Towards Partial Reasoning

Krzysztof Zbigniew Nowak
Computing Department, Macquarie University
Sydney NSW 2109, Australia
Chris.Nowak@mq.edu.au

Abstract

This paper presents a framework for reasoning with partial information provided by multiple agents. On the semantic side, there are partial objects corresponding to equivalence classes of indiscernible objects. Sets of partial objects form partial worlds. On the syntactic side, agents describe their partial worlds, and their description sets are taken as sets of axioms for formal systems with intuitive rules of inference. Given a formal system, its set of theorems forms a consistent partial theory. The set of all theories is equipped with an information ordering and forms a lattice. The lattice structure allows to visualise how theories, and agents, support or contradict each other. The set of description sets provided by agents determines a set of theories believed by the agents, and this set gives rise to a sub-lattice of the set of all theories. The framework is appropriate for dealing with information provided by multiple sources of information, with emphasis on partial and contradictory information.

Introduction

This paper presents a framework for reasoning with partial information provided by multiple agents. We start with a set G of ***partial objects***, refer to any subset of G as a ***context***, denoted K, and introduce "saturated" contexts called ***worlds***.

We propose a language and define ***validity***—given a world W, one can determine sentences valid in W. A set of all sentences valid in a world W is a ***theory of the world***, denoted T_W. However, there are also other theories of interest, theories which fail to be theories *of worlds*—they can be seen as ***partial*** theories of worlds; in such a case, a theory determines a set of alternatives (worlds that the theory so far only partly discribes), rather than a single partial world. The theories we consider are ***consistent***, and form a ***lattice*** $(\boldsymbol{T}, \leq)$, or $(\boldsymbol{T}, \wedge, \vee)$, where $\leq$ is an ***information ordering*** and $\wedge$ and $\vee$ are ***meet*** and ***join*** operations.

Multiple agents provide a (finite) set of ***description sets***, which result in a finite set of theories, referred to as a set $\boldsymbol{B}$ of ***believed theories***, $\boldsymbol{B} \subseteq \boldsymbol{T}$. The set of believed theories determines its ***closure*** $\boldsymbol{C}$ (under $\wedge$, $\vee$), being a sub-lattice of $\boldsymbol{T}$ and a smallest lattice containing $\boldsymbol{B}$. It is claimed that $\boldsymbol{C}$ is an appropriate structure to perform reasoning with partial information provided by multiple agents. The lattice allows to visualise information ordering on theories and shows where theories ***agree*** and which theories are ***contradictory***. The theories of the lattice can also be mapped to (sets of) agents ***supporting*** and ***rejecting*** the theories.

Partial Worlds

Given that a set Σ^+ of all finite, non-empty strings over a finite alphabet Σ can be enumerated, let the set N of natural numbers be the set of ***attributes***. Assume a finite set M of attributes, $M \subseteq N$. If $g \colon M \longrightarrow \{0,1\}$ then g is called a ***partial object***—note that g can be a properly partial function, i.e., it can be undefined for some elements of M. Then $G = \{g \colon M \longrightarrow \{0,1\}\}$ is a set of all partial objects (over M). G can be equipped with an ***information ordering (on objects)*** $\leq$ given by: $g_1 \leq g_2$ if g_2 ***extends*** g_1, i.e., if $\mathrm{dom}(g_2) \supseteq \mathrm{dom}(g_1)$ and g_2, g_1 take the same values at $\mathrm{dom}(g_1)$. Maximal elements of G are called ***total objects***. Let the set of total objects be denoted by G_t. After adding a ***top element*** 1 to G, the ordered set $(G \cup \{1\}, \leq)$ is a ***lattice***.

Any subset K of the set G of partial objects is called a ***context***. A context is a partial and finite representation of a collection of "things," where some of them are ***indiscernible*** w.r.t. the attributes, and therefore they are partitioned into equivalence classes determined by the indiscernibility relation—the equivalence classes correspond to our partial objects.

Let $\boldsymbol{K} = \mathcal{P}(G)$ be the set of all contexts. A context is ***total*** if all its objects are total; let $\boldsymbol{K}_t$ denote the set of all total contexts, $\boldsymbol{K}_t = \mathcal{P}(G_t)$. It seems appropriate to equip $\boldsymbol{K}$ with an ***information ordering (on contexts)*** as follows. $K_1 \leq K_2$ iff:

1. $\forall_{g_1 \in K_1} \exists_{g_2 \in K_2}\ g_1 \leq g_2$, and
2. $\forall_{g_2 \in K_2} \exists_{g_1 \in K_1}\ g_1 \leq g_2$.

