

The Reasonableness of Necessity

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Abstract

The necessity calculus is a familiar adjuvant to the possibility calculus and an uncertain inference tool in its own right. Necessity orderings enjoy a syntactical relationship to some probability orderings similar to that displayed by possibility. In its adjuvant role, necessity may be viewed as bringing possibility closer to achieving the quasi-additive normative desideratum advocated by de Finetti. Nevertheless, there are occasions when one might choose to use possibility without the help of necessity, e.g. when the full range of alternative hypotheses is unknown, or to exploit possibility's distinctive ability simultaneously to express preference as well as credibility ordering. Such situations arise in uncertain domains like the evaluation of scientific or mathematical hypotheses, where professions of belief may reflect aesthetic, utilitarian, and evidentiary considerations as much as the usual notions of credibility.

Introduction

The possibility calculus is a popular tool for uncertainty management which evaluates disjunctions according to the simple and computationally efficient rule

$$\Pi(A \vee B) = \max(\Pi(A), \Pi(B))$$

By convention, a tautology has a possibility of unity and a contradiction has (and typically *only* known falsehoods have) a possibility of zero. In any set of mutually exclusive and collectively exhaustive sentences, at least one of the sentences must also have a possibility of unity, so that the *max* rule yields the desired unit value for the tautology.

Possibility is easily identified as a measure of "credibility" in that

if $A \Rightarrow B$, then B is ranked no less than A

This is a widely-cited criterion of intuitive credibility-likeness identified by Lukasiewicz and revived by Sugeno. Probability is another familiar example of a Lukasiewicz-Sugeno calculus, as is the ordinary Boolean logic.

Perhaps because of its origins in the fuzzy logic community (Zadeh, 1978), possibility has sometimes been viewed as fundamentally different from probabilistic approaches to uncertainty. It has long been known, however, that there was a nexus of similarity among possibility, operations involving probability intervals, and the Dempster-Shafer calculus (Dubois and Prade and Smets, 1996 review this history), itself originally proposed as an innovation in Bayesian technique (Dempster, 1968).

Another similarity was noted in connection with conditional logics and default reasoning (Kraus, Lehmann, and Magidor, 1990). A particular class of probability distributions faithfully emulates the behavior of the default entailment connective (Snow, 1999). These "atomic bound" probabilities are the solutions of the simultaneous constraint system whose typical constraint is

$$p(x) > \sum_{\text{all atoms } y: p(x) > p(y)} p(y) > 0$$

for example, the probabilities over the five exclusive and exhaustive sentences *a* through *e*:

$$p(a) = 16/31, p(b) = 8/31, p(c) = 4/31, \\ p(d) = 2/31, p(e) = 1/31$$

Similar emulation of the default rules is also achieved by "linear" possibility distributions (Benferhat *et al.* 1997), those where each atomic sentence has a distinct possibility value, e.g.,

$$\pi(a) = 1, \pi(b) = 1/2, \pi(c) = 1/4, \\ \pi(d) = 1/8, \pi(e) = 1/16$$

For all mutually exclusive sentences *A* and *B*, atomic bound probabilities follow the possibilistic *max* rule, that is, $p(A) > p(B)$ just when the highest probability atom in $A \vee B$ is in *A*. So, the special probabilities and the linear possibilities are in ordinal agreement for all mutually exclusive sentences. Since default entailment can be understood as the ordering of mutually exclusive sentences, and it, too, follows a version of the *max* rule, either possibility or probability works equally well for emulation.

The relation between atomic bound probabilities and linear possibilities is somewhat closer than that. A property of all probability distributions given special normative emphasis by Bruno de Finetti (1937) is called *quasi-additivity*,

$p(A) \geq p(B) \Leftrightarrow p(A \neg B) \geq p(B \neg A)$
 for arbitrarily related A and B . Since atomic bound probabilities and linear possibilities agree for all mutually exclusive sentences (like $A \neg B$ and $B \neg A$), we have

$p(A) \geq p(B) \Leftrightarrow \Pi(A \neg B) \geq \Pi(B \neg A)$
 Conversely, using a standard possibilistic identity,

$\Pi(A) \geq \Pi(B) \Leftrightarrow \Pi(A) \geq \Pi(B \neg A)$
 we find

$\Pi(A) \geq \Pi(B) \Leftrightarrow p(A) \geq p(B \neg A)$
 again since atomic bound probabilities agree with the linear possibilities for exclusive sentences (A and $B \neg A$ in this case).

