Proceedings of the Twenty-Third AAAI Conference on Artificial Intelligence (2008)

Nonmonotonic Modes of Inference

Victor Jauregui
The University of New South Wales
Sydney, Australia.

Abstract

In this paper we investigate nonmonotonic 'modes of inference'. Our approach uses modal (conditional) logic to establish a uniform framework in which to study nonmonotonic consequence. We consider a particular mode of inference which employs a majority-based account of default reasoning—one which differs from the more familiar preferential accounts—and show how modal logic supplies a framework which facilitates analysis of, and comparison with more traditional formulations of nonmonotonic consequence.

Introduction

There has been much interest in the field of Artificial Intelligence concerning forms of reasoning which are not bound by the stringent requirements of mathematical (deductive) inference; e.g., ampliative, and defeasible types of inference. Particular examples arise in logic programming (negation as finite failure (Clark 1978)), default/commonsense reasoning (circumscription (McCarthy 1980), and default logic (Reiter 1980)), etc. So while deductive inference has undergone long and extensive study, and is relatively well understood, characterising nonmonotonic consequence, by comparison, perhaps by its diverse nature, has proven more elusive.

Mathematical deductive inference is based on the notion of proof: the irrefutable, logical process of establishing truths. Gödel (see (Gödel 1986)) showed that it is possible, in sufficiently expressive theories such as Peano arithmetic, to axiomatise deduction, and hence mathematical inference, within the theory, allowing inference itself to become the formal object of study.

Löb's (1955) subsequent characterisation of mathematical proof led to the realisation (Solovay 1976) that provability (i.e., deductive inference) could be formalised as a modal notion in which the expression $\Box\varphi$ is interpreted as saying that "φ is provable" (in a theory incorporating Peano arithmetic). When formulated in modal logic Löb's derivability conditions yield the following basic properties of mathematical provability (Boolos 1993), (Smorynski 1984), (Solovay 1976):

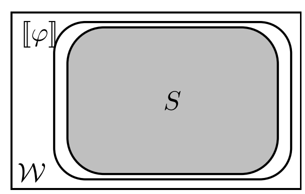
$$RN: \quad \text{if} \vdash \varphi, \text{then} \vdash \Box\varphi$$
$$K: \quad \vdash \Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$$
$$4: \quad \vdash \Box\varphi \rightarrow \Box\Box\varphi$$

Copyright © 2008, Association for the Advancement of Artificial Intelligence (www.aaai.org). All rights reserved.

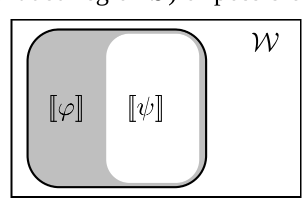
The first says that if φ is established in a theory including the axioms of Peano arithmetic then it can be shown to be provable within that theory. In modal logic this corresponds to the rule of *necessitation*. The second reflects the inference rule of *modus ponens*, or *detachment*: if $\varphi \rightarrow \psi$ and φ are both provable (within the theory), then so is ψ. Notably, properties RN and K combine to characterise mathematical inference as a 'normal' modality (Chellas 1980).

In this way we may regard deduction as a particular *mode of inference* satisfying certain properties. This begs the question: if similar accounts were extended to modes of inference other than that of deduction, e.g., to nonmonotonic modes of inference, what properties would characterise them?

Default reasoning by majorities. To assist in addressing the question posed above, let us consider a particular mode of nonmonotonic inference which we refer to as *majority default inference* (Jauregui 2007). In the modal tradition we supply a possible-worlds semantics in which the basic idea is to infer a formula φ to be true by default if it holds over a 'majority' of possible worlds. For example, only infrequently is my local baker out of bread. One might infer defeasibly that by going to the baker I will be able to acquire bread; an inference based on the observation that 'in most cases' the baker has bread.



Let $\mathcal{W}$ be the set of all possible worlds, and let S be a set which represents a majority of $\mathcal{W}$—defining which sets comprise majorities will be postponed until later. We write $[\![\varphi]\!]$ to denote the set of φ-worlds: the set of worlds in which φ holds. The diagram above represents a model in which, by this account, φ would be inferred by default as φ holds over some majority (the shaded region S) of possible worlds; i.e., $S \subseteq [\![\varphi]\!]$.