For instance, for a singleton set $M = \{n_1\}$ of attributes the set of all contexts (the set of all sets of partial func-

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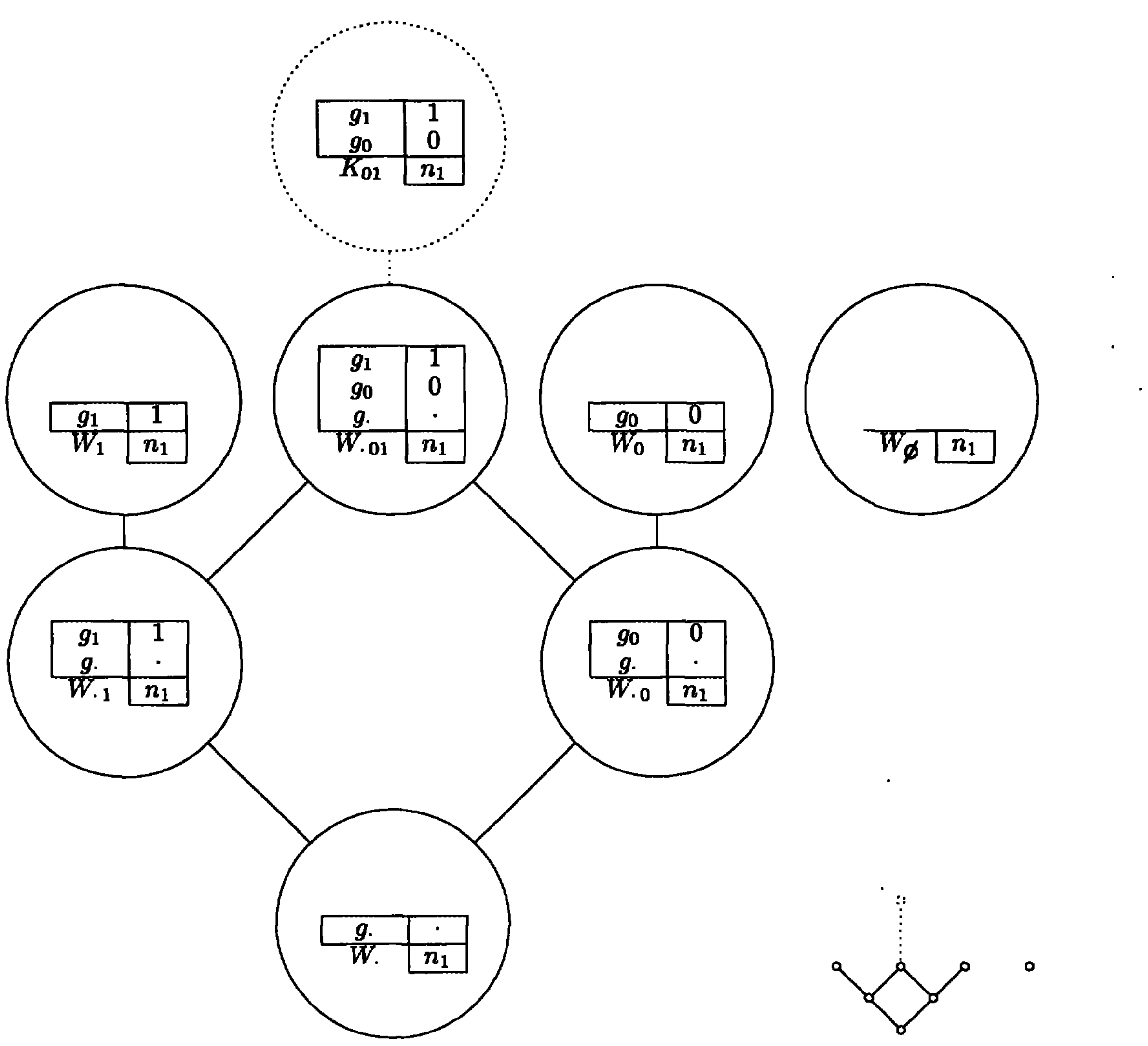

Figure 1: Information ordering on partial worlds

tions from M to $\{0, 1\}$) together with the above ordering is presented in Figure 1.

