$\mathsf{EQD}(x,y,z)$ says that the centres of x and y are equidistant from the centre of z. $\mathsf{Mid}(x,y,z)$ says that the centre of y lies mid-way between the centres of x and z; and $wx \simeq yz$ holds when the distance between the centres of w and x is the same as the distance between the centres of y and z. $\mathsf{Nearer}(w,x,y,z)$ means that the centres of w and x are closer than the centres of y and z.

Finally we define a relation $\mathsf{InI}(s,r)$ that is true when the centre point of sphere s is in the interior of region r:

D20) $\mathsf{InI}(s,r) \equiv_{def} \exists^\circ s'[s' \circledcirc s \wedge \mathsf{P}(s',r)]$

In addition to the usual principles of classical logic and the theory of sets, the system is required to satisfy the following spatial axioms:

A1) $\forall \alpha[\exists x[x \in \alpha] \rightarrow \exists! x[\mathsf{SUM}(\alpha,x)]]$
A2) $\forall r \exists^\circ s[\mathsf{P}(s,r)]$
A3) $\forall^\circ xy[\neg(x \circledcirc y) \rightarrow \exists^\circ s[s \circledcirc x \wedge \forall^\circ z[\mathsf{InI}(z,s) \leftrightarrow \mathsf{Nearer}(x,z,x,y)]]$
A4) $\forall^\circ x \exists^\circ y[\neg(x \circledcirc y) \wedge \forall^\circ z[\mathsf{InI}(z,s) \leftrightarrow \mathsf{Nearer}(x,z,x,y)]]$
A5) $\forall xy[\mathsf{P}(x,y) \leftrightarrow \forall^\circ s[\mathsf{InI}(s,x) \rightarrow \mathsf{InI}(s,y)]]$
A6) $\forall r \forall^\circ s[\exists^\circ s'[s' \circledcirc s \wedge \forall^\circ s''[\mathsf{P}(s'',s') \rightarrow \mathsf{O}(s'',r)]] \rightarrow \mathsf{InI}(s,r)]$
A7) A suitable axiom set for two dimensional geometry formulated in terms of the B and $\simeq$ relations,[1] with the quantifiers restricted to range over spheres and equality replaced by the $\circledcirc$ relation.
A8) $\forall^\circ xyz[(x \circledcirc y \wedge y \circledcirc z) \rightarrow x \circledcirc z]$
A9) $\forall^\circ xx'yzw[(xy \simeq zw \wedge x' \circledcirc x) \rightarrow x'y \simeq zw]$

A1 ensures that every set of regions has a sum. **A2** states that every region has a spherical part (from this and **A1** it can be proved that every region is equal to the sum of its spherical parts). **A3** ensures that for every pair of distinct points x and y there is a sphere centred at one and bounded by the other. **A4** says that all spheres can be constructed in this way. Because Nearer is defined in terms of the purely geometrical concepts (B and $\circledcirc$) and the geometrical axioms are known to be complete relative to the classical interpretation in Cartesian spaces, this means that InI is completely determined for the class of spheres. **A5** means that $\mathsf{P}(x,y)$ holds just in case every interior point of x is an interior point of y (thus P must be symmetric and transitive). This axiom could be used as a definition so that InI was taken as primitive instead of P. **A6** is needed to fully fix the domain of regions by ensuring that they correspond to *regular* sets of points in the intended Cartesian models. Axioms **A8** and **A9** ensure that $\circledcirc$ behaves like equality relative to the geometrical axioms.[2]

In (Bennett *et al.* 2000) it is proved that this theory is *categorical* — all models are isomorphic to standard models defined over classical Cartesian spaces:

Theorem: *A formula is a consequence of* **A1–7** *(together with the definitional formulae) just in case it is true for any assignment to the variables of regular open subsets of* $\Re^2$*, where* P *is interpreted as the subset relation and* $\mathsf{S}(x)$ *holds just in case the set of points denoted by* x *is an open disc.*

For practical applications one would almost certainly want to avoid using set theory, in which case **A1** can be replaced by one of the following, weaker 1st-order formulae:

A1') $\exists x[\phi(x)] \rightarrow \exists! y[\mathsf{SUM}_x(\phi(x):y)]$
A1") $\forall xy \exists! z[z = x + y]$

A1' makes use of the following special syntax to refer to the sum of all regions satisfying a given predicate:

D21) $\mathsf{SUM}_x(\phi(x):y) \equiv_{def} \forall z[\phi(z) \rightarrow \mathsf{P}(z,y)] \wedge \neg\exists z[\mathsf{P}(z,y) \wedge \forall w[\phi(w) \rightarrow \mathsf{DR}(w,z)]]$

Useful Definitions

We now define a number of basic concepts that are very useful for describing spatial situations:

D22) $\mathsf{C}(x,y) \equiv_{def} \exists z[\mathsf{S}(z) \wedge \forall z'[z' \circledcirc z \rightarrow (\mathsf{O}(z',x) \wedge \mathsf{O}(z',y))]]$
D23) $\mathsf{EC}(x,y) \equiv_{def} \mathsf{C}(x,y) \wedge \neg \mathsf{O}(x,y)$
D24) $\mathsf{CON}(x) \equiv_{def} \forall yz[x = y + z \rightarrow \mathsf{C}(y,z)]$
D25) $\mathsf{Comp}(c,r) \equiv_{def} \mathsf{CON}(c) \wedge \neg\exists x[\mathsf{CON}(x) \wedge \mathsf{P}(x,r) \wedge \mathsf{PP}(c,x)]$

