove the following lemma.

**Lemma 6** Let  $\Gamma$  be a set of relations such that  $\text{SAT}_p(\Gamma)$  is tractable and assume there exists a gadget  $\Psi$  from  $\Gamma$  implementing  $\gamma$ , then  $\text{SAT}_p(\Gamma \cup \{\gamma\})$  is tractable.

We also show that independence is preserved under the introduction of gadgets.

**Theorem 7** Let  $\Delta$  be a set of relations independent of  $\Gamma$  and assume there exists a gadget  $\Psi$  from  $\Delta$  implementing the relation  $\delta$ . Then,  $\Delta \cup \{\delta\}$  is independent of  $\Gamma$ .

**Proof:** Let  $\Pi$  be an arbitrary instance of  $\Gamma \cup \Delta \cup \{\delta\}$  and let  $S$  be the subinstance of  $\Pi$  only containing relations from  $\Gamma - \Delta$ . Assume that every subset  $S \cup \{x_1 \delta_1 y_1\}, \dots, S \cup \{x_n \delta_n y_n\}$  of  $\Pi$  where  $\delta_i \in \Delta \cup \{\delta\}$  is satisfiable. Each such set of constraints can be rewritten on the form  $S \cup \Pi_i$

```

1 algorithm A
2 Input: An instance  $\Pi$  of  $\text{SAT}_{\text{po}}(\Gamma_A)$  or  $\text{SAT}_{\text{po}}(\Gamma_B)$ 
3 repeat
4    $\Pi' \leftarrow \Pi$ 
5   for each pair of nodes  $n_1, n_2 \in \Pi$  do
6     if  $n_1 \parallel n_2$  and  $n_1 \leq^* n_2$  then reject
7     if  $n_1 \parallel n_2$  and  $n_1 \leq^* n_2$  then
8       if  $n_1 \neq n_2$  then reject
9       else  $\Pi' \leftarrow \text{contract}(\Pi, n_1, n_2)$ 
10    elseif  $n_1 \leq^* n_2$  and  $n_2 \leq^* n_1$  then
11      if  $n_1 \neq n_2 \in \Pi$  then reject
12      else  $\Pi' \leftarrow \text{contract}(\Pi, n_1, n_2)$ 
13    end if
14  end for
15 until  $\Pi' = \Pi$ 
16 accept

```

Figure 1: The algorithm for solving  $\text{SAT}_{\text{po}}(\Gamma'_A)$  and  $\text{SAT}_{\text{po}}(\Gamma'_B)$

where  $\Pi_i$  is either a constraint from  $\Delta$  or a set of constraints resulting from replacing the  $\delta_i$  constraint by the gadget  $\Psi$ . Obviously each set  $S \cup \Pi_i$  is also satisfiable as well as all the relaxed sets  $S \cup \{\pi_{i,1}\}, \dots, S \cup \{\pi_{i,m}\}$ , where  $\{\pi_{i,1}, \dots, \pi_{i,m}\}$  are the constraints in  $\Pi_i$ . Since  $\Delta$  is independent of  $\Gamma$  and  $\Pi_i$  only contains relations from  $\Delta$ , we know that  $S \cup H$  where  $H = \{\pi_{i,j} \mid i \leq n, j \leq m\}$  is satisfiable. Since  $S \cup H$  is the result of replacing each  $\delta$  constraint in  $\Pi$  by the gadget  $\Psi$ , we have shown that  $\Pi$  is satisfiable and hence, that  $\Delta \cup \{\delta\}$  is independent of  $\Gamma$ .  $\square$

## Tractability Results

We continue by defining four classes of disjunctive relations  $\mathcal{T}_A, \dots, \mathcal{T}_D$  and show that their corresponding satisfiability problems are tractable. The classes are defined as follows:  $\mathcal{T}_A = \Gamma_A \dot{\vee} \Delta_A^*$ ,  $\mathcal{T}_B = \Gamma_B \dot{\vee} \Delta_B^*$ ,  $\mathcal{T}_C = \Gamma_C \dot{\vee} \Delta_C^*$  and  $\mathcal{T}_D = \Delta_D^*$  and the exact definitions of the sets of relations can be found in Table 1. The classes  $\mathcal{T}_A$  and  $\mathcal{T}_B$  are extensions of the algebras  $\mathcal{A}_{14}$  and  $\mathcal{A}_{10}$  as defined by Broxvall and Jonsson (1999). It should be noted that if a problem instance of  $\mathcal{T}_B$  has a model, then it has a model that is a total order. The class  $\mathcal{T}_D$  is trivial since if an instance has a model, then it has a one-point model. The class  $\mathcal{T}_C$  is quite obscure but note that its basic relations are a subset of the basic relations in  $\mathcal{T}_A$ .

