Belief Update Using Graphs

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Abstract

The purpose of this paper is to introduce a form of update based on the minimization of the geodesic distance on a graph. We provide a characterization of this class using set-theoretic operators and show that such operators bijectively correspond to geodesic metrics. As distance is generated by distinguishability, our framework is appropriate in contexts where distance is generated by threshold, and therefore, when measurement is erroneous.

Introduction

It was early noticed by (Keller & Winslett 1985) that updating a belief base differs from revising it. Updating should be performed when the belief base is assumed to reflect the external world and a new contradictory piece of information comes about because the external world has changed. In contrast, revision is the process of accommodating a new (possibly contradictory) piece of information describing an external world which remains fixed. An example of this difference can be demonstrated by the following. Let a, b, c three atomic propositions and suppose that the states we consider possible are $A = \{w_1, w_2\}$, where $w_1 = \{\neg a, b, c\}$ and $w_2 = \{\neg a, \neg b, \neg c\}$. Measure the distance between a pair of states with the number of atomic symbols that they differ. Suppose we need to update A with $a \wedge b$, that is, $B = \{\{a, b, c\}, \{a, b, \neg c\}\}$. The result of the *revision* of A with B is the singleton $\{\{a, b, c\}\}$ because $\{a, b, c\}$ is the closest to A (differs from w_1 by a single atomic proposition). On the other hand, the result of the *update* of A with B is B because $\{a, b, \neg c\}$ is closer to w_2 than $\{a, b, c\}$ is, while $\{a, b, c\}$ is closer to w_1 than $\{a, b, \neg c\}$ is. The reason for this difference is that if we believe that the only states possible are w_1 and w_2, then the change represented by B should have an effect on both states rather than render one of those states impossible. In other words, revision is a result of a global minimization while update is a result of a component-wise one.

The process of updating is better behaved than revision. In particular, revision cannot be defined using the so-called Ramsey test ((Gärdenfors 1986), (Gärdenfors 1987)). More precisely, revision that follows from a set of the postulates stated in (Alchourrón, Gärdenfors, & Makinson 1985) (AGM hereafter) is not expressible with an object language conditional connective. On the other hand, update is compatible with the Ramsey test as it was shown by (Grahne 1991) (See also (Herzig & Rifi 1999)). Therefore, update is interchangeable with conditioning and this interchange makes update a significant process worth studying in depth. Update has not been modeled sufficiently. To our knowledge, apart from specific (quantitative) models, the only qualitative characterization is the characterization using orderings parameterized by interpretations of (Katsuno & Mendelzon 1991) (KM for short). (Katsuno & Mendelzon 1991) provided a set of rules for update that follow closely the AGM rules for revision. Parametrization is somewhat surprising in view of the fact that most concrete quantitative examples used are based on a global metric. Nevertheless, parametrization is widely applied in the semantics of belief change (revision and update). Apart from KM orderings, it is also present in the systems of spheres of (Grove 1988) and the ordinal conditional functions of (Spohn 1987). In such modelings, parametrization occurs when each belief change brings about a new ranking, ordering, sphere system, etc., which in turn will define the next belief change. As a result, the strategy for belief change is not part of the epistemic state; it is described separately. However, unless there is a notion of learning explicitly involved, there is no justification for such exclusion.

The purpose of the present paper is to introduce and characterize a class of update operators that are modeled by a global metric. We consider graphs whose vertices are possible states, and edges represent indistinguishability. Although many authors have argued that indistinguishability is better expressed through equivalence ((Aumann 1976; Hintikka 1962; Fagin, Halpern, & Vardi 1991)), many have also dropped transitivity as early as (Poincaré 1905) (see also (Goodman 1977) for similarity), while others have weakened transitivity, most notable example being the t-norm transitivity of the similarity relation in Fuzzy sets ((Zadeh 1971)). A set equipped with a reflexive symmetric relation has been called a *proximity* space in (Bell 1986).