In particular, in Figure 1 we have that, e.g., $W_{\cdot 01} < K_{01}$. The context $W_\emptyset$, representing an empty world, is non-comparable to any other context (and therefore a maximal and also minimal element of the set of all contexts). The context $W_\cdot$ is a minimal element—it represents a non-empty, but otherwise unknown world.

However, there are two problems with $(\boldsymbol{K}, \leq)$:

1. $\leq$ is not ***antisymmetric***, and hence not a ***partial ordering*** (e.g., consider contexts $\{g_1, g_2, g_3\}$ and $\{g_1, g_3\}$, where $g_1 < g_2 < g_3$),
2. there are contexts such that although $K_1 < K_2$, the contexts are nevertheless "essentially" the same (e.g., K_{01} and $W_{\cdot 01}$ of Figure 1 are such that $W_{\cdot 01} < K_{01}$, but K_{01} represents the same information as $W_{\cdot 01}$).

The remedy is to consider "saturated" contexts.

Firstly, introduce an ordering relation $\leq$ between contexts and total contexts, i.e., $\leq \subseteq \boldsymbol{K} \times \boldsymbol{K}_t$. Let $K \in \boldsymbol{K}$ and $K_t \in \boldsymbol{K}_t$. Then $K \leq K_t$ iff

1. $\forall_{g_t \in K_t} \exists_{g \in K} \; g \leq g_t$
2. $\forall_{g \in K} \exists_{g_t \in K_t} \; g \leq g_t$

Secondly, given the above ordering, we can introduce a mapping "tot" from contexts to sets of total contexts, $\mathrm{tot}\colon \boldsymbol{K} \longrightarrow \mathcal{P}(\boldsymbol{K}_t)$. The mapping allows us to find all total contexts above a given context, and then employ the mapping to define an equivalence relation on the set $\boldsymbol{K}$ of all contexts. A mapping $\mathrm{tot}\colon \boldsymbol{K} \longrightarrow \mathcal{P}(\boldsymbol{K}_t)$ is given by $\mathrm{tot}(K) = \{K_t \in \boldsymbol{K}_t \mid K \leq K_t\}$. Further, an equivalence relation $\approx$ on $\boldsymbol{K}$, i.e., $\approx \subseteq \boldsymbol{K} \times \boldsymbol{K}$ is given by $K_1 \approx K_2$ iff $\mathrm{tot}(K_1) = \mathrm{tot}(K_2)$. Then $\boldsymbol{K}/\!\approx$ is the set of equivalence classes on $\boldsymbol{K}$ w.r.t. $\approx$.

Referring to Figure 1, there are four total contexts (namely, W_1, K_{01}, W_0 and $W_\emptyset$). Two contexts are

$\approx$-equivalent, $K_{01} \approx W_{01}$, as they both have the same set of total contexts above them, namely the singleton set $\{K_{01}\}$. So, $\{K_{01}, W_{01}\}$ is one of the equivalence classes of $\boldsymbol{K}/\approx$—all the other equivalence classes are singleton sets.

Given the above considerations, we could accept equivalence classes of contexts as appropriate semantic entities—however, it would be useful to have single, uniquely determined "representatives" of the equivalence classes. One might decide to "saturate" contexts, and to keep such saturated contexts—call them *partial worlds*—as appropriate semantic entities. For instance, such a saturation mapping would map K_{01} to W_{01}, and would map every other context to itself.

This suggests one condition a saturation mapping should satisfy—if, in a context, there is an object having attributes $\psi \cup \{m\}$ and there is an object having the attributes $\psi \cup \{\overline{m}\}$ then the saturated context should also contain the object that has the attributes ψ. There is another condition saturated contexts should satisfy, the condition of *convexity* w.r.t. the information ordering on objects.

Therefore, a *saturation mapping*, $\mathrm{sat} : \boldsymbol{K} \longrightarrow \boldsymbol{K}$ is defined as follows. Let $K \in \boldsymbol{K}$. Let g_ψ be a partial object with attributes $\psi = \psi_g$, where ψ_g is given by $\{m \mid g(m) = 1\} \cup \{-m \mid g(m) = 0\}$—see the next section. Then $\mathrm{sat}(K)$ is a context (element of $\boldsymbol{K}$) that satisfies the following conditions:

1. $\mathrm{sat}(K) \supseteq K$
2. if $g_1, g_3 \in \mathrm{sat}(K)$ and $g_1 \leq g_2 \leq g_3$ then $g_2 \in \mathrm{sat}(K)$
3. if $g_{\psi \cup \{m\}}, g_{\psi \cup \{\overline{m}\}} \in \mathrm{sat}(K)$ then $g_\psi \in \mathrm{sat}(K)$
4. $\mathrm{sat}(K)$ is $\subseteq$-minimal

We can now introduce *partial worlds* as saturated context, and then define an information ordering on partial worlds in an appropriate, and expected, way.