Consider the model shown in the following diagram. Intuitively worlds are assumed to be uniformly distributed so the number of worlds in a region is determined geometrically by its area. Moreover, for the purposes of this example, we will regard two thirds of the area of $\mathcal{W}$ to constitute a majority. In the diagram, $[\![\psi]\!]$ represents the white region enclosed in $[\![\varphi]\!]$, and $[\![\varphi]\!]$ includes both the shaded region and $[\![\psi]\!]$. From the picture, it follows that $[\![\varphi]\!]$ represents roughly two-thirds (in area) of $\mathcal{W}$, and hence comprises a majority; so φ would be inferred by default as being true over a majority of worlds. Similarly, $[\![\varphi \to \psi]\!]$ is represented by the union of the complement of $[\![\varphi]\!]$, and $[\![\psi]\!]$, which corresponds to all but the shaded region of $\mathcal{W}$. This again represents roughly two-thirds of $\mathcal{W}$, and so $\varphi \to \psi$ would thus be also inferred by default. However, $[\![\psi]\!]$ encompasses only roughly one third of $\mathcal{W}$ (again, by area), and so ψ would not be inferred by default.

It would appear that, we would infer φ and $\varphi \to \psi$, but not ψ; i.e., in this mode of inference the principle of *modus ponens* is not always valid. This poses the question: if there are reasonably intuitive modes of nonmonotonic inference for which basic principles such as *modus ponens* fail, what principles then characterise nonmonotonic inference?

In this paper we will extend the ideas discussed above to investigate the properties of nonmonotonic inference at the object level within the context of modal logic. In particular, because it represents a significant departure from traditional accounts of nonmonotonic inference, we will furnish a logical account of majority default inference which will allow us to compare its properties with those of the more well-known approaches in the literature; e.g., (Gabbay 1985), (Makinson 1988), (Kraus, Lehmann, & Magidor 1990), *et al.*. Although we focus on the majority-based inference, our primary intention is to show how different nonmonotonic forms of inference, mirroring Solovay's account of deductive inference, can be regarded as modalities satisfying different properties.

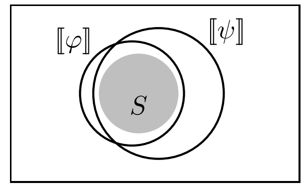
Logical Consequence

The presentation above helps to motivate our approach, however, the use of a unary modality ($\Box$) neglects the central concern in nonmonotonic reasoning: the notion of consequence—a binary notion which, in this paper, will comprise a relation between two propositions. In mathematical logic the notion of provability (theoremhood) is commonly taken as primitive; the notion of consequence is derivative (notably Tarski (1956) and Gentzen (see (Szabo 1969)) took the opposite approach). Usually we define the theorems of a logic and define consequence in terms of those. For example, we say that ψ follows from φ iff $\varphi \to \psi$ is a theorem. Because deductive inference is finitary this means that the material conditional adequately captures, within the language, the desired notion of inference.

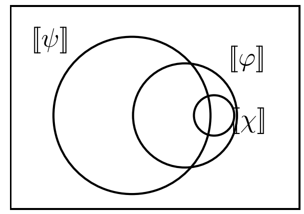
Moreover, deductive inference is monotonic in the sense that, if χ follows from ψ, and φ entails ψ, then χ must follow from φ. In standard notation: if $\psi \vdash \chi$ and $\varphi \vdash \psi$, then $\varphi \vdash \chi$, or if $\vdash \psi \to \chi$, and $\vdash \varphi \to \psi$, then $\vdash \varphi \to \chi$.

In nonmonotonic reasoning, however, we are more often interested in the case where the underlying premises are changing, and we want to know how this affects the inferences we draw. Clearly then the material conditional, being monotonic in the above sense, is no longer adequate. What is required is a counterpart to the material conditional which is faithful to particular nonmonotonic modes of inference.

We adopt what has become standard 'sequent' notation in using $\varphi \mathrel{|\!\sim} \psi$ to represent that ψ follows from φ in a given nonmonotonic mode of inference. In keeping with this, we will introduce a nonmonotonic conditional $\varphi \rightsquigarrow \psi$ which is suggestive of this new purpose. The idea is that $\varphi \mathrel{|\!\sim} \psi$ if $\varphi \rightsquigarrow \psi$ is a theorem of our logic. Intuitively, this is intended to be the case if ψ holds over a majority of φ-worlds (i.e., when a majority of φ-worlds are ψ-worlds).



The following diagram represents this situation. As $[\![\psi]\!]$ contains S, which is intended to comprise a majority of $[\![\varphi]\!]$, this would mean ψ is inferred as a default consequence of φ.



Note that this mode of inference is nonmonotonic in the following sense. In the second diagram χ entails φ, as $[\![\chi]\!] \subseteq [\![\varphi]\!]$. Moreover, intuitively $[\![\psi]\!]$ contains a majority of φ-worlds. However, $[\![\psi]\!]$ encompasses only a small proportion of χ-worlds. So while we would infer that ψ follows from φ, we would not infer that ψ follows from χ, even though χ logically entails φ.