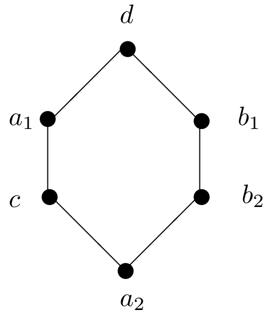

Figure 1:

In the framework of Rough Sets (Pawlak 1991), a reflexive, symmetric, and transitive relation of indistinguishability is called *indiscernibility* and if it is only reflexive and symmetric is called a *tolerance* relation ((Zeeman 1962; Nieminen 1988)). In addition, their logical status has been well studied and give rise to orthologic (Goldblatt 1974), and the modal logic system B (see (Hughes & Cresswell 1984)).

The class of graphs is a very general and intuitive framework for studying belief update as it makes as few assumptions as possible. Examples are easily modeled by a graph, as the indistinguishability relation needs no quantitative information. Metrics is perhaps the easiest way to generate distinguishability; for example, let $x \sim y$ if and only if $d(x, y) < \epsilon$ for some appropriate fixed metric d and non-negative real number ϵ. Other examples follow:

Example 1 Let S be the set of finite binary strings of finite length n. We can say that two strings are indistinguishable when they have the same length and differ in at most one digit. Otherwise, they are distinguishable.

Example 2 Let D be a set of documents (sets of terms) and n a positive integer. Two documents d_1 and d_2 are indistinguishable when they have at least n common terms. That is,

$$d_1 \sim d_2 \quad \text{iff} \quad |d_1 \cap d_2| \geq n.$$

The above examples are important and will be discussed in more detail in the next section.

Every graph comes equipped with a distance map called *geodesic* which is simply defined as the length of the shortest path between two vertices. Minimization over a geodesic is straightforward. We illustrate the process with Figure 1. Let $A = \{a_1, a_2\}$, and $B = \{b_1, b_2\}$. We denote update with $\bullet$ and revision with $*$. Then $A * B = \{b_2\} \neq \{b_1, b_2\} = A \bullet B$. Notice that this example also illustrates the example mentioned earlier.

The idea of using adjacency in order to express indistinguishability is not new. What is new is our idea of using the geodesic to express similarity. In (Georgatos 2000; 2003), we argued that it is possible to view similarity as a derived notion, the more basic concept being distinguishability:

> Our idea rests on the following maxim: two objects are similar when there is a context under which they are indistinguishable. Therefore, similarity can be *measured* with degrees of distinguishability. For example, although two similar houses might appear different in various details when we stand in front of them, they might appear identical if we observe them from an appropriate distance x. Thus, indistinguishability at distance x implies similarity. The smaller the distance x, the more similar the objects are.

This way our approach realizes the Stalnaker idea of conditioning by picking the most similar worlds ((Stalnaker 1968)) . In our framework, the closest inerpretations (according to the geodesic metric) are the most similar, as long as similarity is measured by degress of distinguishability.

The advantage of our framework over that of (Katsuno & Mendelzon 1991) is that our semantics is based on a global structure; KM makes use of parametrized orderings. A graph is easy to describe and can generate a ranking for every node or a subset of nodes in a straightforward manner: given a set of nodes A then the set of adjacent nodes to A, $Adj(A)$, are first in the rank, followed by $Adj(Adj(A))$, etc.

On the other hand, our semantics is a refinement of KM because the KM postulates include distance based operators (a global distance map to a partial order induces a KM model). As a result, there are well-known update operators which satisfy the KM postulates, but they are not based on a geodesic metric. Those include the Possible Models Approach of (Winslett 1988) and the update operators of (Forbus 1989) which are distance based but not geodesic. Other update operators proposed in the literature fail to be geodesic as they do not satisfy the KM postulates, and as a consequence they are not distance based either. These include MCD of (Zhang & Foo 1996), WSS of (Winslett 1990), and WSS^{dep} of (Herzig & Rifi 1998). There is no characterization of distance based update operators, although (Lehmann, Magidor, & Schlechta 2001) have characterized distance based *revision* operators. Distance based revision operators do not correspond bijectively to distance maps, as (Lehmann, Magidor, & Schlechta 2001) has shown (see also (Delgrande 2004)). Geodesic revision operators, on the other hand, correspond exactly to geodesic metrics (see (Georgatos)).