These results say that orderings based upon atomic bound probabilities and those based upon linear possibility are syntactic restatements of one another. If one is “reasonable” by some standard, then so is the other. The “same thought” can be expressed equally well by a probability relation or by a possibility relation.

Since many of the normative arguments advanced by probabilists are ordinal in character, ordinal reasonableness should be enough to moot attempts to portray possibility as normatively inferior to probability (or *vice versa*). That linear possibilities also solve the equations advanced in Cox’s Theorem (Snow, 2001), a classical pillar of probabilist normative argumentation, should secure the point.

None of the existing results briefly reviewed above addresses the status of the other calculus common in the possibilist literature, necessity, which is defined as

$$N(A) \equiv 1 - \Pi(\neg A)$$

Like possibility, the necessity calculus has enjoyed conspicuous success in emulating other approaches to uncertainty management (Dubois and Prade and Smets, 1996).

In the current paper, the concern will be to examine a specific role played by necessity in possibilistic practice, a role which might be called “tie breaking.” That is to say, on some occasions when

$$\Pi(A) = \Pi(B)$$

we find that the necessities of A and B differ. For example, if A is a tautology, and B is an uncertain sentence containing a “top atom” (one whose possibility is unity), then the possibilities tie at unity, but we find that

$$N(A) > N(B)$$

This allows the analyst to distinguish between uncertainties of high possibility and certainties, obviously a useful capability in the management of uncertainty.

In the next section, we shall examine necessity and observe that some aspects of its “tie-breaking” behavior have normative appeal from a probabilist perspective. That said, in the subsequent sections, we consider an uncertain domain where other properties might be desirable.

In this domain, the range of alternative hypotheses (and hence the “ $\neg A$ ” upon which necessity depends) is

typically not fully apprehended, and some subtleties of dynamic belief revision may be especially germane. Possibility values applied within the domain may be representing information about preferences in addition to credibility in the Lukasiewicz-Sugeno sense. To accomplish this dual representation, it may be best to leave possibilistic ties unbroken.

Reasonableness and Tie-breaking

Continuing in an ordinal mode of discourse, it is clear that since necessity’s orderings are the expression of a “thought” in possibilistic terms

$$N(A) \geq N(B) \Leftrightarrow \Pi(\neg B) \geq \Pi(\neg A)$$

When the possibility is linear, then necessity inherits linear possibility’s ability to syntactically restate the atomic bound probabilities’ orderings.

The specific relationship can be easily worked out by an application of De Morgan’s law to the argument-sentences in the equivalences discussed in the introduction, yielding

$$p(C) \geq p(D) \Leftrightarrow N(C \vee \neg D) \geq N(D \vee \neg C)$$

$$N(C) \geq N(D) \Leftrightarrow p(C \vee \neg D) \geq p(D)$$

So, linear necessity orderings are indeed syntactic restatements, and fully expressive restatements, of a special probabilistic ordering. No special analysis is required to confirm the Cox-reasonableness of necessity, since in general, any function of a Cox-reasonable belief representation is itself Cox-reasonable.

In discussing the reasonableness of necessity in the specific application of tie-breaking in concert with possibility, it is convenient to combine the two calculi into a single ordering,

$$A \geq^* B \Leftrightarrow \Pi(A) \geq \Pi(B) \text{ and}$$

$$A >^* B \Leftrightarrow \Pi(A) > \Pi(B) \text{ or } \Pi(\neg B) > \Pi(\neg A)$$

where the symbol “ $>^*$ ” means “is ranked ahead of,” and the meanings of “ \geq^* ” and “ $=^*$ ” are parallel.

A simple way to realize this ordering is to create a composite function,

$$f(A) = [\Pi(A) + N(A)] / 2$$

This composition is possible since the necessity of all sentences whose possibility is less than unity is zero, a well-known feature of the relationship between necessity and possibility. Since the function is invertible, no information about the original possibility or necessity values is lost in the composition.

$$f(A) < 1/2 \Rightarrow \Pi(A) = 2f(A); N(A) = 0$$

$$\text{otherwise } \Pi(A) = 1; N(A) = 2[f(A) - 1/2]$$

So conceived, it is easy to verify that $f()$ is a Lukasiewicz-Sugeno credibility function. The $f()$ ordering is also reasonable in the two senses discussed, ordinally restating and Cox-reasonable.

