

Can We Work Around Numerical Methods? An Insight

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Introduction

Self Sustaining Oscillators are of interest to researchers given the fact that they provide for excellent test beds to study the nuances and intricacies involved in typical oscillatory systems. Numerous such oscillations have been studied using traditional numerical methods in the past, contrary to the propositions of this article. Majority of study of van der Pol equation like oscillatory systems are directed towards deducing a stability criterion.

This paper outlines a non-conventional study of the stability standards of the van der Pol Equation. The equation simulates a typical RLC circuit i.e. as a resistor for higher currents, but as a negative resistor for lower currents. Such a behavior is termed *Relaxation Oscillation* [Nanjundiah 1958]. The van der Pol equation may be defined as an ODE describing oscillations which amplifies smaller oscillations and dampens large oscillations on the other side. It may be obtained by differentiating the Rayleigh Equation and setting $y = y'$. Detailed analysis of van der Pol equation and its applicability may be found in [Wiggins 1990, Buonomo 1999, Kneubuehl and Kneubhyl 2001]. One of prevalent forms of the van der Pol equation may be given as [Buonomo 1999, Kneubuehl and Kneubhyl 2001];

$$y'' - \mu(1 - y^2)y' + y = 0 \quad (0)$$

Preliminary analysis of the above equation leads to a conclusion that the coefficient of the first order derivative of y plays an important role in the behavior and stability of the oscillations; the system equation may be then given as,

$$Y'' = \varepsilon(1 - Y^2)Y' - Y \quad (1)$$

The Van der Pol equation describes systems characterized by an unstable fixed / equilibrium point, but with a steady limit cycle. It is expected that the behavior of the equation is dependent upon the values of ' ε '. The objective of this work is to approximate the function parameters and predict the behavior of the system without the use of, but in comparison to the conventional numerical methods with a Neural Network based system.

Theory and Related Mathematics

The van der Pol equation (Equation.1) can be discretized as,

$$X' = \varepsilon(1 - Y^2)X - Y \quad (2)$$

$$\text{with, } Y' = X \quad (3)$$

The transformed set of equations has the fixed point at $(0, 0)$, but it is seen that the aforesaid fixed point is not stable. Thus one should expect the plot for the aforementioned equation to be a closed curve and such a closed curve surrounding the fixed (unstable) point is called the '*limit cycle*'. The above equations have been subjected to numerical methods (both formulated and standard) to arrive logically at a stability criterion. Further, observations with respect to the behavior of the equation for changing ' ε ' and '**step – size**' (h) values have been made at consistent intervals. Two implicit and one second order method have been formulated to analyze the '*van der Pol*' equation. Standard *Euler* and the *Fourth order Runge – Kutta* methods were also employed to compare the efficiency of the formulated methods. Further, Neural Network models, based on the mathematical framework of the formulated numerical methods, have been developed. Such a non-conventional computing method has been adopted in order to generalize the approximation capability of the formulated methods. The formulated methods and the neural network models have been tested and all the properties of the solutions have been tabulated thereon. The stability of the van der Pol equation is analyzed at the fixed point $(0, 0)$. Now re-writing equations (2) and (3);

$$G_1(X, Y) = \varepsilon(1 - Y^2)X - Y \quad (4)$$

$$G_2(X, Y) = X \quad (5)$$

Eigen values are given from the characteristic equation; i.e.

$$\lambda^2 - \varepsilon\lambda + 1 = 0 \quad (6)$$

For the above system to be stable we know the condition is $|\lambda| < 1$; and further conditional to $\varepsilon > 0$. In the given case, $\varepsilon < 0$, hence rendering an unstable fixed point. After the preliminary study, we have the following systems;

First Order Implicit Method – I; (Formulated)

$$X_{n+1} = \frac{X_n - h Y_n + \varepsilon h X_n}{1 + \varepsilon h Y_n^2} \quad (7)$$

$$\text{and, } Y_{n+1} = Y_n + h X_{n+1} \quad (8)$$

First Order Implicit Method – II; (Formulated)

$$X_{n+1} = \frac{X_n(1 + 2 h \varepsilon - 2 h \varepsilon X_n^2) - h^2 X_n}{1 + \varepsilon h (1 - X_n^2)} \quad (9)$$

$$Y_{n+1} = Y_n + h X_{n+1} \quad (10)$$

Second Order Implicit Method; (Formulated)

$$X_{n+1} = \frac{X_n [2 - \epsilon(1 - Y_n^2) + 2\epsilon - 2Y_n^2 \epsilon - h - 2\epsilon Y_n] - 2Y_n(1 + 2\epsilon Y_n)}{2 - h\epsilon(1 - Y_n^2) + h/2 + hX_n Y_n \epsilon} \quad \text{--- (11)}$$

$$Y_{n+1} = Y_n + \frac{h}{2} X_{n+1} + \frac{h}{2} X_n \quad \text{(12)}$$

Following are the plots for the formulated methods; post the analysis for stability at the fixed point; (Figure. 1 – 5).