First, we discuss graphs and their geodesics and present examples. Then, we define geodesic update, and present our postulates and associated results. The main characterization result is Corollary 17. We conclude with an axiomatization of the class of selection functions corresponding to the geodesic update operators.

Graphs and Their Geodesics

Given a graph $G = (V, E)$ its geodesic metric is the map d_G from $V \times V$ to $Z^+ \cup \{\infty\}$ where $d_G(u, v)$ between two vertices u and v is the length of a shortest (u, v)-path (counting the number of edges of the path) if there is a path, and equals to ∞ otherwise. When the distinguishability graph is connected then the range of the geodesic restricts to Z^+ (the set of non-negative integers) and the geodesic metric is a topological metric; that is, it satisfies identity, symmetry and the triangle inequality. It is important to note that a geodesic metric is an integer metric. Distance on a graph and therefore the values of the geodesic metric is determined by adjacency. The results of this paper depend heavily on this property which can be described with: for all $x, y \in V$ such that $d_G(x,y) = n$ with $1 < n < \infty$ there is $z \in V$ with $z \neq x, y$ such that $d_G(x,y) = d_G(x,z) + d_G(z,y)$. In particular, we can choose z so that $d_G(x,z) = 1$.

The geodesic distance extends to distance between subsets with

$$d(A,B) = \begin{cases} \min\{d(x,y) \mid x \in A, y \in B\} \\ \quad \text{if there are } x \in A, y \in B \ d(x,y) \neq \infty \\ \infty \quad \text{otherwise} \end{cases}$$

Observe that the definition sets $d(A,B) = \infty$ when $A = \emptyset$ or $B = \emptyset$. We shall also write $d(x,A)$ for $d(\{x\},A)$. Similarly for $d(A,x)$. The following observation will be useful.

Lemma 3 *We have $d_G(A, A^c) = 1$ or ∞.*

We turn now to examples of distance functions that can or cannot be expressed using a graph. First consider the *hamming distance*, defined as the number of symbols where two valuations differ (see (Dalal 1988) and (Forbus 1989)). In general, a set of valuations equipped with hamming distance is not geodesic: consider two propositional atoms and a domain made of two valuations 00 and 11 whose hamming distance is 2. There is no valuation between those two and this space cannot be represented with a graph. This counterexample indicates that a hamming distance based space is geodesic if it has "enough" elements. It is enough to add a third valuation 10 to the above domain to turn the distance into a geodesic one. In short, a hamming distance based space may not be geodesic itself but it can be embedded into a geodesic one. In a similar fashion, arbitrary integer metrics are not geodesic but they can be embedded to an appropriate geodesic space.

We shall now discuss a more interesting construction of a geodesic space given an arbitrary metric space. This construction is a straightforward modeling of the concept of threshold which was the original motivation of defining updates on graphs. Let (X,d) be a a metric space and let $e > 0$ be a real number representing a threshold. Then (X, d_e) is called the e-threshold space of (X,d), with $d_e(x,y) = d_G(x,y)$, where $G = (V,E)$ is a graph with $V = X$ and $(x,y) \in E$ iff $d(x,y) < e$. By definition every e-threshold space is geodesic.

We will now present two concrete examples of a geodesic space. The first example models degrees of separation. Let V be the set of computer science authors and $(A_1, A_2) \in E$ if A_1 and A_2 are joint authors (of the same paper). Clearly, the associated geodesic models degrees of (collaborative) separation between two authors.