*Contractions* are needed in order to understand the algorithms presented in this section. Let  $n_1, n_2$  be two variables in a problem instance  $\Pi$  with variable set  $V$  and constraints  $C$  and let  $\Pi'$  be a problem instance with variables  $V - \{n_2\}$  and constraints  $(C - \{n_2 R x, x R n_2 \mid x \in V\}) \cup \{n_1 R x \mid n_2 R x \in C\} \cup \{x R n_1 \mid x R n_2 \in C\}$ . We say that  $\Pi'$  is obtained by *contracting*  $n_1, n_2$ , that is, we identify the nodes  $n_1, n_2$  by  $n_1$ . Note that this operation may introduce constraints of the form  $n_1 R n_1$ .

**Lemma 8**  $\text{SAT}_{\text{po}}(\Gamma'_A)$ ,  $\text{SAT}_{\text{po}}(\Gamma'_B)$  and  $\text{SAT}_{\text{po}}(\Gamma'_C)$  are tractable.

```

1 algorithm C
2 Input: An instance  $\Pi$  of  $\text{SAT}_{\text{po}}(\Gamma_C)$ 
3 repeat
4    $\Pi' \leftarrow \Pi$ .
5   if exists  $n_1, n_2$  such that  $n_1 = n_2$  in  $\Pi$  then    $\Pi \leftarrow$ 
     contract( $\Pi$ )
6   until  $\Pi' = \Pi$ 
7   if exists node  $n$  such that  $n \neq n$  or  $n < n$  in  $\Pi$  then
     reject
8   if  $n_1 <^* n_2$  and  $n_1 R n_2$  in  $\Pi$  where  $< \notin R$  then
     reject
9   accept

```

Figure 2: The algorithm for solving  $\text{SAT}_{\text{po}}(\Gamma'_C)$

**Proof:** An algorithm for  $\text{SAT}_{\text{po}}(\Gamma'_A)$  and  $\text{SAT}_{\text{po}}(\Gamma'_B)$  which is a slightly modified version of the algorithm for  $\mathcal{A}_{14}$  and  $\mathcal{A}_{10}$  proven correct and polynomial by Broxvall and Jonsson (1999) is presented in figure 1.

A polynomial-time algorithm for  $\text{SAT}_{\text{po}}(\Gamma'_C)$  is presented in figure 2. Obviously algorithm *C* rejects only unsatisfiable instances of  $\text{SAT}_{\text{po}}(\Gamma'_C)$ .

Assume algorithm *C* accepts a certain instance  $\Pi$  and let  $\Pi'$  be the instance resulting from the contractions made by the algorithm in lines 3-6. Observe that  $\Pi'$  does not contain the relation  $(=)$  and if  $\Pi'$  is satisfiable, then  $\Pi$  is satisfiable. Let  $I$  be the interpretation given by the function  $f(x) = x$  and the partial order  $\langle V, \{(x, y) \mid x <^* y \text{ in } \Pi'\} \rangle$  where  $V$  is the set of variables appearing in  $\Pi'$ .

Each  $\neq$  and  $<$  constraint is satisfied by  $I$  and if there exists a constraint  $x \parallel < y$  in  $\Pi'$  then either  $x < y$  or  $x \parallel y$  under  $I$  which guarantees that each such constraint is satisfied so  $I$  is a model and  $\Pi$  is satisfiable. Thus,  $\text{SAT}_{\text{po}}(\Gamma'_C)$  is tractable.  $\square$

Next, we show a number of independence results.

**Lemma 9**  $\Delta'_A, \Delta'_B, \Delta'_C$  are independent of  $\Gamma'_A, \Gamma'_B$  and  $\Gamma'_C$ , respectively.

**Proof:** We show that  $\Delta'_A$  is independent of  $\Gamma'_A$ ; the proofs of the other two cases are similar.

Let  $\Pi$  be an instance of  $\text{SAT}_{\text{po}}(\Gamma'_A \cup \Delta'_A)$  and assume that every instance  $\Pi'$  such that  $\Pi'$  only contains constraints present in  $\Pi$  and at most one constraint from  $\Delta'_A$  is satisfiable. We prove  $\Pi$  to be satisfiable.