Let $\boldsymbol{K}$ be a set of all contexts (over M). A set $\boldsymbol{W}$ of *partial worlds* is the set of all *saturated contexts*, i.e., $\boldsymbol{W} = \{\mathrm{sat}(K) \mid K \in \boldsymbol{K}\}$. An *information ordering* on partial worlds is introduced as follows. $W_1 \leq W_2$ iff the following conditions hold:

1. $\forall_{g_2 \in W_2} \exists_{g_1 \in W_1}\ g_1 \leq g_2$,
2. $\forall_{g_1 \in W_1} \exists_{g_2 \in W_2}\ g_1 \leq g_2$.

This finishes our considerations on information ordering on contexts, or rather partial worlds. Referring again to Figure 1, the context K_{01} is not a partial world, and therefore should be ommitted from the set of semantic entities (worlds) and the ordering on them. [When theories of contexts are introduced, then the context K_{01} and its saturated version, the world $W_{01} = \mathrm{sat}(K_{01})$, both map—as one would expect—to the same theory, see Figure 3.]

Partial Theories

Let $g \in G$; then the corresponding *term* ψ_g is given by $\{m \mid g(m) = 1\} \cup \{-m \mid g(m) = 0\}$ and so ψ_g is a subset of the set of integers $\{m, -m \mid m \in M\}$. The set of all terms is $\Psi = \{\psi_g \mid g \in G\}$. For instance, consider the set of attributes $\{1, 2, 3\}$ and an object g given by $g(1) = 0$, $g(2)$ undefined and $g(3) = 1$; we can identify such a function, if the set of attributes is fixed, with the sequence of values it takes, saying that $g = (0, \cdot, 1)$. Then $\psi_g = \{-1, 3\}$, and the term lists which attributes the object has / does not have: g has the attribute 3, but does not have the attribute 1, and remains "partial" w.r.t. the attribute 2. After allowing negative integers as attributes we can say that -1 and 3 are the attributes of g; in general, $\psi_g = \mathrm{attrs}(g)$, the set of attributes of g.

The set of all *formulae* is $\Phi = \{\ominus\psi, \oplus\psi \mid \psi \in \Psi\}$, i.e., formulae are terms preceded by the $\oplus$ or the $\ominus$ sign. The intention is to use such formulae to provide information about worlds, by saying whether an object with the given set of attributes is in the world. We now introduce *validity*. Let $W \in \boldsymbol{W}$ be a world. Define:

$$W \models \oplus\psi \text{ iff } \exists_{g \in W} \forall_{m \in \psi}\ m \in \psi_g,$$

$$W \models \ominus\psi \text{ iff } \forall_{g \in W} \exists_{m \in \psi_g}\ -m \in \psi.$$

Consider again the set of attributes $\{1, 2, 3\}$ and an object $g = (0, \cdot, 1)$, and let $W = \{g\}$. We have that, for instance, $W \models \oplus\{-1, 3\}$, but also $W \models \ominus\{-1\}$, and $W \models \oplus\{\}$. Note that $\oplus\{\}$ is valid in W, but the corresponding object (undetermined w.r.t. all the attributes 1, 2 and 3) is *not* in the world, as its presence would invalidate $\ominus\{1\}$, but $W \models \ominus\{1\}$. Define a *negation* operator $\neg : \Phi \longrightarrow \Phi$ by requesting $\neg \oplus \psi = \ominus\psi$ and $\neg \ominus \psi = \oplus\psi$. We can have that neither φ nor $\neg\varphi$ is valid, as is the case for $\varphi = \ominus\{2\}$ in W.