The second example shows that if language is used as a relevance criterion then the resulting space is geodesic. So geodesic update is the appropriate form of update when a change in the outside world has an impact only on those parts of the knowledge base where one can establish a linguistic link. Let $Atom$ be a set of atomic symbols and $\mathcal{L}$ a propositional language build on $Atom$. For each formula $a \in \mathcal{L}$, let $\mathcal{L}_a$ be the subset of $Atom$ occurring in a. Then the *relevance graph* of $\mathcal{L}$ is the graph $G = (\mathcal{L}, E)$ with $(a,b) \in E$, where $a, b \in \mathcal{L}$, iff $\mathcal{L}_a \cap \mathcal{L}_b \neq \emptyset$. The resulting geodesic metric d_G models the following idea. Let A and B be two subsets of $\mathcal{L}$, representing parts of the knowledge base pertaining to two topics t_1 and t_2. Then the distance $d_G(t_1, t_2)$ between those topics can be defined as the minimum number of propositions one can use to link those topics. If there is no such sequence of propositions, then a linguistic link cannot be established, the two topics are disconnected (i.e., irrelevant to each other), and their distance is ∞.

Table 1: Geodesic Update Postulates

1. $A \bullet B \subseteq B$
2. If $A \subseteq B$, then $A \bullet B = A$
3. $(A \bullet B) \cap C \subseteq A \bullet (B \cap C)$
4. $\cup_{i \in I}(A_i \bullet B) = (\cup_{i \in I} A_i) \bullet B$
5. If $A \subseteq B \bullet B^c$, then $A \bullet B = (A \bullet A^c) \cap B$
6. If $B \subseteq A^c$, then $A \bullet B = (A \bullet (B \bullet B^c)) \bullet B$
7. If $B \subseteq A^c$, then $(A \bullet A^c) \cap B \neq \emptyset$ iff $(B \bullet B^c) \cap A \neq \emptyset$
8. If $A \bullet B = \emptyset$, $C \subseteq A$, and $D \subseteq B$, then $C \bullet D = \emptyset$
9. If $A \neq \emptyset$ and $B \neq \emptyset$, then $A \bullet B \neq \emptyset$

Update Based on a Geodesic

Given a graph, there is a straightforward notion of conditioning based on its geodesic. Since we have a notion of distance then we can choose the closest vertices. This follows, of course, Lewis' basic idea of giving semantics to conditionals ((Lewis 1973)).

Definition 4 Given a graph $G = (V,E)$ and $x \in V$, $B \subseteq V$ then the *induced simple update of x by B* is defined with

$$x \bullet_G B = \begin{cases} \{y \in B \mid d_G(x,y) = d_G(x,B)\} \\ \quad \text{if } d_G(x,B) \neq \infty \\ \emptyset \quad \text{otherwise} \end{cases}$$

The simple update generalizes to an update operator between subsets of V with

$$A \bullet_G B = \bigcup_{x \in A} x \bullet_G B.$$

Next we state some observations regarding the interaction of an update operator of a graph with respect to its geodesic.

Lemma 5 *Suppose $x \notin A$, then*

1. *$d_G(A,x) = 1$ if and only if $x \in A \bullet_G A^c$.*
2. *$A \bullet_G B = \emptyset$ iff $A = \emptyset$, or $B = \emptyset$, or A and B are disconnected.*
3. *We have $d_G(A,x) = n$, for $1 < n < \infty$, if and only if, $d_G(A \bullet_G A^c, x) = n - 1$.*

The postulates for geodesic update appear in Table 1. Note that the postulates could be presented using a propositional language with a finite alphabet exactly as in (Katsuno