Assume to the contrary that  $\Pi$  is not satisfiable. Then, there exists at least one constraint  $x R y \in \Pi$  where  $R$  is  $\neq$  or  $\parallel$  which causes algorithm *A* to reject. Let  $\Pi'$  denote the problem instance containing all constraints from  $\Pi$  of the form  $z R' w$  where  $R'$  is  $\leq$  or  $\parallel$  plus the constraint  $x R y$ . Note that  $\Pi'$  causes the algorithm to reject. Contradiction, since  $\Pi'$  is a problem instance previously assumed to be satisfiable and the algorithm correctly solves  $\text{SAT}_{\text{po}}(\Gamma'_A \cup \Delta'_A)$ . Hence,  $\Delta'_A$  is independent of  $\Gamma'_A$ .  $\square$

We can now show that the four classes are tractable.

**Theorem 10**  $\text{SAT}_{\text{po}}(\mathcal{T}_A)$ ,  $\text{SAT}_{\text{po}}(\mathcal{T}_B)$ ,  $\text{SAT}_{\text{po}}(\mathcal{T}_C)$  and  $\text{SAT}_{\text{po}}(\mathcal{T}_D)$  are tractable.

Table 1: Tractable classes

	$\Gamma'_A$	$\Delta'_A$	$\Gamma_A$	$\Delta_A$	$\Gamma'_B$	$\Delta'_B$	$\Gamma_B$	$\Delta_B$	$\Gamma'_C$	$\Delta'_C$	$\Gamma_C$	$\Delta_C$	$\Delta_D$
$<$			•				•		•		•		
$\leq$	•		•		•		•						•
$\langle \rangle$							•	•					
$\langle \Rightarrow \rangle$					•	•	•	•					•
$\parallel$	•	•	•	•							•	•	
$\perp$	•		•								•	•	•
$=$			•				•		•		•		•
$\neq$	•	•	•	•	•	•	•	•	•	•	•	•	
$< \parallel$			•	•							•	•	
$\leq \parallel$			•						•	•	•	•	•

**Proof:** That  $\text{SAT}_{\text{po}}(\Gamma'_A \check{\Delta}'_A)$  is tractable follows from Lemmata 8, 9 and Theorem 4. That  $\Gamma'_A$  implements  $\Gamma_A$  is proven by Broxvall and Jonsson (1999). Let  $\Pi$  be an arbitrary instance of  $\text{SAT}_{\text{po}}(\mathcal{T}_A)$ . We will show how every  $\parallel <$  relation can be replaced by relations in  $\Gamma'_A \check{\Delta}'_A$  which implies the tractability of  $\text{SAT}_{\text{po}}(\mathcal{T}_A)$ . Choose an arbitrary disjunction  $\gamma = x R_1 y \vee \dots \vee x_n R_n y_n$  in  $\Pi$  where  $R_1 = (\parallel <)$ . Introduce a fresh variable  $t_{x,y}$ , add the constraint  $x \leq t_{x,y}$  and replace  $\gamma$  by the disjunction  $\gamma' = t_{x,y} \parallel y \vee \dots \vee x_n R_n y_n$ . Repeat this transformation until no  $(\parallel <)$  remains—a process which clearly takes polynomial time. Let  $\Pi'$  denote the resulting instance. Since the gadget  $x \leq t_{x,y}, t_{x,y} \parallel y$  implements the relation  $< \parallel$ , it follows that  $\Pi'$  is satisfiable iff  $\Pi$  is satisfiable. Hence,  $\text{SAT}_{\text{po}}(\mathcal{T}_A)$  is tractable.

The tractability of  $\text{SAT}_{\text{po}}(\Gamma'_B \check{\Delta}'_B)$  follows from Lemmata 8, 9 and Theorem 4. That  $\Gamma'_B$  implements  $\Gamma_B$  is proven by Broxvall and Jonsson (1999). The gadget  $x \neq y, x \langle \Rightarrow \rangle y$  implements  $x \langle \rangle y$  which shows that  $\Delta'_B$  implements  $\Delta_B$ . This fact combined with Theorem 7 proves that  $\text{SAT}_{\text{po}}(\mathcal{T}_B)$  is tractable.