$\mathsf{C}(x,y)$ is the connection relation and behaves in a similar fashion to the primitive of (Randell, Cui, & Cohn 1992). The definition is based on the observation that for every point of contact (or overlap) between two regions, any sphere centred on that point overlaps both regions. It is easy to prove that C is symmetric and reflexive. EC is the relation of External Connection. $\mathsf{CON}(x)$ means that x is self-connected and $\mathsf{Comp}(c,r)$ means that c is a maximal self-connected Component of r.

Generalised Betweenness and Collinearity

The *bounding sphere* for a region is the smallest sphere of which the region is a part. Since our domain includes unbounded regions, not every region has a bounding sphere. We define

D26) $\mathsf{BS}(x,x') \equiv_{def} \mathsf{P}(x,x') \wedge \mathsf{S}(x') \wedge \neg\exists y[\mathsf{P}(x,y) \wedge \mathsf{S}(y) \wedge \mathsf{PP}(y,x')]$

We can generalise betweenness to a relation that can hold among any bounded regions:

D27) $\mathsf{BSB}(x,y,z) \equiv_{def} \exists x'y'z'[\mathsf{BS}(x,x') \wedge \mathsf{BS}(y,y') \wedge \mathsf{BS}(z,z') \wedge \mathsf{B}(x',y',z')]$

For convenience we also define a macro expression $\mathsf{Inline}[r_1,\ldots,r_n]$ to stand for the conjunction of the relations $\mathsf{Between}(r_{k-1},r_k,r_{k+1})$, for each k such that $1 < k < n$. Sometimes, we want to say that regions are collinear but don't care about their order:

D28) $\mathsf{Collin}(x,y,z) \equiv_{def} (\mathsf{B}(x,y,z) \vee \mathsf{B}(y,x,z) \vee \mathsf{B}(x,z,y))$

The macro $\mathsf{Collin}\{r_1,\ldots,r_n\}$ is used to specify a set of collinear regions and is defined analogously to Inline. $\circledcirc\{s_1,\ldots s_n\}$ specifies a set of concentric spheres.

The relation $\mathsf{Aligned}(x,y;z,w)$ holds when x,y,z,w are collinear and the direction from x to y is the same as the direction from z to w:

[1] E.g. (Tarski 1956). See also (Bennett 2000).

[2] Reflexivity and symmetry are implicit in the definition of $\circledcirc$.

$$\textbf{D29)}\ \mathsf{Aligned}(x,y;z,w) \equiv_{def} \mathsf{Collin}\{x,y,z,w\} \wedge \exists e e_1 e_2 e_3 e_4[\circledcirc\{e,e_1,e_2,e_3,e_4\} \wedge \mathsf{P}(e_1,e_2) \wedge \mathsf{P}(e_2,e_3) \wedge \mathsf{CB}(x,e_1) \wedge \mathsf{CB}(y,e_2) \wedge \mathsf{CB}(x,e_1) \wedge \mathsf{CB}(y,e_2)]$$

Congruence and Isometry

Borgo, Guarino, & Masolo (1996) showed how a *congruence* relation between arbitrary regions could be defined in terms of the S predicate. This relation is true of two regions if one can be transformed into the other by the operations of translation, rotation and taking a mirror image. The movement of physical bodies can be described in terms of translations and rotations but they do not ordinarily undergo mirror inversion. Thus, for our kinematic application we employ a congruence relation $\mathsf{CG}(x,y)$ which excludes the case where x is a mirror image of y, unless x (and therefore also y) is in itself symmetric. The more general case including mirror transforms we call *isometry* and write $\mathsf{Iso}(x,y)$.[3]

We first define congruence between spheres and pairs of spheres (SCG). We can then give a definition of isometry (Iso) that makes use of the concept of a '*scalene* sum of spheres' (SSS) (Borgo, Guarino, & Masolo 1996). This is a region whose Components are spheres and are all of different sizes:

$$\textbf{D30)}\ \mathsf{SCG}(x,y) \equiv_{def} \mathsf{S}(x) \wedge \mathsf{S}(y) \wedge \exists s_1 s_2[s_1 \circledcirc s_2 \wedge \mathsf{ET}(x,s_1) \wedge \mathsf{ET}(y,s_1) \wedge \mathsf{IT}(x,s_2) \wedge \mathsf{IT}(y,s_2)]$$

$$\textbf{D31)}\ \mathsf{SCG}(x,y;x',y') \equiv_{def} \mathsf{SCG}(x,x') \wedge \mathsf{SCG}(y,y') \wedge ((\mathsf{ET}(x,y) \wedge \mathsf{ET}(x',y')) \vee \exists z z'[\mathsf{SCG}(z,z') \wedge \mathsf{ED}(x,y,z) \wedge \mathsf{ED}(x',y',z')] \vee \exists z z'[\mathsf{SCG}(z,z') \wedge \mathsf{IT}(z,x) \wedge \mathsf{IT}(z,y) \wedge \mathsf{IT}(z',x') \wedge \mathsf{IT}(z',y')])$$