The ordering based on $f()$ is transitive, and so linear or “one dimensional.” This should not be interpreted as a refutation of the common characterization (*e.g.*, Dubois

and Prade, 1994) of combining necessity and possibility as offering “two dimensions” of information about uncertainties. Distinct information is indeed offered by the two contributors to $f()$, different “thoughts” if you will. The two strands are linearly consonant, however, which is a different matter.

It is interesting from a probabilist-normative point of view to consider which possibilistic ties are resolved by necessity. The tie-breaking $f()$ ordering is a kind of quasi-additive extension of the strict orderings in the underlying possibility distribution. That is,

$$f(A) > f(B) \Rightarrow \Pi(A \neg B) > \Pi(B \neg A)$$

To see this, it is obvious that all the strict inequalities in possibility are echoed in $f()$, and

$$\Pi(A) > \Pi(B) \Rightarrow \Pi(A \neg B) > \Pi(B)$$

since order is determined by the *max* rule (something in A and not in B must beat B). But $B \neg A$ implies B , so by Lukasiewicz-Sugeno and transitivity,

$$\Pi(A) > \Pi(B) \Rightarrow \Pi(A \neg B) > \Pi(B \neg A)$$

For the ties which are broken by necessity,

$$\Pi(\neg B) > \Pi(\neg A) \Leftrightarrow$$

$$\Pi(A \neg B \vee \neg A \neg B) > \Pi(B \neg A \vee \neg A \neg B)$$

and by similar reasoning to the previous case based on the *max* rule, Lukasiewicz-Sugeno, and transitivity,

$$\Pi(\neg B) > \Pi(\neg A) \Rightarrow \Pi(A \neg B) > \Pi(B \neg A)$$

Of course, necessity breaks only some of the ties in the underlying possibility ordering. In general, a complete transitive ordering based on *max* cannot be quasi-additive for all sentence pairs. For example, consider the ordering

$$A =^* B >^* C >^* D$$

If quasi-additivity obtained in a *max* calculus, then we would have the cycle

$$A \vee C =^* B \vee C \text{ (by } \max) >^* B \vee D \text{ (by quasi-additivity)} \\ =^* A \vee C \text{ (by } \max)$$

Nevertheless, to the extent that necessity breaks possibilistic ties, it does so in a way which is consonant with a probabilist’s intuition about good credal ordering.

The Polya Domain

There is an interesting domain of uncertain reasoning in which necessity may experience difficulties, yet possibility itself may be an especially attractive reasoning tool.

A pioneer in the exploration of the domain is the mathematician George Polya (1954). It concerns reasoning about the possible truth of mathematical hypotheses, particularly as one’s opinion in the matter might be affected by the discovery of analogous, implied, or otherwise logically or intuitively related facts. In the course of his studies, Polya noted that the domain was similar to other inferential arenas, notably the development of scientific theories and historical questions such as determination of guilt in criminal investigations.

Perhaps of most immediate relevance to necessity is that in this domain, “not X ” is often unavailable for credal evaluation with any useful specificity. For example, it is easy enough to think about the import of finding the defendant’s DNA at the scene of a crime in relation to the hypothesis that she is guilty. But in relation to “the” contrary hypothesis? Would that be “not guilty, but someone with innocent access to the scene,” or “not guilty, and contamination occurred,” or something else?

Although Polya developed his analyses along probabilistic lines, total probability (that $p(X) + p(\neg X) = 1$) plays very little role in his work apart from algebraic formalities. In fact, his probabilism is at least skeptical about any important use for additivity within the domain, itself suggestive of a role for a non-additive calculus like possibility here.

Naturally, his viewpoint is subject to criticism from Bayesians (notably de Finetti, 1949 in response to a journal exposition of part of Polya’s ideas). Nevertheless, Polya pursued his work with a high level of normative sophistication (including some rebuttal to de Finetti, politely unreferenced, in the 1954 book).

This lack of access to a usefully specific “ $\neg X$ ” would justify some thought about a divorce of probability from necessity, at least in this domain. In itself, though, the negation problem would not forestall what might be called “contingent necessity,” the complement of the possibility of all *known* alternatives. Much of the earlier discussion of reasonableness and perhaps the considerations of quasi-additive extension might be marshaled on its behalf.