The Conditional Language

Recapitulating: we want to define a conditional $\varphi \rightsquigarrow \psi$ which carries nonmonotonic inference in a similar way that $\varphi \to \psi$ carries deductive inference. In particular, we will look to give an account of majority default consequence as described above. Our conditional language $\mathcal{L}$ is defined inductively from a set of primitive propositional symbols P as follows:

- $P \subseteq \mathcal{L}$
- $\bot \in \mathcal{L}$
- if $\varphi, \psi \in \mathcal{L}$, then $\varphi \to \psi, \varphi \rightsquigarrow \psi \in \mathcal{L}$

The language $\mathcal{L}$ comprises the smallest set of formulae obtained by the application of these rules.

The last condition says that we admit two conditionals: the standard material conditional, as well as the conditional which carries our nonmonotonic inference. Other notation is introduced by definition: $\neg\varphi$ is $\varphi \to \bot$, $\top$ is $\neg\bot$, $\Box\varphi$ is $\top \rightsquigarrow \varphi$, $\Diamond\varphi$ is $\neg\Box\neg\varphi$, etc. [1]

A Simple Account of Majorities

In what follows we will look to give a formal account of majority default inference. But first, in order to make this mode of inference precise, we need to make explicit what we mean by a 'majority'. Suppose we are given a set A; the following properties constitute our basic account of majorities:

First, if S represents a majority of A then any subset of A which is a superset of S would also be considered a majority of A. Secondly, if S represents a majority of A, then that

[1]The author would like to thank James Delgrande for his helpful suggestions regarding the modalities $\Box\varphi$ and $\Diamond\varphi$.

part of A which is left over ($A - S$) would, intuitively, represent a relatively small set in comparison; i.e., would not be considered to be a majority.

Formally, we wish to define collections of subsets of A which we consider to be majorities of A. Let $\kappa(A)$ represent the collection of majority subsets of A (i.e., $\kappa(A) \subseteq 2^A$). We give the following requirements on $\kappa(A)$:

Definition 1. *Let $A \subseteq \mathcal{W}$ be a set and $\kappa : 2^{\mathcal{W}} \to 2^{2^{\mathcal{W}}}$. A non-empty collection $\kappa(A)$ represents a collection of majorities of A iff:*

1. *if S is a majority of A, and $S \subseteq T \subseteq A$, then T is a majority of A: i.e., if $S \in \kappa(A)$, and $S \subseteq T \subseteq A$, then $T \in \kappa(A)$ (majorities are closed under supersets)*
2. *if A is non-empty, and S is a majority of A, then $A - S$ is not a majority: i.e., if $A \neq \varnothing$, and $S \in \kappa(A)$, then $A - S \notin \kappa(A)$*

The properties are reasonably intuitive for a basic account of majorities, particularly when A is finite. Some issues arise when A is infinite: e.g., if A is enumerably infinite, then, given any enumeration, the elements that occupy even places comprise a subset of the same cardinality as A, and so might be reasonably argued to comprise a majority. However, the same argument applies to its complement—the set comprising those elements occupying the odd places. It would seem counterintuitive to admit one as a majority but not the other.

For the moment, to avoid some of these issues, we consider majorities only for finite sets. The extent to which we are justified in doing this will be shown later when it is revealed that the ensuing logic has the 'finite model property'.

Properties of majorities. The following properties can be shown to follow from the account of majorities given above:

- $A \in \kappa(A)$
- if $A \neq \varnothing$, then $\varnothing \notin \kappa(A)$
- for $A \neq \varnothing$, if $S, T \in \kappa(A)$, then $S \cap T \neq \varnothing$

The first says that the entire set A is a majority of itself. The second states that we do not regard the empty set to comprise a majority of any non-empty set. The last reflects the view that for any two non-empty sets to comprise majorities they must overlap at some point. Again, these properties are reasonably intuitive for a simple, finite account of majorities.

Majority Frame Semantics

As is typical in modal/conditional logics, our semantics is based on the notion of a *frame*. The frames we will use are generalisations of the 'standard' frames more commonly encountered in conditional logic (Chellas 1975). A *majority frame* is a pair $(\mathcal{W}, \kappa)$ consisting of a non-empty set of possible worlds $\mathcal{W}$, and a mapping κ which gives, for a set of worlds, its majorities. A frame is said to be *finite* if $\mathcal{W}$ is finite, and *infinite* otherwise.

We add one further generalisation, we will make κ world-dependent; i.e., the majorities of a set A are defined relative to a world $w \in \mathcal{W}$ (see (Ben-David & Ben-Eliyahu-Zohary 2000) for an account of why such context-dependency is desirable). So now $\kappa : 2^{\mathcal{W}} \times \mathcal{W} \to 2^{2^{\mathcal{W}}}$, and for $A \subseteq \mathcal{W}$, and $w \in \mathcal{W}$, $\kappa(A, w) \subseteq 2^A$ is the collection of majorities of A at world w. That is, each $\kappa(A, w)$ satisfies the properties in Definition 1.