& Mendelzon 1991). In a finite setting, propositions are faithfully represented by the sets of valuations that satisfy them and, therefore, a translation is straightforward. We do not deviate significantly from the theory of update of (Katsuno & Mendelzon 1991). In fact, postulates 1, 2, 3, 4, and 9 correspond to (U1), (U2), (U5), (U8), and (U3) of (Katsuno & Mendelzon 1991), respectively. The rest of the KM postulates are valid in geodesic update modulo the syntactic translation. Postulates 5, 6, 7, and 8 are new and their function is to ensure that update depends on a global metric. It is well known that the important case of both revision and update is when the piece of new information to be incorporated is inconsistent with the existing base. We postulate that the update with inconsistent information depends on the update with the negation of the existing theory. In particular, update with the negation should be performed inductively. The forward inductive step is postulate 5. The backward inductive step is postulate 6. We make sure that update is symmetric with postulate 7. Rule 8 postulates (dis)connectedness. Observe that Postulate 4 is infinite and ensures that update distributes union. Postulate 7 was chosen because it implies the following containment, which in turn, implies symmetry.

Lemma 6 *Postulate 7 implies*

$$(\cup_{i \in I} A_i) \bullet (\cup_{i \in I} A_i)^c \subseteq \cup_{i \in I}(A_i \bullet A_i^c).$$

Definition 7 An update operator that satisfies 1–8 of the table 1 will be called *centered geodesic*. If in addition satisfies 9 will be called *connected geodesic*.

Showing that the postulates are sound on update operators is straightforward.

Proposition 8 *Let G be a graph. Then,*

1. *The operation $\bullet_G$ is centered geodesic update.*
2. *If G is connected then the operation $\bullet_G$ is a connected geodesic update.*

Next we study the conditions under which a geodesic update operator defines a graph. To this end suppose V is a set and $\bullet$ is an operation on its subsets. Define a relation $E_\bullet$ on V with $(x, y) \in E_\bullet$ if and only if $y \in \{x\} \bullet \{x\}^c$. Denote $(V, E_\bullet)$ with $G_\bullet$.

Lemma 9 *Suppose $\bullet$ is geodesic. Then*

1. *If $\bullet$ satisfies Postulate 1 then $E_\bullet$ is irreflexive.*
2. *If $\bullet$ satisfies Postulate 7 then $E_\bullet$ is symmetric.*

The definition of $E_\bullet$ can be restated as follows

Lemma 10 *Suppose $\bullet$ is centered geodesic. Then, for all $x, y \in V$ and $A \subseteq V$ such that $(x, y) \in E_\bullet$, $x \in A$ and $y \in A^c$, we have $y \in A \bullet A^c$.*

Lemma 11 *Suppose $\bullet$ is centered geodesic. Then, $d_{G_\bullet}(A, x) = 1$ iff $x \in A \bullet A^c$.*

Corollary 12 $A \bullet A^c = \{x \mid d_{G_\bullet}(A, x) = 1\} = A \bullet_{G_\bullet} A^c$

Lemma 13 *If $\bullet$ is a connected geodesic then $G_\bullet$ is connected.*

To reach a set from a given point we should necessarily step on a point belonging to its complement. This will form the basis of the induction proof and is expressed with the following lemma.

Table 2: Selection Function Postulates

1.	$f(x, A) \subseteq A$
2.	If $x \in A$, then $f(x, A) = \{x\}$
3.	$f(x, A) \cap B \subseteq f(x, A \cap B)$
4.	If $f(x, A) \cap B \neq \emptyset$, then $f(x, A \cap B) \subseteq f(x, A) \cap B$
5.	$f(x, A) = \bigcup_{y \in f(x, \bigcup_{z \in A} f(z, \{z\}^c))} f(y, A)$
6.	If $f(x, A) = \emptyset$ and $B \subseteq A$, then $f(x, B) = \emptyset$
7.	If $y \in f(x, A)$, then $f(y, A^c) \neq \emptyset$
8.	If $y \in f(x, A)$ and $z \in f(y, A^c)$, then $z \in f(y, \{y\}^c)$

Lemma 14 *Suppose $\bullet$ is centered geodesic. Then we have, for $1 < n < \infty$, $d_{G_\bullet}(x, A) = n$ iff $d_{G_\bullet}(x, A \bullet A^c) = n - 1$.*

The following lemma shows that an update results in an empty set when we update with nodes whose distance from the given set is infinite.