To show that  $\mathcal{T}_C$  is tractable we begin by noting that the following gadgets from  $\Delta'_C$  implements  $\Delta_C$ :  $x \leq \parallel y, y \leq \parallel x$  implement  $x \parallel y$ ,  $x \leq \parallel y, y \leq \parallel x, x \neq y$  implement  $x \parallel y$  and  $x \leq \parallel y, x \neq y$  implement  $x < \parallel y$ . Hence,  $\Delta_C$  is independent of  $\Gamma'_C$  by Lemma 9 and Theorem 7. That  $\text{SAT}_{\text{po}}(\Gamma'_C \check{\Delta}'_C)$  is tractable follows from Theorem 4. The theorem follows since  $\Gamma_C = \Gamma'_C \cup \Delta_C$  which implies that  $\mathcal{T}_C = \Gamma'_C \check{\Delta}'_C = \Gamma_C \check{\Delta}'_C$ .

Finally, the tractability of  $\text{SAT}_{\text{po}}(\mathcal{T}_D)$  is easy to show since each relation present in  $\Gamma_D$  contains only the relations  $=$  and/or  $\perp$ . Consequently, it is sufficient to check whether an instance  $\Pi$  contains a disjunction containing only the  $\perp$  relation. If this is the case, the instance is not satisfiable. Otherwise,  $\Pi$  is satisfied by the model mapping every variable to a single node.  $\square$

## Intractability Results

This section contains the NP-completeness results which are needed in the classification theorem.

**Lemma 11** The following problems are NP-complete:

1.  $\text{SAT}_{\text{po}}(\{\langle \rangle\}, R)$  if  $R \in \{\parallel, < \parallel, \leq \parallel\}$ ;
2.  $\text{SAT}_{\text{po}}(\{\langle \rangle\} \vee \{\langle \rangle\})$ ;
3.  $\text{SAT}_{\text{po}}(\{R\} \cup \{\leq\} \vee \{\leq\})$  if  $R \in \{\neq, \langle \rangle\}$ ;
4.  $\text{SAT}_{\text{po}}(\{R\} \cup \{=\} \vee \{=\})$  if  $R \in \{\langle, \langle \rangle, \neq, \parallel, \parallel \langle\}$ ;
5.  $\text{SAT}_{\text{po}}(\{R_1, R_2\} \cup \{\parallel\} \vee \{\parallel\})$  if  $R_1 \in \{\leq, \langle \Rightarrow \rangle\}$  and  $R_2 \in \{\langle, \langle \rangle, \neq, \parallel \langle\}$ .

**Proof:** Case 1 is shown in Broxvall and Jonsson (1999). We only prove case 2 since the proofs of the other cases are similar. The proof is by a reduction from the NP-complete problem MONOTONE 3SAT:

INSTANCE: Set  $U$  of variables, collection  $C$  of clauses over  $U$  such that for each  $c \in C$  has  $|c| = 3$  and each clause contain either only positive or negative literals.

QUESTION: Is there a satisfying truth assignment for  $C$ ?

Given an arbitrary instance  $P$  of MONOTONE 3SAT with variables  $U = \{u_i, \dots, u_m\}$  and clauses  $C = \{c_1, \dots, c_n\}$ , construct incrementally an instance  $\Pi$  of  $\text{SAT}_{\text{po}}(\{\langle \rangle\} \vee \{\langle \rangle\})$  by the following steps.

Begin with the variables in the set  $U$  and a fresh variable  $a$  and add the constraints  $(u_i < a) \vee (u_i > a)$ ,  $1 \leq i \leq m$ . For each positive clause  $c_i \in C$  of the form  $x \vee y \vee z$  add a fresh variable  $t_i$  and the constraints:

$$x > a \vee t_i > a, y > t_i \vee z > a.$$

For each negative clause  $c_i \in C$  of the form  $\neg x \vee \neg y \vee \neg z$  add a fresh variable  $t_i$  and the constraints:

$$x < a \vee t_i < a, y < t_i \vee z < a.$$

We show that the problem instance constructed is satisfiable iff the original MONOTONE 3SAT problem instance is satisfiable. For the if-direction, assume that  $P$  is satisfiable and that  $\mathcal{I}$  is a truth assignment satisfying all clauses in  $C$ . Construct an interpretation  $M = \langle f, \langle T, \leq \rangle \rangle$  of  $\Pi$  as follows: let  $T = \{a_i \mid 1 \leq i \leq 6\}$ ,  $\leq = \{\langle a_i, a_j \rangle \mid i \leq j\}$  and