$$\textbf{D32)}\ \mathsf{SSS}(r) \equiv_{def} \forall x[\mathsf{Comp}(x,r) \rightarrow (\mathsf{S}(x) \wedge \neg\exists y[x \neq y \wedge \mathsf{Comp}(y,r) \wedge \mathsf{CG}(x,y)])]$$

$$\textbf{D33)}\ \mathsf{IsoSSS}(x,y) \equiv_{def} \mathsf{SSS}(x) \wedge \mathsf{SSS}(y) \wedge \forall^\circ st[(\mathsf{Comp}(s,x) \wedge \mathsf{Comp}(t,x)) \rightarrow \exists^\circ s't'[\mathsf{Comp}(s',y) \wedge \mathsf{Comp}(t',y) \wedge \mathsf{SCG}(s,t;s',t')]] \wedge \forall^\circ st[(\mathsf{Comp}(s,y) \wedge \mathsf{Comp}(t,y)) \rightarrow \exists^\circ s't'[\mathsf{Comp}(s',x) \wedge \mathsf{Comp}(t',x) \wedge \mathsf{SCG}(s,t;s',t')]]$$

$$\textbf{D34)}\ \mathsf{Iso}(x,y) \equiv_{def} \forall r[\mathsf{SSS}(r) \rightarrow ((\mathsf{P}(r,x) \rightarrow \exists r'[\mathsf{P}(r',y) \wedge \mathsf{IsoSSS}(r,r')]) \wedge (\mathsf{P}(r,y) \rightarrow \exists r'[\mathsf{P}(r',x) \wedge \mathsf{IsoSSS}(r,r')]))]$$

To distinguish (non-mirrored) congruence from isometry we define a predicate that identifies whether two triangular configurations of spheres are mirror images of one another. This makes use of the fact that the mid-points of the three segments connecting corresponding corners of mirror image triangles are collinear (see Fig. 1).[4]

$$\textbf{D35)}\ \mathsf{MirrorTs}(x_1,y_1,z_1;x_2,y_2,z_2) \equiv_{def} \exists^\circ x'y'z'[\mathsf{Mid}(x_1,x',x_2) \wedge \mathsf{Mid}(y_1,y',y_2) \wedge \mathsf{Mid}(z_1,z',z_2) \wedge \mathsf{Collin}(x',y',z') \wedge \neg(x' \circledcirc y' \wedge y' \circledcirc z')]$$

$$\textbf{D36)}\ \mathsf{CGTs}(x_1,y_1,z_1;x_2,y_2,z_2) \equiv_{def} \mathsf{SCG}(x_1,y_1;x_2,y_2) \wedge \mathsf{SCG}(y_1,z_1;y_2,z_2) \wedge \mathsf{SCG}(z_1,x_1;z_2,x_2) \wedge \neg\mathsf{MirrorTs}(x_1,y_1,z_1;x_2,y_2,z_2)$$

CGTs gives us a non-mirror congruence relation between triangles which we use to define a general congruence relation:

[3] Bennett *et al.* (2000) show how S can in fact be defined from Iso, so Iso could be taken as the morphological primitive of **RBG**.

[4] Whether isosceles are counted as mirrors depends on the ordering of the defining circles given as arguments to $\mathsf{MirrorTs}$.

$$\textbf{D37)}\ \mathsf{CGSSS}(s,t) \equiv_{def} \mathsf{SSS}(s) \wedge \mathsf{SSS}(t) \wedge \forall^\circ s_1,s_2,s_3[(\mathsf{Comp}(s_1,s) \wedge \mathsf{Comp}(s_2,s) \wedge \mathsf{Comp}(s_3,s)) \rightarrow \exists^\circ t_1,t_2,t_3[\mathsf{Comp}(t_1,t) \wedge \mathsf{Comp}(t_2,t) \wedge \mathsf{Comp}(t_3,t) \wedge \mathsf{CGTs}(s_1,s_3,s_3;t_1,t_2,t_3)]] \wedge \forall^\circ t_1,t_2,t_3[(\mathsf{Comp}(t_1,t) \wedge \mathsf{Comp}(t_2,t) \wedge \mathsf{Comp}(t_3,t)) \rightarrow \exists^\circ s_1,s_2,s_3[\mathsf{Comp}(s_1,s) \wedge \mathsf{Comp}(s_2,s) \wedge \mathsf{Comp}(s_3,s) \wedge \mathsf{CGTs}(s_1,s_3,s_3;t_1,t_2,t_3)]]$$

$$\textbf{D38)}\ \mathsf{CG}(x,y) \equiv_{def} \forall r[\mathsf{SSS}(r) \rightarrow ((\mathsf{P}(r,x) \rightarrow \exists r'[\mathsf{P}(r',y) \wedge \mathsf{CGSSS}(r,r')]) \wedge (\mathsf{P}(r,y) \rightarrow \exists r'[\mathsf{P}(r',x) \wedge \mathsf{CGSSS}(r,r')]))]$$