A *majority structure* is a tuple $\mathcal{M} = (\mathcal{W}, \kappa, \nu)$ with valuation $\nu : P \to 2^{\mathcal{W}}$ over a majority frame $(\mathcal{W}, \kappa)$. We write $w \models_{\mathcal{M}} \varphi$ to denote that formula φ holds (is true) at world w in structure $\mathcal{M}$. We define our semantics as follows:

$$
\begin{array}{rll}
w \models_{\mathcal{M}} p & \text{iff} & w \in \nu(p) \qquad\qquad \text{for } p \in P \\
w \not\models_{\mathcal{M}} \bot & & \\
w \models_{\mathcal{M}} \varphi \to \psi & \text{iff} & w \not\models_{\mathcal{M}} \varphi \text{ or } w \models_{\mathcal{M}} \psi \\
w \models_{\mathcal{M}} \varphi \rightsquigarrow \psi & \text{iff} & \text{there exists some } S \in \kappa([\![\varphi]\!], w) \\
& & \text{such that } S \subseteq [\![\psi]\!]
\end{array}
$$

The last deserves some mention. It states that $\varphi \rightsquigarrow \psi$ is true at world w in structure $\mathcal{M}$ if there is some majority of φ-worlds ($S \in \kappa([\![\varphi]\!], w)$) over which ψ holds ($S \subseteq [\![\psi]\!]$).

For the derived formulae this gives:

$$
\begin{array}{rll}
w \models_{\mathcal{M}} \neg\varphi & \text{iff} & w \not\models_{\mathcal{M}} \varphi \\
w \models_{\mathcal{M}} \top & & \\
w \models_{\mathcal{M}} \Box\varphi & \text{iff} & \text{there exists } S \in \kappa(\mathcal{W}, w)\ S \subseteq [\![\varphi]\!] \\
w \models_{\mathcal{M}} \Diamond\varphi & \text{iff} & \text{for all } S \in \kappa(\mathcal{W}, w)\ S \cap [\![\varphi]\!] \neq \varnothing
\end{array}
$$

In particular, as $\kappa(\mathcal{W}, w)$ is non-empty, this implies that $w \models_{\mathcal{M}} \Diamond\varphi$ only if $[\![\varphi]\!] \neq \varnothing$.

Following the usual accounts in the literature (Chellas 1975), (Bull & Segerberg 1984), etc., a formula φ is deemed *true* in a structure $\mathcal{M}$ iff it is true in every world in the structure; i.e., $w \models_{\mathcal{M}} \varphi$, for every $w \in \mathcal{W}$. If formula φ is true in structure $\mathcal{M}$, we write $\models_{\mathcal{M}} \varphi$, and say that structure $\mathcal{M}$ is a *model* of φ. The formulae we are interested in, in particular, are those which are true in all structures—in this case in all majority structures. These formulae comprise the 'logic' of interest. Let M represent the class of all majority structures. We write $\models_{\mathsf{M}} \varphi$ to denote that φ is *valid* in all majority structures; i.e., that $\models_{\mathcal{M}} \varphi$, for every $\mathcal{M} \in \mathsf{M}$. In this case we say that φ is *majority valid*.

The Majority Default Conditional

In this section we investigate some of the properties of the majority default conditional.

Let $\mathcal{L}_{\mathsf{M}}$ be the 'logic' of majority; i.e., the set of all formulae valid in all majority structures: $\mathcal{L}_{\mathsf{M}} = \{\varphi \in \mathcal{L} \mid\ \models_{\mathsf{M}} \varphi\}$. The following are some properties which are majority valid.

$$
\begin{array}{rl}
CN: & \chi \rightsquigarrow \top \\
CDD: & \Diamond\chi \to ((\chi \rightsquigarrow \varphi) \to \neg(\chi \rightsquigarrow \neg\varphi)) \\
RCM: & \dfrac{\varphi \to \psi}{(\chi \rightsquigarrow \varphi) \to (\chi \rightsquigarrow \psi)} \\
CQ: & \Diamond\chi \to \neg(\chi \rightsquigarrow \bot)
\end{array}
$$

A number of these have natural counterparts under modal readings. For example, CDD is a weakening of CD: $(\chi \rightsquigarrow \varphi) \to \neg(\chi \rightsquigarrow \neg\varphi)$, which is the conditional version of the more familiar deontic modal axiom D: $\Box\varphi \to \Diamond\varphi$. When $\Box\varphi$ is interpreted as 'φ is inferred' (from $\top$ in this case) this represents a consistency property: if φ is inferred then $\neg\varphi$ is not inferred; or if $\mid\!\sim \varphi$, then $\not\mid\!\sim \neg\varphi$.