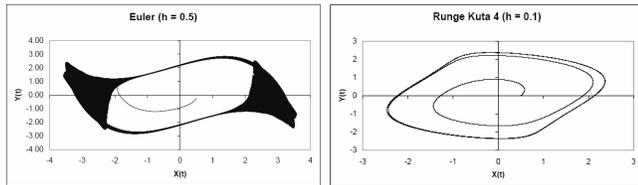


Figure 1. Euler Method

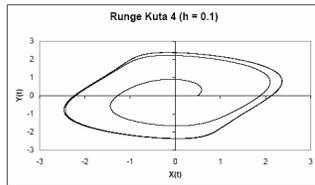


Figure 2. Runge Kutta Method

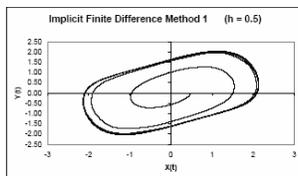


Figure 3. First Order Method – I

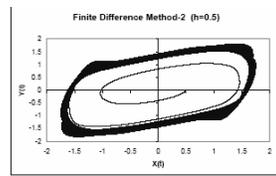


Figure 4. First Order Method – II

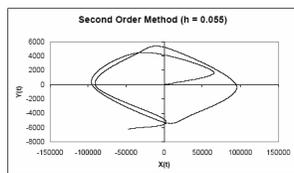


Figure 5. Second Order Method

Figures 1 – 5 depict the stability limit cycles for the five methods.

The Neural Network & the Justification: Despite of the superior performance (considering the results and that the formulated methods have outperformed the standard ones); given time and effort, many more such numerical methods may be formulated and tested iteratively (usually in a random fashion). Such a practice is not feasible in real time systems and thus a novel system based on a special neural architecture has been developed and generalized over the aforementioned methods in order to suit the varied parameters and conditions of a van der Pol system. The formulated three methods along with the standard Euler & Runge Kutta methods have been used to as the basis for designing the structure and type of the Neural Network. Further to the design, some training & testing data was generated using a random combination algorithm, thus inculcating trends of all the five methods into one data stream. A Feed Forward Neural Network was specifically chosen along with a hybrid learning algorithm [Jang 1993]. The discrete parameter set from the above equations is estimated in a two tier fashion i.e. the higher order parameters are estimated using the Least Squares Estimator in the backward run and first order parameters are estimated in the forward run using the gradient descent method. Following is the results table;

TABLE – 1
Comparison of Analysis by Numerical & Neural Methods

Step size (h)	IMPLICIT METHOD # 1	EULER METHOD	NEURAL NETWORK
0.01	SLC	SLC	SLC
0.05	SLC	SLC	SLC
0.30	SLC	SLC	SLC
0.5	SLC	Overflow	SLC
0.55	SLC	Overflow	SLC
0.6	Overflow	Overflow	SLC
0.65	Overflow	Overflow	SLC

SLC – Stable Limit Cycle.

Discussion & Conclusions: The above helps clearly identify that the trained neural network possesses good approximation capabilities. Pending extensive testing, it can be stated that the presented work shows light into a prospective application area for Neural Networks. Not only from the application point of view, but even, with respect to the ease of numerical analysis, the presented work evaluates unconventional computing favorably to classic numerical solutions. We have analyzed an augmented form of the van der Pol equation using standard numerical methods and obtained respective solution sets. Upon comparison with the standard Euler method, one implicit method (implicit method 1) was stable for longer than the Euler until a step size of 0.55 for $X(0) = 0.5, Y(0) = 0.0$. The first implicit method, unexpectedly, has not overflowed until $h = 0.55$. And the neural network based method remained stable past a step size of 0.6 until 0.65. With increasing step size, the system becomes more and more unstable until the critical step size where the system overflows. Thus we can finally assert that for a given ϵ value there would be a step size (also called the ‘critical step size’), beyond which the solution set shall overflow (regardless of the values); and such a conclusion is in accordance to the previous studies [Buonomo 1999]. The developed system is advantageous in the sense that, it has the generalized approximation capability of all the five methods; it also has an improved performance. Such a trait may be attributed to the fact that the hybrid system is not pruned to one kind of the parameter set but is generalized over an array of sets i.e. the new system avoids over fitting the methodology onto a single instance of van der Pol system and in turn learns to equally fit all available sets.

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