$$f = \begin{aligned} & \{ \langle x, a_5 \rangle \mid x \text{ true in } \mathcal{I} \} \cup \\ & \{ \langle x, a_1 \rangle \mid x \text{ false in } \mathcal{I} \} \cup \\ & \{ \langle t_i, a_0 \rangle \mid c_i = x \vee y \vee z \text{ and } \mathcal{I}(x) = \text{true} \} \cup \\ & \{ \langle t_i, a_4 \rangle \mid c_i = x \vee y \vee z \text{ and } \mathcal{I}(x) = \text{false} \} \cup \\ & \{ \langle t_i, a_2 \rangle \mid c_i = \neg x \vee \neg y \vee \neg z \text{ and } \mathcal{I}(x) = \text{true} \} \cup \\ & \{ \langle t_i, a_6 \rangle \mid c_i = \neg x \vee \neg y \vee \neg z \text{ and } \mathcal{I}(x) = \text{false} \} \cup \\ & \{ \langle a, a_3 \rangle \} \end{aligned}$$

That each constraint is satisfied by  $M$  can trivially be verified so  $M$  is a model of  $\Pi$  and  $\Pi$  is satisfiable if  $P$  is satisfiable. For the only-if direction, assume that  $\Pi$  is satisfiable and that  $M$  is a model of  $\Pi$  with mapping  $f$ . Construct a truth assignment  $\mathcal{I}$  over the variables  $U$  assigning a variable  $x$  the value true if  $f(x) > f(a)$  and otherwise the value false. Assume that there exists a clause  $c_i$  not satisfied by  $\mathcal{I}$ . The case when  $c_i$  is a positive clause  $x \vee y \vee z$  leads to  $f(x) < f(a), f(y) < f(a), f(z) < f(a)$  so  $f(t_i) > f(a)$  and  $f(y) > f(t_i)$  which contradicts that  $f(y) < f(a)$ . The case where  $u$  is a negative clause can be ruled out in a similar manner. Hence, no unsatisfied clause exists and  $\mathcal{I}$  is a satisfying truth assignment. Since the reduction can be performed in polynomial time, the result follows.  $\square$

## Maximality

This section contains the main result of this paper, namely that  $\mathcal{T}_A, \mathcal{T}_B, \mathcal{T}_C$  and  $\mathcal{T}_D$  are the only maximal tractable sets of the disjunctive point algebra for partially ordered time. However, we need some auxiliary results first.

To reduce the number of NP-completeness results needed in the classification, we will employ a *closure* operator  $\mathcal{C}(\cdot)$  which is defined with the aid of the standard operators converse, intersection ( $\cap$ ) and composition ( $\circ$ ) together with a number of rules for handling disjunctions. These rules are defined and their tractability-preserving properties are stated in the next lemma. The straightforward proof is omitted.

**Lemma 12** Let  $R_i \in \mathcal{PA}$  and  $\Gamma_j \subseteq \mathcal{PA}$ .

- (1) if  $\text{SAT}_{\text{po}}(\Gamma_1 \cup \{<\} \check{\vee} \Gamma_2)$  is tractable, then  $\text{SAT}_{\text{po}}(\Gamma_1 \cup \{<\} \check{\vee} \Gamma_2 \cup \Gamma_2 \check{\vee} \Gamma_2)$  is tractable;
- (2) assume  $<, <>, \neq$  or  $< \parallel$  is in  $\Gamma_1$ . If  $\text{SAT}_{\text{po}}(\Gamma_1 \cup \{=\} \check{\vee} \Gamma_2)$  is tractable, then  $\text{SAT}_{\text{po}}(\Gamma_1 \cup \{=\} \check{\vee} \Gamma_2 \cup \Gamma_2 \check{\vee} \Gamma_2)$  is tractable.
- (3) assume  $\oplus \in \{\cap, \circ\}$ . If  $\text{SAT}_{\text{po}}(\Gamma_1 \cup \{R_1 \vee R_3, R_2 \vee R_3\})$  is tractable, then  $\text{SAT}_{\text{po}}(\Gamma_1 \cup \{R_1 \vee R_3, R_2 \vee R_3, (R_1 \oplus R_2) \vee R_3\})$  is tractable.
- (4) if  $\text{SAT}_{\text{po}}(\Gamma_1 \cup \{R_1 \vee R_2\})$  is tractable, then  $\text{SAT}_{\text{po}}(\Gamma_1 \cup \{R_1, R_2, R_1 \vee R_2\})$  is tractable.