Given a world, *any* set of sentences valid in the world generates a theory (and *any* such set can be *communicated* by an *agent*). Such a set of sentences is called a *description set*, denoted by D, and the corresponding theory T is a logical closure of D, i.e., $T = \mathrm{Cn}(D)$, where the inference rules are the following:

1. if $W \models \oplus\psi_2$ and $\psi_2 \supseteq \psi_1$ then $W \models \oplus\psi_1$.
2. if $W \models \ominus\psi_1$ and $\psi_1 \subseteq \psi_2$ then $W \models \ominus\psi_2$.
3. if $W \models \ominus\psi \cup \{m\}$ and $W \models \ominus\psi \cup \{-m\}$ then $W \models \ominus\psi$,
4. if $W \models \oplus\psi$ and $W \models \ominus\psi \cup \{m\}$ then $W \models \oplus\psi \cup \{-m\}$.

For instance, if $D = \{\oplus\{m_1\}, \ominus\{m_1, m_2, m_3\}, \ominus\{m_1, m_2, -m_3\}\}$ then $T = \mathrm{Cn}(D) = \{\oplus\{m_1, -m_2\}, \oplus\{m_1\}, \oplus\{-m_2\}, \oplus\{\}, \ominus\{m_1, m_2\}, \ominus\{m_1, m_2, m_3\}, \ominus\{m_1, m_2, -m_3\}\}$

A theory is *consistent* if it does not contain both φ and $\neg\varphi$; a description set is *consistent* if its theory is. It is easy to see that the set $\boldsymbol{T}$ of all consistent theories, after adding a *top element* 1 representing inconsistency (0 is used to denote an empty theory, $\{\}$), is a lattice, i.e., $(\boldsymbol{T}, \leq)$ is a *lattice*, where $\leq$, an *information ordering (on theories)*, is simply $\subseteq$. The $\wedge$ and $\vee$ operations are defined by $T_1 \wedge T_2 = T_1 \cap T_2$, and $T_1 \vee T_2 = \mathrm{Cn}(T_1 \cup T_2)$, if consistent, or 1, otherwise.