Other valid properties include:

$$\begin{array}{rl} ID: & \varphi \rightsquigarrow \varphi \\ ABS: & (\varphi \rightsquigarrow \psi) \rightarrow (\varphi \rightsquigarrow (\varphi \wedge \psi)) \\ RCEA: & \dfrac{\varphi \leftrightarrow \psi}{(\varphi \rightsquigarrow \chi) \leftrightarrow (\psi \rightsquigarrow \chi)} \\ RID: & \dfrac{\varphi \rightarrow \psi}{\varphi \rightsquigarrow \psi} \end{array}$$

Concerning RID: this represents what Makinson (2005) refers to as 'supraclassicality'; i.e., the conditional sanctions all classical consequences, and possibly others.

Counter properties. As was alluded to in the introduction, one property which is not majority valid is the axiom:

$$CK : (\chi \rightsquigarrow (\varphi \rightarrow \psi)) \rightarrow ((\chi \rightsquigarrow \varphi) \rightarrow (\chi \rightsquigarrow \psi))$$

As indicated previously, this is the conditional counterpart to the normality axiom K which corresponds to the rule of *modus ponens*. Another property which fails, as a consequence, is:

$$CC : \quad ((\chi \rightsquigarrow \varphi) \wedge (\chi \rightsquigarrow \psi)) \rightarrow (\chi \rightsquigarrow (\varphi \wedge \psi))$$

Semantically, this means that majorities need not be closed under finite intersections. It may be that the φ-worlds and ψ-worlds both represent a majority of χ, but enough worlds are excluded in their intersection that it no longer constitutes a majority.

Finally, consider the rule (for every $n \in \mathbb{N}$):

$$RCK : \quad \frac{(\varphi_1 \wedge \cdots \wedge \varphi_n) \rightarrow \varphi}{((\chi \rightsquigarrow \varphi_1) \wedge \cdots \wedge (\chi \rightsquigarrow \varphi_n)) \rightarrow (\chi \rightsquigarrow \varphi)}$$

The rule RCK also fails for majority validity—RCK alone characterises normal conditional logics.

Axiomatisation

It turns out that the logic outlined above ($\mathcal{L}_{\mathsf{M}}$) can be axiomatised as the smallest logic Λ containing the axioms:

$$\begin{array}{rl} ID: & \varphi \rightsquigarrow \varphi \\ ABS: & (\varphi \rightsquigarrow \psi) \rightarrow (\varphi \rightsquigarrow (\varphi \wedge \psi)) \\ CDD: & \Diamond\chi \rightarrow ((\chi \rightsquigarrow \varphi) \rightarrow \neg(\chi \rightsquigarrow \neg\varphi)) \end{array}$$

and closed under the rules:[2]

$$\begin{array}{rl} RCEA: & \dfrac{\varphi \leftrightarrow \psi}{(\varphi \rightsquigarrow \chi) \leftrightarrow (\psi \rightsquigarrow \chi)} \\ RCM: & \dfrac{\varphi \rightarrow \psi}{(\chi \rightsquigarrow \varphi) \rightarrow (\chi \rightsquigarrow \psi)} \end{array}$$

The members of Λ are called the *theorems* of Λ. We write $\vdash_\Lambda \varphi$ if formula φ is in the logic Λ; i.e., is a theorem of Λ.

Note that, somewhat subtly, RCM features in the logic but neither CK nor RCK do; we will return to this fact later. That each of the axioms ID, ABS, and CDD is majority valid can be readily verified (we omit the proof). Similarly, it can be shown that the rules $RCEA$ and RCM all preserve majority validity. Consequently:

[2]Note that a logic is already assumed to contain the tautological axioms and be closed under Detachment. See e.g., (Chellas 1980).

Theorem 1 (Soundness)**.** *Every theorem of Λ is majority valid; i.e., if $\vdash_\Lambda \varphi$, then $\models_{\mathsf{M}} \varphi$.*

To show the converse we construct a canonical model for Λ and show that the canonical model is a majority structure. From this it follows that:

Theorem 2 (Completeness)**.** *Every majority-valid formula is a theorem of Λ; i.e., if $\models_{\mathsf{M}} \varphi$, then $\vdash_\Lambda \varphi$.*

The Majority Consequence Relation

We now turn the logic developed previously into a corresponding finitary consequence relation to facilitate comparison with other approaches in the literature. The basic idea is to define ψ as being a *majority consequence* of φ when $\varphi \rightsquigarrow \psi$ is a theorem of Λ; i.e., we construct the consequence relation $\varphi \mid\!\sim_{\mathbb{M}} \psi$ if $\vdash_\Lambda \varphi \rightsquigarrow \psi$.