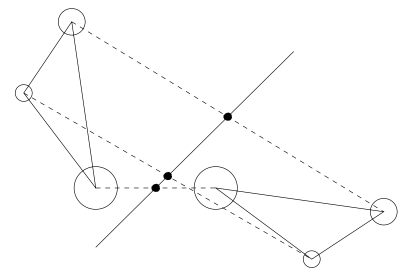

Figure 1: Mirror image triangles

Rectangular Disc Configurations

Later we shall define coordinate systems in terms of spheres. This will make use of the relation that holds when three spheres are arranged so that their centre points subtend a right angle. We use the following definition (explained by Fig. 2):

$$\textbf{D39)}\ \mathsf{RECT}(a,b,c) \equiv_{def} \exists b'c'dd'[b' \circledcirc b \wedge \mathsf{CG}(c',c) \wedge \mathsf{CG}(d',d) \wedge \mathsf{ED}(c,c',b') \wedge \mathsf{ED}(a,d,c) \wedge \mathsf{ED}(a,d',c')]$$

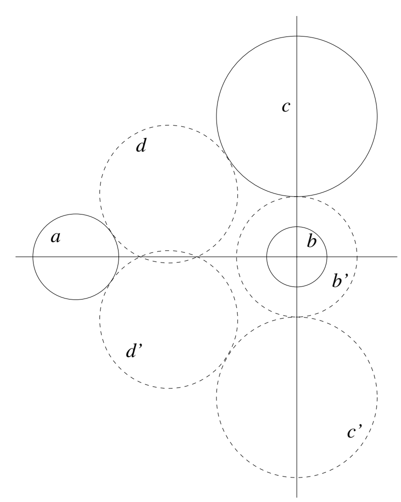

Figure 2: Rectangular configuration of discs a, b, c

Half-Planes

Both for expressing qualitative conditions and for interfacing with geometrical representations, the concept of a half-plane is extremely useful. We note that any two externally tangent discs d_1 and d_2 define a unique tangent line passing through their point of contact. Moreover, the set of discs containing d_1 but discrete from d_2 all lie on one side of the tangent whereas those containing d_2 but discrete from d_1 lie on the other side. These considerations lead to the following definition of a predicate $\mathsf{HP}(x)$, which says that the region x is a half-plane:

D40) $\mathsf{HP}(h) \equiv_{def}$
$$\forall^{\circ} d_1 d_2[(\mathsf{ET}(d_1,d_2) \wedge \mathsf{P}(d_1,h) \wedge \mathsf{DR}(d_2,h)) \rightarrow (\forall^{\circ} d_3[\mathsf{P}(d_1,d_3) \wedge \mathsf{DR}(d_2,d_3)) \rightarrow \mathsf{P}(d_3,h)] \wedge \forall^{\circ} d_4[\mathsf{DR}(d_1,d_4) \wedge \mathsf{P}(d_2,d_4)) \rightarrow \mathsf{DR}(d_4,h)])]$$

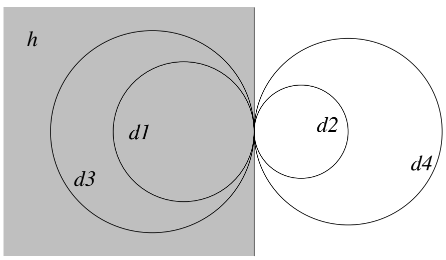

Figure 3: Defining a half-plane in terms of spheres

Congruent Pairs

It is very useful to be able to specify that a given pair of regions $\langle a, b\rangle$ is congruent to another pair $\langle a', b'\rangle$. By this we mean that $\mathsf{CG}(a, a')$ and $\mathsf{CG}(b, b')$ and also that the position of a relative to b is the same as the position of a' relative to b'. If a and b are discrete then this is true just in case $\mathsf{CG}((a + b),(a' + b'))$; but in general we have to take care of cases where a and b overlap. Thus we define:

D41) $\mathsf{CG}(a, b; c, d) \equiv_{def}$
$$(\mathsf{CG}(a, a') \wedge \mathsf{CG}(b, b') \wedge \mathsf{CG}((a + b),(a' + b'))) \wedge$$
$$(\mathsf{PP}(a, b) \rightarrow \exists xy[\mathsf{Diff}(b, a, x) \wedge \mathsf{Diff}(b', a'y) \wedge \mathsf{CG}(x, y)]) \wedge$$
$$(\mathsf{PP}(b, a) \rightarrow \exists xy[\mathsf{Diff}(a, b, x) \wedge \mathsf{Diff}(a', b', y) \wedge \mathsf{CG}(x, y)]) \wedge$$
$$(\mathsf{PO}(a, b) \rightarrow \exists xyzw[\mathsf{Diff}(a, b, x) \wedge \mathsf{Diff}(b, a, y) \wedge \mathsf{Diff}(a', b', z) \wedge \mathsf{Diff}(b', a', w) \wedge \mathsf{CG}(x + y, z + w)])$$

Disc-Based Coordinate Systems

Because Cartesian fields over real numbers provide canonical models for point-based Euclidean geometry, configurations of points and lines are very often described in terms of numerical coordinates. At first sight it may seen that the idea of a coordinate system is alien to region-based geometries. However, as we shall see, something very similar can in fact be set up. Fig. 4 illustrates how three distinguished discs (orig, xunit and yunit) fix a coordinate system. These discs must satisfy the axiom:

A10)
$$\mathsf{S}(\mathsf{orig}) \wedge \mathsf{S}(\mathsf{xunit}) \wedge \mathsf{S}(\mathsf{yunit}) \wedge \mathsf{xunit} \neq \mathsf{yunit} \wedge \mathsf{IT}(\mathsf{xunit}, \mathsf{orig}) \wedge \mathsf{IT}(\mathsf{yunit}, \mathsf{orig})$$
$$\wedge \exists u[\mathsf{CG}(u, \mathsf{xunit}) \wedge \mathsf{EC}(u, \mathsf{xunit}) \wedge \mathsf{ID}(\mathsf{xunit}, u, \mathsf{orig})]$$
$$\wedge \exists u[\mathsf{CG}(u, \mathsf{yunit}) \wedge \mathsf{EC}(u, \mathsf{yunit}) \wedge \mathsf{ID}(\mathsf{yunit}, u, \mathsf{orig})]$$

The x and y 'coordinates' of an arbitrary disc are given by the following defined relation:

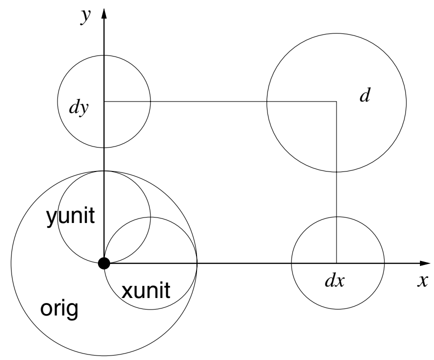

Figure 4: A disc-based coordinate system: x is a disc with coordinates dx and dy: $\mathsf{Coords}(x, dx, dy)$

D42) $\mathsf{Coords}(d, x, y) \equiv_{def}$
$$\mathsf{Collin}(\mathsf{orig}, \mathsf{xunit}, x) \wedge \mathsf{RECT}(\mathsf{orig}, x, d) \wedge \mathsf{Collin}(\mathsf{orig}, \mathsf{yunit}, y) \wedge \mathsf{RECT}(\mathsf{orig}, y, d)$$

Since we have distinguished discs of 'unit length', one could go on to define metrical predicates.

Motion

In order to specify motions of rigid objects we first specify the simple motions of linear translation and rotation about the centre point of some sphere.

Linear Translation

To specify simple linear motions we define the translation of a region x_1 to the congruent region x_2 along a vector defined by the centre points of two discs d_1 and d_2 as follows (see Fig. 5):

D43) $\mathsf{TAV}(x_1, x_2, d_1, d_2) \equiv_{def} \exists d[d \circledcirc d_2 \wedge \mathsf{CG}(x_1, d_1; x_2, d)$

We can also define translation part-way along a vector:

D44) $\mathsf{PTAV}(x_1, x_2, d_1, d_2) \equiv_{def}$
$$\exists d[\mathsf{B}(d_1, d, d_2) \wedge \mathsf{CG}(x_1, d_1; x_2, d)$$

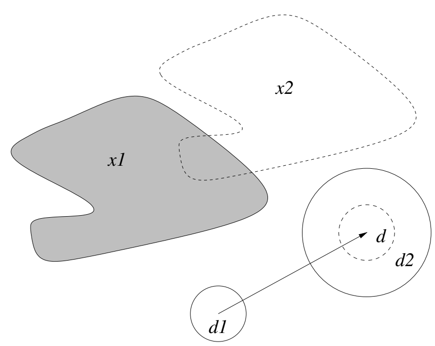

Figure 5: Translation along a vector

Rotation

One region is a bounding-sphere-centred rotation of another iff they are congruent and share the same bounding sphere.

D45) $\mathsf{BSC\text{-}Rot}(x, y) \equiv_{def}$
$(\mathsf{CG}(x, y) \wedge \exists s[\mathsf{BS}(x, s) \wedge \mathsf{BS}(y, s)])$

More generally we can consider the rotation of a region about the centre point of any arbitrary sphere.

D46) $\mathsf{Rot}(x, y, s) \equiv_{def} (\mathsf{S}(s) \wedge \mathsf{CG}(x, s; y, s))$

Later we shall consider rotations of an object within some confining environment and to do this we shall consider those positions which are occupied by a region as it undergoes a continuous rotation between two orientations. We define $\mathsf{RotOrd}(a, b, c, s)$ to mean that a, b and c are rotations of a region about the centre point of a sphere s, such that: if a region r congruent to the three regions is rotated continuously (about the centre of s) from a to c in the direction that requires the smallest angular rotation, then r passes through b.

D47) $\mathsf{RotOrd}(a, b, c, s) \equiv_{def} \mathsf{Rot}(a, b, s) \wedge \mathsf{Rot}(b, c, s) \wedge$
$\exists a'b'c'[\mathsf{CG}\{a', b', c', s\} \wedge \mathsf{EC}(a', s) \wedge \mathsf{EC}(b', s) \wedge$
$\mathsf{EC}(c', s) \wedge \mathsf{CG}(a, a'; b, b') \wedge \mathsf{GC}(b, b'; c, c') \wedge$
$\exists t[\mathsf{S}(t) \wedge \mathsf{EC}(t, s) \wedge \mathsf{IT}(b', t) \wedge \mathsf{O}(t, a) \wedge \mathsf{O}(t, c)]]$

The construction employed in this definition is shown in Fig. 6, where for the sake of clarity we have chosen a case where s is the bounding circle of a, b and c.