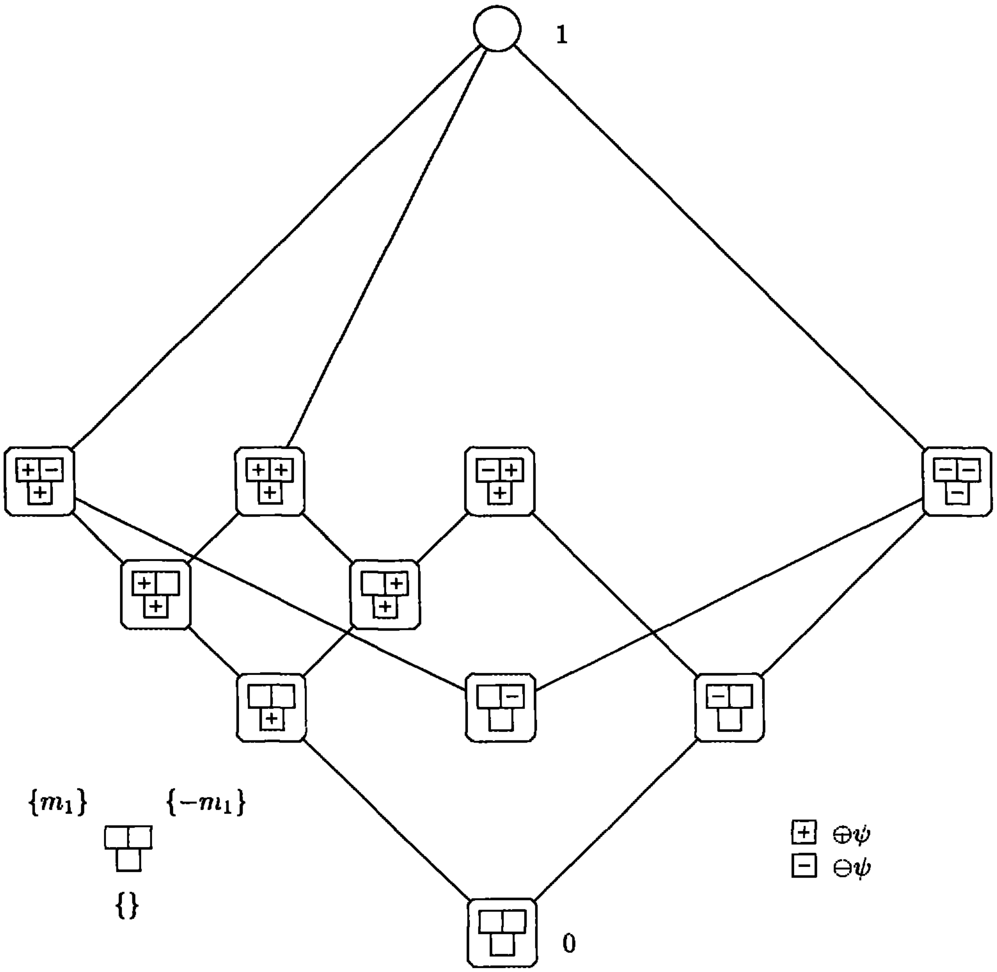

Figure 2: Information ordering on theories

The set of all theories (over $M = \{m_1\}$) is presented in Figure 2. In this case, there are three possible terms, namely $\{m_1\}$, $\{-m_1\}$ and $\{\}$. Therefore there are six possible formulae, $\oplus\{m_1\}$, $\oplus\{-m_1\}$, $\oplus\{\}$. $\ominus\{m_1\}$, $\ominus\{-m_1\}$ and $\ominus\{\}$. Finding a theory accounts to marking some terms with $\ominus$ and some other ones with $\ominus$, as govern by the logical closure operator, Cn. The resulting theories are presented in Figure 2, where formulae are marked with '+' and '−', as shown in the figure. Note the top element 1 (representing inconsistency), and the bottom element 0, the empty theory (no term is marked with $\oplus$ or $\ominus$, i.e., the set of formulae is empty), or "total ignorance" state.

Given that Figure 1 presents all partial worlds (and one context that is not a world) for the same set of attributes, we can see how worlds can be mapped to their theories—this is shown in Figure 3 (note that the contexts K_{01} and $W_{\cdot 01}$ are mapped to the same theory).

For instance, the world W_1 is mapped to its theory $T_{W_1} = \{\ominus\{m_1\}, \oplus\{\}, \ominus\{-m_1\}\}$. However, there is no world that maps to the theory $\{\ominus\{m_1\}\}$, the theory $\{\ominus\{-m_1\}\}$, or the empty theory $0 = \{\}$. Note that e.g., the theory $\{\ominus\{m_1\}\}$ is saying that there are no objects with the attribute m_1—this information alone is not sufficient to determine the world the theory describes, as it is not even known whether or not the world is empty.

Multiple Agents

Let S be a finite set of ***agents*** providing their ***description sets***, $\{D_s\}_{s \in S}$. Define $s_1 \approx s_2$ iff $\mathrm{Cn}(D_{s_1}) = \mathrm{Cn}(D_{s_2})$, and $\Xi = S/\approx$. Then $\boldsymbol{B} = \{\mathrm{Cn}(D_s)\}_{s \in S}$ is in fact $\boldsymbol{B} = \{B_\xi\}_{\xi \in \Xi}$ and the elements of $\boldsymbol{B}$ are called ***believed theories***. As $\boldsymbol{B} \subseteq \boldsymbol{T}$, a ***closure*** $\boldsymbol{C}$ of $\boldsymbol{B}$ under $\wedge$ and $\vee$ can be found, i.e., $\boldsymbol{C} = \mathrm{Cl}(\boldsymbol{B})$ satisfies the conditions: $\boldsymbol{C} \supseteq \boldsymbol{B}$, and if $C_1, C_2 \in \boldsymbol{C}$ then $C_1 \wedge C_2, C_1 \vee C_2 \in \boldsymbol{C}$. Clearly, $(\boldsymbol{C}, \leq)$ is a ***sublattice*** of $(\boldsymbol{T}, \leq)$. Figure 4 shows examples of sets of believed theories (marked with filled circles) and their closures. Considering the example (d) of Figure 4,