If  $\Gamma$  is a set of relations, we define  $\mathcal{C}(\Gamma)$  as the least set  $X$  such that  $\Gamma \subseteq X$  and  $X$  is closed under converse, intersection and composition on point relations and the expansion rules in the previous lemma. A composition table for the point relations can be found in Broxvall and Jonsson (1999). By using the lemma, the  $\check{\vee}$ -closure property and the properties of converse, intersection and composition, it is a routine verification to show that if  $\text{SAT}_{\text{po}}(\Gamma)$  is tractable, then  $\text{SAT}_{\text{po}}(\mathcal{C}(\Gamma))$  is also tractable.

Now, we introduce a construction which simplifies the proof by allowing us to only consider a small number of disjunctions; this is shown in the next lemma.

**Definition 13** Let  $\mathcal{T} = \Gamma \check{\vee} \Delta^*$  and  $\Delta \subseteq \Gamma$ . We define  $\overline{\mathcal{T}}$  as  $(\mathcal{PA} - \Gamma) \cup (\Gamma - \Delta) \check{\vee} (\Gamma - \Delta)$ .

**Lemma 14** If  $\mathcal{T} = \Gamma \check{\vee} \Delta^*$ ,  $\Delta \subseteq \Gamma$  and  $\mathcal{T}' \not\subseteq \mathcal{T}$ , then there exists  $C \in \overline{\mathcal{T}}$  such that  $C \in \mathcal{T}'$ .

**Proof:** Arbitrarily choose a  $C \in \mathcal{T}'$  such that  $C \notin \mathcal{T}$  and choose  $C_1, \dots, C_n \in \mathcal{PA}$  such that  $C_1 \vee \dots \vee C_n = C$ . Assume first that there exists some  $i$  such that  $C_i \notin \Gamma$ . The definition of  $\overline{\mathcal{T}}$  implies that  $C_i \in \overline{\mathcal{T}}$  and the  $\check{\vee}$ -closure property implies that  $C_i \in \mathcal{T}'$ .

Assume instead that  $C_1, \dots, C_n \in \Gamma$ . Since  $\Gamma \subseteq \mathcal{T}$  and  $C \notin \mathcal{T}$  we know that  $n > 1$ . If all or all but one  $C_i \in \Delta$ , then we know that  $C \in \mathcal{T}$ . Hence, there exists at least two  $i, j$  such that  $C_i, C_j \notin \Delta$ . Now,  $C_i \vee C_j \in \mathcal{T}'$  by the  $\check{\vee}$ -closure property and the definition of  $\overline{\mathcal{T}}$  gives that  $C_i \vee C_j \in \overline{\mathcal{T}}$ .  $\square$

Using the tractability and NP-completeness results given in previous sections as well as the previous definition and results we can now by a simple computer assisted case analysis prove the main result of this paper.

**Theorem 15**  $\mathcal{T}_A, \mathcal{T}_B, \mathcal{T}_C$  and  $\mathcal{T}_D$  are the only maximal tractable subclasses of  $\text{SAT}_{\text{po}}$ .

**Proof:** Suppose to the contrary that there exists another maximal tractable algebra  $\mathcal{T}$ . From the previous lemma it follows that there exists  $\gamma_A, \dots, \gamma_D$  in  $\mathcal{T}$  such that  $\gamma_A \in \overline{\mathcal{T}}_A, \dots, \gamma_D \in \overline{\mathcal{T}}_D$ . Note that there exists only a finite number of possible values for  $\gamma_A, \dots, \gamma_D$ .

To prove the result, a machine-assisted case analysis of the following form was performed: each admissible choice of  $\gamma_A, \dots, \gamma_D$  was generated and  $X = \mathcal{C}(\gamma_A, \dots, \gamma_D)$  was computed. Each such set  $X$  was examined and it was found that at least one of the NP-complete sets of Lemma 11 was a subset of  $X$ . Thus,  $\text{SAT}_{\text{po}}(\mathcal{T})$  is NP-complete and the theorem follows.  $\square$

## Totally Ordered Time

We will now identify the maximal tractable subclasses of  $\text{SAT}_{\text{to}}$ , i.e.,  $\text{SAT}_{\text{po}}$  restricted to total orders. The classification will rely on the results in the previous sections but the main theorem is shown without the use of a computer-assisted case analysis.