Necessity has another difficulty, however. While it is not specific to the Polya domain, the difficulty acquires some urgency here and is worthy of mention in any case.

Much of the belief revision which occurs in Polya’s approach consists of the elimination of known alternatives. For example, it is easy to imagine that a DNA test might eliminate one or more suspects, and this elimination would have some significance for the assessment of guilt of the remaining suspects.

The difficulty may be illustrated by recalling the four exclusive hypotheses whose possibilities are in the order

$$A =^* B >^* C >^* D$$

Obviously, we have

$$B \vee C =^* B \vee D$$

which tie cannot be broken because the liveliness of A forces a tie in necessity. If, however, we were then to learn that A is untrue, but the possibility ordering of the remaining hypotheses is unchanged, then we arrive at

$$B \vee C >^* B \vee D$$

in the new $f()$ ordering, since A no longer defeats the quasi-additive conclusion based on C ’s advantage over D .

At which point, we have an impasse in intuitions. The idea that the elimination of an alternative can change the credal order among the surviving assertions is acceptable to some. It is, for example, the nub of the famous “Peter, Paul, and Mary” case (Dubois, Prade, and Smets, 1996).

A point made in advancing that case, however, is precisely that probabilist intuition differs. That is, barring something special about how A was eliminated, its demise should leave the survivors in the same order as before in typical probabilist accounts of belief change.

This divergence of opinion will not be settled here. It is offered as a factor to consider, and as a needed counterbalance to the earlier recitation of probabilist-reasonableness results, all of which relate to what might be called the static cogency of the $f()$ ordering.

A final point about necessity in the Polya domain is not a matter of reasonableness or normative dispute, but concerns rather a unique property of the possibility calculus which it does not share with necessity, nor with the $f()$ composition. The property will be the subject of the next section; its motivation belongs here with Polya.

A distinctive feature of Polya's work is an ambivalence about the goals of inference. Obviously, the mathematician, scientist, and jurist are all concerned with the discovery of truth. But that is not the whole of their jobs.

Polya also wishes to counsel mathematicians about what problems are worthwhile to work on. Discovering the truth of some implication of a conjecture not only encourages belief in its truth, but also establishes that the conjecture has consequences, that it might explain those consequences, that it is *interesting*.

For their part, jurists self-consciously adopt rules of evidence which incorporate notions of fairness as well as probative value. A revealing hearsay may be ruled out of court not because it is uninformative, but because to admit it is to compromise a defendant's right to cross-examine witnesses against him.

Scientists sometimes engage in especially nuanced inferential episodes. Theories may be judged on their tractability and elegance along with their fidelity to the experimental record. Beauty may also be a factor in scientific thought (McAllister, 1998), both as a value in its own right, and as a heuristic guide to truth.

The conclusions scientists arrive at may also be complicated, as evidenced by the survival in practice of Newtonian mechanics and the simultaneous "acceptance" of incompatible theories (wave and particle models of light as the occasion demands, or the unresolved discrepancies between general relativity and quantum mechanics). This complexity is unsurprising in an enterprise which aspires to *approach* the truth, rather than (solely) to *attain* it.

Throughout the domain, then, a "good" conclusion is not necessarily determined by the truth of the matter. Merit is more a matter of preference (interestingness, usefulness, fairness, ...), unrebutted by the evidence rather than proven by it.

If this characterization of expert goal-setting practice is accurate, then there would seem to be a role for an inferential calculus which served both preferential and more ordinary credibility inference in the style of Lukasiewicz-Sugeno.

A Distinctive Feature of the Possibility Calculus

The value of the Lukasiewicz-Sugeno insight is that it captures some of the intuitive force of what people mean when they speak of credibility, while at the same time preserving a high degree of generality. Suppose one set out to look at preferential reasoning with a similar goal.

If we have a domain of mutually exclusive rewards, we see immediately that preferential reasoning is unlike Lukasiewicz-Sugeno. Offered the choice between

a commitment to be paid \$5

a commitment to be paid \$5 or else \$1

it is the stronger, rather than the weaker, commitment which is preferred. If the issue were credibility (perhaps an onlooker wondering how much money will change hands), then Lukasiewicz-Sugeno favors the weaker sentence (weakly).

But preference does not always follow logical strength, since between

a commitment to be paid \$1

a commitment to be paid \$5 or else \$1

things are more complicated. Realistically, it would depend on what you thought about the prospects for actually getting the \$5 if you opted for the disjunctive commitment.