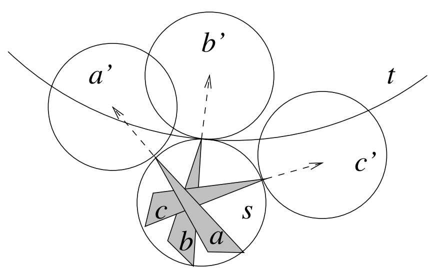

Figure 6: Illustration of the definition of RotOrd

Movement in a Constraining Environment

We now apply our theory to provide a model of physical environments that may be useful for reasoning about motions of rigid objects. This model might support practical applications such as the control of robots.

The fundamental problems of kinematics are of the forms: can a rigid body move between two locations within a confining environment? (the piano-movers problem (Schwartz & Sharir 1983)) and, if so, what is a possible path between the two locations? A simple example is illustrated in Fig. 7, where we want to know whether an object can move between two rooms *via* a narrow corridor.

Our idea is to model a possible movement within an environment as a series of translations and rotations, during each of which the area occupied by a (rigid) object must always lie within a region of free space.

First we define a *linear translation within* as a translation from x_1 to x_2 along some vector such that all translations part way along the vector lie within a confining region y:

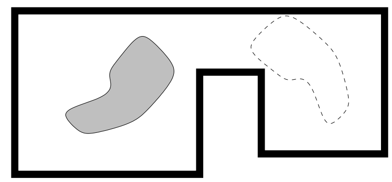

Figure 7: A problem of constrained movement

D48) $\mathsf{LTW}(x_1, x_2, y) \equiv_{def} \exists d_1 d_2[\mathsf{TAV}(x_1, x_2, d_1, d_2) \wedge$
$\forall x[\mathsf{PTAV}(x_1, x, d_1, d_2) \rightarrow \mathsf{P}(x, y)]]$

We define a 'short' (less than $180°$) rotation of a region between two orientations x_1 and x_2, such that during the rotation the region remains within some hosting environment y as follows:

D49) $\mathsf{SRotW}(x_1, x_2, y) \equiv_{def} \exists s[\exists x'[\mathsf{RotOrd}(x_1, x', x_2, s)]$
$\wedge \forall x'[\mathsf{RotOrd}(x_1, x', x_2, s) \rightarrow \mathsf{P}(x', y)]$

An arbitrary rotation within is just defined as the concatenation of two short rotations:

D50) $\mathsf{RotW}(x_1, x_2, y) \equiv_{def}$
$\exists x'[\mathsf{SRotW}(x_1, x', y) \wedge \mathsf{SRotW}(x', x_2, y)]$

Simple move within:

D51) $\mathsf{SMW}(x_1, x_2, y) \equiv_{def}$
$(\mathsf{LTW}(x_1, x_2, y) \vee \mathsf{RotW}(x_1, x_2, y))$

We now axiomatise a predicate $\mathsf{MoveWithin}(x_1, x_2, y)$ to say that a rigid body can move from region x_1 to region x_2, while remaining within region y:

A11) $\mathsf{MoveWithin}(x_1, x_2, y) \leftrightarrow \mathsf{SMW}(x_1, x_2, y) \vee$
$\exists x'[\mathsf{SMW}(x_1, x', y) \wedge \mathsf{MoveWithin}(x', x_2, y)]$

We could restrict this so that a move was a strictly alternating sequence of LTW and RotW operations, which reduces the search space (though it is still infinite).

MoveWithin as a Finite Sequence of SMWs

In fact, although axiom **A11** is correct it is not strictly adequate since it does not actually force MoveWithin to coincide with the transitive closure of SMW. However it is likely to be sufficient for a large class of constructive proofs of whether $\mathsf{MoveWithin}(x_1, x_2, y)$ follows from (or is consistent with) some given constraints.

To get a more complete charaterisation of MoveWithin we would first have to define the notion of a finite sequence. Within our framework this can be modelled by (e.g.) a region σ that is a sum of discrete spheres, such that there is a smallest sphere s_1 and a largest sphere s_n and for all other spheres there is a sphere of half the diameter and a sphere of twice the diameter (thus the spherical Components are stricly ordered). We can then define $\mathsf{MoveWithin}(a, b, r)$ by saying that for each sphere s where $\mathsf{Comp}(s, \sigma)$ there is a pair of regions $\langle x_s, y_s \rangle$ related by $\mathsf{SMW}(x_s, y_s, r)$, such that $x_{s_1} = a$ and $y_{s_n} = b$ and moreover if t is a component of σ with twice the diameter of s then $y_s = x_t$. Given such a definition **A11** would not be needed.

General Continuous Motions

The definition of MoveWithin also assumes that any motion could be reduced to a series of linear translations and rotations. We call such a motion a *TR-motion*. However, in general there are many continuous motions that cannot be reduced in this way. For instance an object might trace a parabolic curve. On the other hand, it is clear that if an object moves within a confining environment such that it always has more than one degree of freedom, then for every continuous motion there is a TR-motion with the same start and end locations. Hence we believe that for most robotic applications only TR-motions need to be considered.