The following are derived rules of Λ which we use to define majority consequence relations $\mid\!\sim_{\mathbb{M}}$ as consequence relations satisfying the following properties:[3]

$$\begin{array}{rl} \mathcal{MC}: & \dfrac{\varphi \rightsquigarrow \psi}{\varphi \mid\!\sim_{\mathbb{M}} \psi} \\ \mathcal{ABS}: & \dfrac{\varphi \mid\!\sim_{\mathbb{M}} \psi}{\varphi \mid\!\sim_{\mathbb{M}} \varphi \wedge \psi} \\ \mathcal{CD}: & \dfrac{\not\mid\!\sim_{\mathbb{M}} \neg\varphi \qquad \varphi \mid\!\sim_{\mathbb{M}} \psi}{\varphi \not\mid\!\sim_{\mathbb{M}} \neg\psi} \\ \mathcal{RCEA}: & \dfrac{\varphi \leftrightarrow \psi \qquad \varphi \mid\!\sim_{\mathbb{M}} \chi}{\psi \mid\!\sim_{\mathbb{M}} \chi} \\ \mathcal{RCM}: & \dfrac{\varphi \rightarrow \psi \qquad \chi \mid\!\sim_{\mathbb{M}} \varphi}{\chi \mid\!\sim_{\mathbb{M}} \psi} \end{array}$$

To distinguish these inference rules from their conditional counterparts they are labelled in calligraphic font and inherit the name of the conditional axiom/rule from which they were derived. In addition to those sequents obtained from $\mathcal{MC}$, we may have additional sequents which encode defeasible inferences.

Comparison with Related Work

We now proceed to show how the consequence relations defined above compare with those which appear in the literature on nonmonotonic reasoning; e.g., the proposals of Gabbay (1985), Makinson (1988; 2005), Delgrande (1987), Kraus *et al.* (1990), and more recently Schlechta (2004), and Ben-David and Ben-Eliyahu-Zohary (2000).

Consider the system of Gabbay (1985) as it appears in Kraus *et al.* (1990) who refer to it as the *cumulative system* $\mathbb{C}$:

$$\begin{array}{rl} \mathcal{R}: & \varphi \mid\!\sim_{\mathbb{C}} \varphi \\ \mathcal{LLE}: & \dfrac{\varphi \leftrightarrow \psi \qquad \varphi \mid\!\sim_{\mathbb{C}} \chi}{\psi \mid\!\sim_{\mathbb{C}} \chi} \\ \mathcal{RW}: & \dfrac{\varphi \rightarrow \psi \qquad \chi \mid\!\sim_{\mathbb{C}} \varphi}{\chi \mid\!\sim_{\mathbb{C}} \psi} \\ \mathcal{CT}: & \dfrac{\varphi \mid\!\sim_{\mathbb{C}} \psi \qquad \varphi \wedge \psi \mid\!\sim_{\mathbb{C}} \chi}{\varphi \mid\!\sim_{\mathbb{C}} \chi} \\ \mathcal{CM}: & \dfrac{\varphi \mid\!\sim_{\mathbb{C}} \psi \qquad \psi \mid\!\sim_{\mathbb{C}} \chi}{\varphi \wedge \psi \mid\!\sim_{\mathbb{C}} \chi} \end{array}$$

[3]Where $\mid\!\sim_{\mathbb{M}} \varphi$ is an abbreviation for $\top \mid\!\sim_{\mathbb{M}} \varphi$.

The first thing to note is that the rules $\mathcal{R}$, $\mathcal{LLE}$, and $\mathcal{RW}$, coincide with $\mathcal{ID}$, $\mathcal{RCEA}$, and $\mathcal{RCM}$, respectively. These rules, which are firmly entrenched in the literature on nonmonotonic consequence relations, can be seen to be the counterparts to the conditional properties ID, $RCEA$, and RCM, respectively. Note that 'Right Weakening' (rule $\mathcal{RW}$) is the counterpart of RCM, rather than RCK, as claimed by Kraus *et al.* (1990); in particular, the rule RCK, which characterises normal conditional logics, is not valid in majority default logic (which is not normal), whereas RCM is, and consequently its counterpart $\mathcal{RW}$ (alias $\mathcal{RCM}$) is one of the rules of our system.

Normal modes. In conditional logic a conditional is said to be *normal* (Chellas 1975) if it satisfies $RCEA$, CN, CC, and RCM. Moreover, to be normal a logic must satisfy CK. From the discussion above, CC fails (as does CK), hence the majority conditional is not normal.

Chellas (1975) showed that, semantically, normal logics can be characterised by generalised conditional frames which are based on filters; i.e., where the $\kappa(A,w)$'s are filters.[4] This is readily seen: CN gives non-emptiness, CC gives closure under finite intersection, and RCM gives closure under supersets.