Note that the basic relation  $\parallel$  is unnecessary when dealing with total orders so we only have three basic relations ( $<, =$  and  $>$ ) and eight possible disjunctions of these relations. Let  $\mathcal{PA}_{\text{to}}$  denote the set of these eight relations and define  $\mathcal{X}_1 = \mathcal{PA}_{\text{to}} \check{\vee} \{\neq\}^*$  and  $\mathcal{X}_2 = \Delta^*$  where  $\Delta = \{r \in \mathcal{PA}_{\text{to}} \mid r = \perp \text{ or } \{=\} \subseteq r\}$ . As we will see later on,  $\mathcal{X}_1$  and  $\mathcal{X}_2$  are the only maximal tractable subclasses of  $\text{SAT}_{\text{to}}$ .

**Lemma 16**  $\text{SAT}_{\text{to}}(\mathcal{X}_i), 1 \leq i \leq 2$ , are tractable problems.

**Proof:** Tractability of  $\mathcal{X}_1$  has been proved by Jonsson and Bäckström (1998) and Koubarakis (1996) while the tractability of  $\mathcal{X}_2$  is shown in Theorem 10.  $\square$

The NP-completeness results for  $\text{SAT}_{\text{to}}$  are all based on the previously presented NP-completeness results; interestingly, many of these results hold even when restricted to total orders.

**Lemma 17**  $\text{SAT}_{\text{to}}(\mathcal{N}_i)$ ,  $1 \leq i \leq 5$ , is NP-complete where

$$\begin{aligned} \mathcal{N}_1 &= \{ \langle \vee \langle \rangle \} & \mathcal{N}_2 &= \{ \neq, (\leq \vee \leq) \} \\ \mathcal{N}_3 &= \{ \langle, (= \vee =) \} & \mathcal{N}_4 &= \{ \neq, (= \vee =) \} \\ \mathcal{N}_5 &= \{ \langle, (\leq \vee \leq) \}. \end{aligned}$$

**Proof sketch:** We begin by examining the proof of Lemma 11 which shows that  $\text{SAT}_{\text{po}}(\{ \langle \rangle \vee \{ \langle \rangle \})$  is NP-complete. Obviously, the proof holds even if we restrict the possible partial orders to total orders which implies that  $\text{SAT}_{\text{to}}(\{ \langle \rangle \} \checkmark \{ \langle \rangle \})$  is NP-complete. By analyzing the proofs of the other cases, it follows that  $\text{SAT}_{\text{to}}(\mathcal{N}_i)$ ,  $2 \leq i \leq 4$ , is NP-complete. The NP-completeness of  $\text{SAT}_{\text{to}}(\mathcal{N}_5)$  follows from the NP-completeness of  $\text{SAT}_{\text{to}}(\mathcal{N}_3)$  plus the fact that  $\{ \leq \} \checkmark \{ \leq \}$  implements  $\{ = \} \checkmark \{ = \}$   $\square$

We are now ready to prove the main theorem for the point algebra for totally ordered time.

**Theorem 18**  $\mathcal{X}_1$  and  $\mathcal{X}_2$  are the only maximal tractable subclasses of  $\text{SAT}_{\text{to}}$ .

**Proof:** Assume that there exists a maximal tractable algebra  $\mathcal{X}$  such that  $\mathcal{X} \not\subseteq \mathcal{X}_1$  and  $\mathcal{X} \not\subseteq \mathcal{X}_2$ . By Lemma 14, there exists  $\gamma_1, \gamma_2$  in  $\mathcal{T}$  such that  $\gamma_1 \in \overline{\mathcal{X}_1}$  and  $\gamma_2 \in \overline{\mathcal{X}_2}$ . It is easy to see that  $\overline{\mathcal{X}_1} \subseteq \{r_1, \dots, r_6\}$  where

$$\begin{aligned} r_1 &= (\langle \vee \langle \rangle) & r_2 &= (\leq \vee \leq) \\ r_3 &= (= \vee =) & r_4 &= (\leq \vee =) \\ r_5 &= (\langle \vee =) & r_6 &= (\langle \vee \leq) \end{aligned}$$