Nevertheless, for the sake of having a Comprehensive ontology one might want to define arbitrary continuous motions. In fact we have been able to define set-theoretically the notion of a set of unit discs whose centre points form a continuous non-fractal path; and by using $\mathsf{CG}(x, y; z, w)$ one can easily construct from this a definition of a set of regions lying on a continuous (non-fractal) path. But such a definition is of only theoretical importance and will not be given here.

Movable Obstacles

So far we have considered the case of a single rigid object moving in a constraining environment; but in general a situation will involve multiple movable objects.

We need a way of representing a collection of rigid bodies in terms of a fixed finite number of regions. This would enable the situation to be included in the argument places or ordinary predicates. A naïve representation would be to sum the bodies and then identify objects with Components of this sum. But, if bodies may be EC to each other this does not work. However, we can appeal to the famous *four colour theorem* to see that any configuration of 2-dimensional objects can be described in terms of the Components of four regions.

Thus, we represent a *situation* by a 4-tuple $\sigma = \langle r_1, r_2, r_3, r_4 \rangle$ of mutually DR regions. The following definitions enable us to describe the movement of one amongst a collection of movable objects:

D52) $\mathsf{Sit}(\langle r_1, r_2, r_3, r_4 \rangle) \equiv_{def} \mathsf{DR}(r_1, r_2) \wedge \mathsf{DR}(r_1, r_3) \wedge \mathsf{DR}(r_1, r_4) \wedge \mathsf{DR}(r_2, r_3) \wedge \mathsf{DR}(r_2, r_4) \wedge \mathsf{DR}(r_3, r_4)$

D53) $\mathsf{SitComp}(c, \langle s_1, s_2, s_3, s_4 \rangle) \equiv_{def} \mathsf{Comp}(c, s_1) \vee \mathsf{Comp}(c, s_2) \vee \mathsf{Comp}(c, s_3) \vee \mathsf{Comp}(c, s_4)$

D54) $\mathsf{SitEQ}(\sigma, \sigma') \equiv_{def} \mathsf{Sit}(\sigma) \wedge \mathsf{Sit}(\sigma') \wedge \forall x[\mathsf{SitComp}(x, \sigma) \leftrightarrow \mathsf{SitComp}(x, \sigma')]$

D55) $\mathsf{MoveOne}(\sigma, \sigma', f) \equiv_{def} \exists r_1 r_2 r_3 r_4 o o' f' [\mathsf{Diff}(f, r_1 + r_2 + r_3 + r_4, f') \wedge \mathsf{MoveWithin}(o, o', f') \wedge \mathsf{SitEQ}(\sigma, \langle r_1 + o, r_2 r_3 r_4 \rangle) \wedge (\mathsf{SitEQ}(\sigma', \langle r_1 + o', r_2, r_3, r_4 \rangle) \vee \mathsf{SitEQ}(\sigma', \langle r_1, r_2 + o', r_3, r_4 \rangle))$

$\mathsf{MoveOne}(\sigma, \sigma', f)$ holds just in case: σ and σ' are situation tuples representing configurations of objects; f is the region of free space in which the objects are situated; and, σ can be transformed into σ' by moving just one of the objects. The definition requires some explanation: $\langle r_1, r_2, r_3, r_4 \rangle$ represents the situation of all the objects except the one that is moved; without loss of generality we can assume that the initial situation σ can be obtained by adding the starting location o of the moved object; the final situation is obtained by adding a congruent region o' to either r_1 or r_2 corresponding to cases where a 'four-colouring' of the configuration would have the start and finish locations coloured the same or differently.

To determine whether one situation can be transformed into another by a series of movements of movable objects we can axiomatise a general MoveTrans relation as follows.[5]

A12) $\mathsf{MoveTrans}(\sigma, \sigma', f) \leftrightarrow (\mathsf{MoveOne}(\sigma, \sigma', f) \vee \exists \sigma'' [\mathsf{MoveOne}(\sigma, \sigma'', f) \wedge \mathsf{MoveTrans}(\sigma'', \sigma', f)])$

Pushing Obstacles

In order to model the action of an auto-motive object such as a robot pushing a movable obstacle we have to know something about the resistance of obstacles to forces. Since we assume that we know which obstacles can and cannot be moved we are not concerned with the magnitude of an obstacle's resistance but only its direction of action. Thus, we model a movable obstacle by a pair $\langle r, c \rangle$, where r is the region it occupies and c is its *centre of resistance*. If a force acts on an obstacle through its centre of resistance it will move in a straight line; if it acts otherwise the force will cause a rotation.