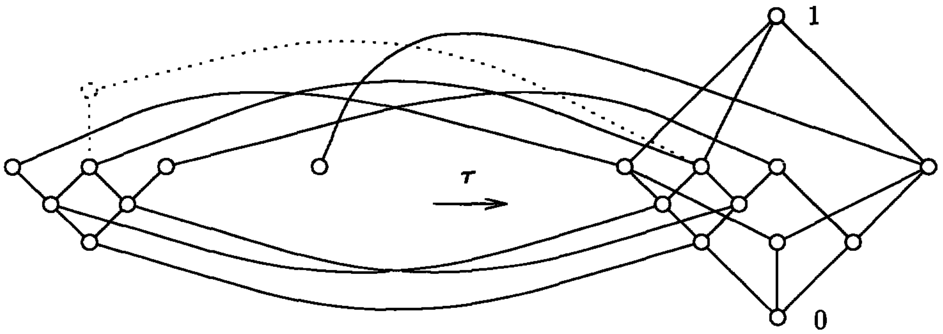

Figure 3: Mapping worlds to theories

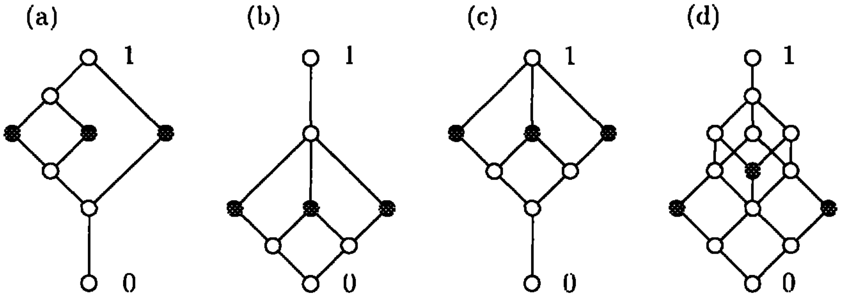

Figure 4: Some believed theories and their closures

we have $\boldsymbol{B}_d = \{B_{d.1}, B_{d.2}, B_{d,3}\}$, these are the believed theories, and are marked with filled circles. The theories $B_{d.1}, B_{d,2}, B_{d.3}$ are obtained from the description sets $D_{d,1} = \{\oplus\{m_1\}\}$, $D_{d.2} = \{\oplus\{\}, \ominus\{m_2, m_3\}\}$, $D_{d.3} = \{\ominus\{m_2, m_4\}\}$. In the case of $\boldsymbol{B}_d$, its closure $\boldsymbol{C}_d = \mathrm{Cl}(\boldsymbol{B}_d)$ does not contain the "inconsistent theory" 1 (because the theories $B_{d,1}, B_{d.2}$ and $B_{d,3}$ join consistently, i.e., below 1), but it does contain the empty theory 0 (because the theories $B_{d.1}, B_{d,2}$ and $B_{d,3}$ have an empty meet, or intersection, and so they meet at 0). Figure 4 (d) includes the element 1, i.e., it shows $(\boldsymbol{C}_d \cup \{1\}, \leq)$. An obvious advantage of locating theories in a lattice of consistent theories is that it shows where theories meet, and whether they can be joined consistently.

Further, with a $C \in \boldsymbol{C}$ associate $v_C = (v_C^+, v_C^-)$, where $v_C^+ = \{\xi \mid B_\xi \geq C\}$, and $v_C^- = \{\xi \mid B_\xi \vee C = 1\}$. The set $\{v_C\}_C$ represents how (collective) agents in Ξ support (v_C^+) and reject (v_C^-) theories in $\boldsymbol{C}$. It is claimed that $(\boldsymbol{C}, \leq)$ together with $\{v_C\}_C$ are the right representation of partial information provided by multiple agents.

Conclusion

The *partial reasoning* framework presented here addresses two stages of reasoning. Firstly, information provided by the agents is translated to theories. Secondly, the set $\boldsymbol{C}$ of "most interesting" theories is found, it is a lattice, and the ordering represents informational value of theories. By inspecting *meets* and *joins* of theories, one can see where theories *agree* (at their *meets*) and which theories *contradict* each other (those which *join* at 1). Apart from the information ordering $\leq$ on theories, the set $\{v_C\}_C$ evaluates *truthness* of the theories, c.f., (Ginsberg 1988). Partial contexts and worlds are related to contexts of *Formal Concept Analysis* (FCA), see (Ganter & Wille 1996). The *rough sets theory* (Pawlak 1991) also emphasises *partial*, or *boundary* objects. In the area of databases, (Motro 1995) has a similar focus.

References

Ganter, B., and Wille, R. 1996. *Formale Begriffsanalyse. Mathematische Grundlagen.* Springer.

Ginsberg, M. L. 1988. Multivalued logics: a uniform approach to reasoning in artificial intelligence. *Computational Intelligence* 4:265 316.

Motro, A. 1995. Multiplex: A formal model for multidatabases and its implementation. Technical report, Dept. of Inf. and Soft. Syst. Eng., George Mason Univ.

Pawlak, Z. 1991. *Rough Sets. Theoretical Aspects of Reasoning about Data.* Kluwer.