Both the cumulative system $\mathbb{C}$, and the stronger *preferential system* $\mathbb{P}$ of Kraus *et al.* (1990), entail $\mathcal{AND}$ (in fact, $\mathcal{CT}$, and $\mathcal{CM}$ entail $\mathcal{AND}$) which is the counterpart of CC. Ben-David and Ben-Eliyahu-Zohary effectively replace $\mathcal{CT}$ and $\mathcal{CM}$ with $\mathcal{AND}$ and propose this new system, which they call $\mathbb{F}$ (for 'filter'), as a more basic system. Moreover, $\mathcal{R}$ is the counterpart of ID, which entails CN. Therefore, the system $\mathbb{F}$ satisfies CN, $RCEA$, RCM, and CC, which, as noted above, makes it a normal conditional logic. It is not surprising then that Ben-David and Ben-Eliyahu-Zohary's system employs a semantics of 'filter-based models', in accordance with Chellas' (1975; 1980) filter characterisation of normal conditional logics. In fact, Ben-David and Ben-Eliyahu-Zohary's system is precisely the logic CK + ID of Chellas (1975).

As the rule ID appears to be more pertinent to the study of nonmonotonic consequence, we use it in place of CN, and define a *normal consequence relation* to be one which satisfies the conditions:

$$\mathcal{ID}: \quad \varphi \mid\!\sim \varphi$$

$$\mathcal{RCEA}: \quad \frac{\varphi \leftrightarrow \psi \qquad \varphi \mid\!\sim \chi}{\psi \mid\!\sim \chi}$$

$$\mathcal{RCM}: \quad \frac{\varphi \rightarrow \psi \qquad \chi \mid\!\sim \varphi}{\chi \mid\!\sim \psi}$$

$$\mathcal{AND}: \quad \frac{\chi \mid\!\sim \psi \qquad \chi \mid\!\sim \psi}{\chi \mid\!\sim \varphi \wedge \psi}$$

Normal consequence relations are precisely those characterised by Ben-David and Ben-Eliyahu-Zohary's logic.

Majorities and filters. Ben-David and Ben-Eliyahu-Zohary's approach was motivated by the majority-interpretation of default reasoning, and in particular the work of Pearl (1990). However, in contrast to the account we gave, the collections which they consider as majorities comprise filters. We argue that $\mathcal{AND}$ (alias CC)—which amounts to majorities being closed under intersection—and the failure to rule out the empty set as a majority of a non-empty set (property CQ) are properties which do not accord well with our conceptualisation of the notion of majority—especially for finite sets. Similar reservations regarding CC, and consequently, regarding normal modes of inference for representing majority reasoning, are held by Schlechta (2004) and Pacuit and Salame (2004). CC in particular is violated by simple probabilistic and combinatorial accounts of nonmonotonic inference (as discussed by Gabbay (1985), and Makinson (2005))

Finite versus infinite. Returning now to the issue of finite versus infinite sets. Using the technique of filtrations (Chellas 1975; 1980) it is possible to show that if $\mathcal{M} \in \mathsf{M}$ is a model of formula φ then there is a finite structure $\mathcal{M}^* \in \mathsf{M}$ which is a model of φ. It follows that:

Theorem 3 (Finite model property)**.** *The logic Λ is determined by (i.e., is sound and complete for) the class of finite majority structures.*

Corollary 1. *The logic Λ is decidable.*

It is because of Theorem 3 that, as stated earlier, we can justify ignoring infinite sets altogether and the issues that arise with our account for majorities of infinite sets.

It should be added that, while filters give a good account of majorities for infinite sets (Ben-David and Ben-Eliyahu-Zohary's motivations for using filters involve examples from measure theory and topology where the most interesting cases are for infinite spaces), for finite cases the same does not always hold: e.g., for a finite set, closure under intersections means that including the co-singleton subsets—which intuitively would be regarded as majorities of any set of cardinality much greater than one—counterintuitively implies acceptance of the empty set as a majority. Similar issues apply for combinatorial probabilities. In particular, for commonsense notions (such as default reasoning), finite sets are arguably more representative than infinite ones.

Discussion and Conclusions

It is interesting to note that the study of nonmonotonic consequence relations has focussed largely on normal consequence relations (the systems $\mathbb{C}$, and $\mathbb{P}$ of Gabbay, Makinson, and Kraus *et al.*, and $\mathbb{F}$ of Ben-David and Ben-Eliyahu-Zohary). One reason could be the prominent status of the preferential semantics of nonmonotonic logics furnished by Shoham (1988) and Delgrande (1987) (Delgrande's account, incidentally, also involves a conditional logic which, in contrast to the majority semantics, employs a relation of preference between worlds) in which nonmonotonic inference is characterised by inferring what holds at the most preferred worlds—ideas which can be traced back in conditional logic at least to Stalnaker (1968) and Lewis (1973).

Frames which employ a binary relation between worlds (e.g., a preference relation) are called *standard frames* (Chellas 1980), or *relational frames*, and they characterise

[4] Recall that a filter is a non-empty collection of sets which is closed under finite intersections and supersets (Sikorski 1964).

normal modal logics. As such, approaches to nonmonotonic reasoning based on preference can be expected to feature normal modes of inference. This indicates why the more familiar accounts of nonmonotonic consequence, in particular, those characterised by Kraus *et al.* (1990) all turn out to furnish normal consequence relations.