D56) $\mathsf{Push}(r, r', \langle x, c \rangle, \langle x' c' \rangle) \equiv_{def} \exists d_1 d_2 [\mathsf{EC}(d_1, d_2) \wedge \mathsf{P}(d_1, r) \wedge \mathsf{P}(d_2, x) \wedge \mathsf{Aligned}(d_1, d_2; c, c')] \wedge \mathsf{CG}(r, x; r', x') \wedge \mathsf{CG}(x, c; x', c')$

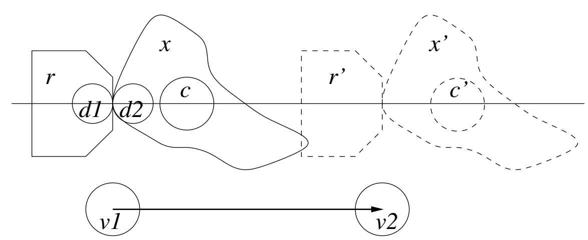

Figure 8: Robot r pushes an object from x to x'

It is fairly straightforward to modify the definitions of MoveOne and MoveTrans in order to describe situation transforms that can be achieved by a series of push operations. However, space does not permit us to give full details here.

Conclusion

We have explored the use of an expressive region-based geometry for describing spatial situations and reasoning about the movements of rigid bodies. Our formalism provides a rigorous ontological foundation for use in AI systems that need to process spatial information.

For most practical applications one will require tractable sublanguages of our very expressive formalism. (Cristani,

[5]This suffers from the same limitations as axiom **A11** for MoveWithin given above, and has the same possible solution.

Cohn, & Bennett 2000) investigate the Complexity of reasoning with a combination of mereological and morphological relations and proves tractability of a significant constraint language, which is a fragment of our formalism. Recursive definitions such as that for MoveWithin could be implemented by search algorithms. We envisage architectures whereby search is combined with one or more constraint solvers which determine when branches can be closed. Inductive theorem proving techniques may also be useful.

We believe that the formalism of region-based geometry will be extremely useful as a framework within which more Computationally oriented representations can be embedded.

References

Bennett, B.; Cohn, A.; Torrini, P.; and Hazarika, S. 2000. Region-based qualitative geometry. Technical Report 2000.07, University of Leeds, LS2 9JT, UK.

Bennett, B. 2000. Research note on Tarski's elementary geometry. Technical report, University of Leeds, School of Computer Studies. Available from web at `http://www.scs.leeds.ac.uk/pub/doc/srg/tarski_geom.ps`.

Borgo, S.; Guarino, N.; and Masolo, C. 1996. A pointless theory of space based on strong congruence and connection. In Aiello, L. C., and Doyle, J., eds., *Principles of Knowledge Representation and Reasoning, Proceedings of the 5th International Conference, KR96*. Morgan Kaufmann.

Cristani, M.; Cohn, A.; and Bennett, B. 2000. Spatial locations via morpho-mereology. In *Proceedings of KR'2000*. To appear.

Davis, E. 1987. A framework for qualitative reasoning about solid objects. In Rodriguez, G., ed., *Proceedings of the Workshop on Space Telerobotics*, 369–375. NASA and JPL. Reprinted in (Weld & De Kleer 1990).

Davis, E. 1988. A logical framework for commonsense predictions of solid object behavior. *AI in Engineering* 3(3):125–138.

Faltings, B. 1987. Qualitative kinematics in mechanisms. In *Proceedings IJCAI-87*, 436–442.

Forbus, K.; Nielsen, P.; and Faltings, B. 1987. Qualitative kinematics: A framework. In *Proceedings IJCAI-87*, 430–436.

Forbus, K. D.; Nielsen, P.; and Faltings, B. 1991. Qualitative spatial reasoning: The clock project. *Artificial Intelligence* 51:417–471.

Guarino, N., ed. 1998. *Formal ontology in information systems: Proceedings of the 1st international conference (FOIS-98)*, volume 46 of *Frontiers in Artificial Intelligence and Applications*. Trento, Italy: Ios Press.

Lenat, D. 1995. CYC: a large-scale investment in knowledge infrastructure. *Communications of the ACM* 38(11).

Mukerjee, A., and Bhatia, P. 1995. A qualitative discretization for two-body contacts. In Mellish, C. S., ed., *Proceedings of the 14th International Joint Conference on Artificial Intelligence (IJCAI-95)*, volume 1, 915–921.

Randell, D. A.; Cui, Z.; and Cohn, A. G. 1992. A spatial logic based on regions and connection. In *Proc. 3rd Int. Conf. on Knowledge Representation and Reasoning*, 165–176. San Mateo: Morgan Kaufmann.

Schwartz, J., and Sharir, M. 1983. On the 'piano movers ' problem, II: General techniques for computing topological properties of real algebraic manifolds. In *Advances in Applied Mathematics*, volume 4. Academic Press.

Tarski, A. 1929. Les fondaments de la géométrie des corps. *Ksiega Pamiatkowa Pierwszego Polskiego Zjazdu Matematycznego* 29–33. A suplement to *Annales de la Société Polonaise de Mathématique*. English translation, 'Foundations of the Geometry of Solids', in A. Tarski, *Logic, Semantics, Metamathematics*, Oxford Clarendon Press, 1956.

Tarski, A. 1956. Foundations of the geometry of solids. In *Logic, Semantics, Metamathematics*. Oxford Clarendon Press. chapter 2. trans. J.H. Woodger.

Weld, D. S., and De Kleer, J., eds. 1990. *Readings in Qualitative Reasoning About Physical Systems*. San Mateo, Ca: Morgan Kaufman.