and  $\overline{\mathcal{X}_2} \subseteq \{ \langle, \neq, (\langle \vee \langle \rangle), (\langle \vee \neq), (\neq \vee \neq) \}$ . By Lemma 11, we know that  $\text{SAT}_{\text{to}}(\{ \langle \vee \langle \rangle \})$  is NP-complete so the disjunction  $r_1 = (\langle \vee \langle \rangle)$  can be excluded from further consideration. We will show that if  $\gamma_1$  equals one of  $r_2, \dots, r_6$  and  $\gamma_2$  equals  $\langle$  or  $\neq$ , then  $\text{SAT}_{\text{to}}(\{\gamma_1, \gamma_2\})$  is NP-hard. By noting that  $(\langle \vee \neq)$  implements the relation  $\langle$  (by the construction  $x < y \vee x \neq x$ ) and  $(\neq \vee \neq)$  implements  $\neq$  (by  $x \neq y \vee x \neq y$ ), this result implies that  $\text{SAT}_{\text{to}}(\{\gamma_1, \gamma_2\})$  is NP-hard for all  $\gamma_1 \in \overline{\mathcal{X}_1}$  and  $\gamma_2 \in \overline{\mathcal{X}_2}$ . This contradicts our initial assumptions and proves the theorem.

1.  $\gamma_1 \in \{r_2, r_3\}$ . Then, the NP-completeness of  $\text{SAT}_{\text{to}}(\{\gamma_1, \gamma_2\})$  is an immediate consequence of Lemma 6.2.
2.  $\gamma_1 = r_4$ . Note that  $r_4$  implements  $r_3$  so  $\text{SAT}_{\text{to}}(\{\gamma_1, \gamma_2\})$  is NP-complete by case 1.
3.  $\gamma_1 = r_5$  and  $\gamma_2 = \{ \langle \}$ . We show how to implement the disjunction  $a < b \vee c < d$  with  $\gamma_1$  and  $\gamma_2$ : introduce an auxiliary variable  $t$  and the following two relations:  $a < t$  and  $t = b \vee c < d$ . NP-completeness follows from case 1.
4.  $\gamma_1 = r_5$  and  $\gamma_2 = \{ \neq \}$ . The relation  $a < b$  can be implemented by the relations  $t_1 \neq t_2$  and  $t_1 = t_2 \vee a < b$  where  $t_1$  and  $t_2$  are auxiliary variables. NP-completeness follows from the previous case.

5.  $\gamma_1 = r_6$ . The NP-completeness of  $\text{SAT}_{\text{to}}(\{\gamma_1, \gamma_2\})$  is a consequence of cases 3 and 4 and the observation that  $r_6$  implies  $r_5$ .  $\square$

## Acknowledgements

This research has been supported by the ECSEL graduate student program and by the *Swedish Research Council for the Engineering Sciences* (TFR) under grant 97-301.

## References

- Allen, J. F. 1983. Maintaining knowledge about temporal intervals. *Communications of the ACM* 26(11):832–843.
- Anger, F. 1989. On Lamport’s interprocessor communication model. *ACM Transactions on Programming Languages Systems* 11(3):404–417.
- Broxvall, M., and Jonsson, P. 1999. Towards a complete classification of tractability in point algebras for nonlinear time. In *Proceedings of the 5th International Conference on Principles and Practice of Constraint Programming (CP-99)*, 448–454.
- Cohen, D.; Jeavons, P.; and Koubarakis, M. 1997. Tractable disjunctive constraints. In *Proceedings of the 3rd International Conference on Principles and Practice for Constraint Programming*, 478–490.
- Jonsson, P., and Bäckström, C. 1998. A unifying approach to temporal constraint reasoning. *Artificial Intelligence* 102(1):143–155.
- Koubarakis, M. 1996. Tractable disjunctions of linear constraints. In *Proceedings of the 2nd International Conference on Principles and Practice for Constraint Programming*, 297–307.
- Lamport, L. 1986. The mutual exclusion problem: Part I—a theory of interprocess communication. *Journal of the ACM* 33(2):313–326.
- Nebel, B., and Bürckert, H.-J. 1995. Reasoning about temporal relations: A maximal tractable subclass of Allen’s interval algebra. *Journal of the ACM* 42(1):43–66.