Ben-David and Ben-Eliyahu-Zohary's (2000) more recent account of default inference based on majorities ostensibly is a departure from this approach. Nevertheless, one of the conclusions which emerged from the analysis above was that their account still yields a normal logic/consequence relation, suggesting that the mode of nonmonotonic inference they furnish remains, in principle, within the scope of preferential-style approaches. Moreover, in the event that Ben-David and Ben-Eliyahu-Zohary's logic satisfies the finite model property,[5] their system can be represented using standard conditional frames (Chellas 1975) where the selection function f is such that $f(A, w)$ represents the closest/minimal/most preferred worlds in A with respect to w under preference relation R.

Conditional logic enables us to investigate the properties of nonmonotonic inference in a uniform framework, facilitating comparison between different accounts (such as preferential accounts which yield normal consequence, and majority based notions of consequence). It appears to be a good candidate in which to analyse various modes of nonmonotonic inference, both symbolically and through their possible worlds semantics, providing natural accounts of both preferential, as well as majority-based reasoning. It would be interesting to see what other modes of nonmonotonic inference lend themselves to this analysis and whether they are amenable to a natural possible-worlds semantics (e.g., Reiter's default logic).

References

Ben-David, S., and Ben-Eliyahu-Zohary, R. 2000. A modal logic for subjective default reasoning. *Artificial intelligence* 116:217–236.

Boolos, G. 1993. *The logic of provability*. Cambridge university press.

Bull, R., and Segerberg, K. 1984. Basic modal logic. In Gabbay, D., and Guenthner, F., eds., *Handbook of philosophical logic*, volume 2. Kluwer. 1–89.

Chellas, B. F. 1975. Basic conditional logic. *Journal of philosophical logic* 4:133–153.

Chellas, B. F. 1980. *Modal logic: an introduction*. Cambridge university press.

Clark, K. 1978. Negation as failure. *Logic and databases* 293–322.

Delgrande, J. P. 1987. An approach to default reasoning based on a first-order conditional logic. In *Proceedings, AAAI-87*, 340–345.

Gabbay, D. 1985. Theoretical foundations for nonmonotonic reasoning in expert systems. In Apt, K. R., ed., *NATO advanced study institute on logics and models of concurrent systems*, 439–457. Springer.

Gödel, K. 1986. *Collected works: Kurt Gödel*, volume 1. Oxford university press.

Jauregui, V. 2007. The 'majority' and 'by default' modalities. In Orgun, M., and Thornton, J., eds., *Proc. 20th Australian joint conference on artificial intelligence*, 263–272. Springer.

Kraus, S.; Lehmann, D.; and Magidor, M. 1990. Nonmonotonic reasoning, preferential models and cumulative logics. *Artificial intelligence* 44:167–207.

Lewis, D. 1973. *Counterfactuals*. Blackwell.

Löb, M. H. 1955. Solution of a problem of Leon Henkin. *Journal of symbolic logic* 20:115–118.

Makinson, D. 1988. General theory of cumulative inference. In Reinfrank, M.; de Kleer, J.; Ginsberg, M.; and Sandewall, E., eds., *2nd international workshop on nonmonotonic reasoning*, 1–18. Springer.

Makinson, D. 2005. *Bridges from classical to nonmonotonic logic*. Kings college.

McCarthy, J. 1980. Circumscription—a form of nonmonotonic reasoning. *Artificial intelligence* 13:27–39.

Pacuit, E., and Salame, S. 2004. Majority logic. In Dubois, D.; Welty, C. A.; and Williams, M.-A., eds., *KR*, 598–605. AAAI Press.

Pearl, J. 1990. System Z: a natural ordering of defaults with tractable applications to nonmonotonic reasoning. In *TARK-90*, 121–135.

Reiter, R. 1980. A logic for default reasoning. *Artificial intelligence* 13:81–132.

Schlechta, K. 2004. *Coherent systems*. Elsevier.

Shoham, Y. 1988. *Reasoning about change*. MIT press.

Sikorski, R. 1964. *Boolean algebras*. Springer-Verlag, 2nd edition.

Smorynski, C. 1984. Modal logic and self reference. In Gabbay, D., and Guenthner, F., eds., *Handbook of philosophical logic*, volume 1. Kluwer. 441–496.

Solovay, R. 1976. Provability interpretations of modal logic. *Israel journal of mathematics* 25:287–304.

Stalnaker, R. 1968. A theory of conditionals. In Rescher, N., ed., *Studies in logical theory*. Blackwell. 98–112.

Szabo, M. 1969. *Collected papers of Gerhard Gentzen*. North-Holland.

Tarski, A. 1956. *Logic, semantics, metamathematics : papers from 1923 to 1938*. Clarendon Press.

[5] It is not clear from their work whether it does or not although we strongly suspect it does.