Nevertheless, this homely example suggests one candidate for a general description of a preference ordering among sentences describing outcomes:

if $A \Rightarrow B$, then $B \succ^* A$ implies $B \neg A \succ^* A$
or equivalently,

if $A \Rightarrow B$, then $A \succeq^* B \neg A$ implies $A \succeq^* B$

As with Lukasiewicz-Sugeno, we might expect more from a practical calculus, but it is plausible that we would not be content with less.

Although derived from an elementary observation about preference, the relationship also echoes something of what is found in orderings of evidentiary support using ordinary conditional probabilities. It is easily verified that

if $A \Rightarrow B$, then $p(e | B) > p(e | A)$ implies
 $p(e | B \neg A) > p(e | A)$

It is also interesting that even though the criterion is not offered as a description of credibility, nevertheless, it states an ordinal property of the ordinary Boolean logic, just as Lukasiewicz-Sugeno does.

In comparing the two kinds of ordering criteria

Lukasiewicz-Sugeno: if $A \Rightarrow B$, then $B \succeq^* A$
preference-support: if $A \Rightarrow B$, then $A \succeq^* B \neg A$
implies $A \succeq^* B$

it is straightforward that a *max* rule for disjunctions satisfies *both* criteria. It is only a bit more work to show that *max* is the only rule which is a function of its

disjuncts and which relates stronger and weaker sentences as required in a transitive ordering.

Sketch proof. If $A \Rightarrow B$, then $B \geq^* A$ by Lukasiewicz-Sugeno. If $B >^* A$, then $B \neg A >^* A$ immediately from preference-support. If $B =^* A$ and $B \neg A >^* A$, then since $B \geq^* B \neg A$ by Lukasiewicz-Sugeno, by transitivity $B >^* A$, contrary to the hypothesis.

In finite domains, transitivity is easily shown to impose the max rule for implications on all comparisons of disjunctions.

Sketch proof. If a is the top atom in A , and b is the top atom in B , then $a =^* A$ and $b =^* B$ by the max rule for implications. Whatever order obtains between a and b will obtain between A and B : $A = a$?? $b = B$.

Possibility, the max calculus, stands alone as the only calculus which is both Lukasiewicz-Sugeno and preference-support. It is easily confirmed that the $f()$ composite ordering is not preference-support, and thus the combination of calculi does not have the distinction that possibility alone has.

Of course, any mechanism for tie-breaking would force the new combined calculus to be either Lukasiewicz-Sugeno alone or preference-support alone (or perhaps neither). The ties are how the possibility calculus manages to walk the tightrope between the all-but-conflicting criteria.

Although it is not the purpose of this short paper to offer alternatives to necessity, it is interesting to note in passing that if one wished to break possibilistic ties in the preference-support direction, one would expect that something very much like “quasi-additivity” would be a plausible ingredient. The notion that two sentences A and B would be compared based on how they differ (that is, how $A \neg B$ stands with respect to $B \neg A$) is a widely-shared intuition about how preferences work.

In any case, possibility’s twin aspect combining credibility reasoning and preference-support reasoning may offer a promising vehicle for excursions into Polya’s realm of incompletely formulated alternatives and ambiguous inferential goals. It may even provide a useful alternate formulation of Polya’s already hardly-additive “probabilistic” account of the territory.

Conclusions

The normative case in favor of necessity, both in its own right and as a tie-breaking mechanism for possibility, is considerable. That a portion of that case arises from outside the fuzzy-possibility community is especially noteworthy, and this portion forms the bulk of the case presented here.

The paper is also frank about a principled disagreement among scholars concerning an important feature of belief revision which the combination of necessity and possibility falls on one side of, while possibility alone can

be placed (in at least some interpretations) on the other side.

Possibility and only possibility, unassisted by any mechanism for tie-breaking, gives rise to an uncertainty ordering which combines credibility and preference elements. Thus, unadorned possibility seems especially well-suited for exploring an interesting and important domain which engaged George Polya.

That domain’s hostility to evaluation based on complementation further diminishes the appeal of necessity there, echoing Polya’s own conclusion about the usefulness of some aspects of conventional probability.

The possibility calculus’ ties are what supports the ability to mirror the ambiguity of the usual notions of merit within the domain. On at least some occasions, then, the use of possibility without necessity has some attraction.

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