Lemma 15 *Suppose $\bullet$ is centered geodesic and let $A^* = \{y \mid d(x, y) = n < \infty, x \in A\}$. Then $A^* \bullet (A^*)^c = \emptyset$.*

We can now show that the simple update operator is characterized by the postulates.

Proposition 16 *Given a centered geodesic update $\bullet$, we have $\{x\} \bullet A = \{x\} \bullet_{G_\bullet} A$.*

The above proposition extends to arbitrary subsets. This is possible because the above update operator distributes union by Postulate 4.

Corollary 17 *Given a centered geodesic update $\bullet$, we have $\bullet = \bullet_{G_\bullet}$.*

The above corollary shows that the set of postulates characterizes the class of geodesic metrics. Note that it also implies that, for a given graph G, we have $G = G_{\bullet_G}$ as, clearly, distinct graphs induce distinct update operators.

Selection Functions and Updates

Observe that the simple update defines, in effect, a selection function. By selection function we mean a function that given an element x of the domain and a subset A, the function selects those elements of the subset A that are the most preferred (closest) to the element x. This concept is the cornerstone of the Lewis-Stalnaker semantics for conditional logic.

Definition 18 Given a graph $G = (V, E)$ the selection function $f_G : V \times \mathcal{P}(V) \to \mathcal{P}(V)$ generated by G is defined by

$$f_G(x, A) = x \bullet_G A.$$

Definition 19 A selection function that satisfies the postulates of Table 2 will be called *geodesic*.

Proposition 20 *Let G be a graph, then*

1. *f_G is geodesic.*
2. *the map $G \mapsto f_G$ is bijective.*

A selection function defines a graph as follows:

Definition 21 Given a set X and a function $f : X \times \mathcal{P}(X) \to \mathcal{P}(X)$ then the graph $G_f = (X, E_f)$ is the graph generated by f, where $(x, y) \in E_f$ iff $y \in f(x, \{x\}^c)$.

Corollary 22

$$f(z, \{z\}^c) = \{z \in V \mid d_{G_f}(y, z) = 1\} = f_{G_f}(z, \{z\}^c)$$

Proposition 23 *The maps $f \to G_f$ and $G \to f_G$ are inverses of each other.*

Summary and Conclusion

We have introduced new semantics for the process of belief update based on the minimization of a geodesic metric of a graph. We also characterized the class of associated set-theoretic update operators and selection functions.

There is a distinct advantage to our approach: it can be used to integrate a variety of epistemic functions. Using the same semantics, we have shown that one may define an associated revision operator. Therefore, revision can be combined with the update. The integration of revision was recently recognized as necessary for handling feedback ((Boutilier 1998; Shapiro & Pagnucco 2004; Hunter & Delgrande 2005)). Similarly, an equivalence relation is generated from the transitive closure of the graph edge relation. Equivalence relations are models of the modal logic $\mathbf{S5}$, which is recognized as the right framework for modeling (resource free) knowledge. This way, one can combine update operators with knowledge, i.e. facts, as in (Baral & Zhang 2005). However, our framework omits actions, and incorporating those is necessary for a satisfactory modeling of planning. In particular, a more dynamic representation, where actions may change the graph model, might be better suited (see (Lang 2007; Gabbay))

Finally, the characterization of the class of selection functions defined on a graph points to a conditional logic that corresponds to update. Such a formulation will allow us to define a logical system where update is expressed within the language. This is a significant departure from the Katsuno-Mendelzon and subsequent formulations, where, although $a \circ b$ is part of the language, a and b are not allowed to contain occurrences of $\circ$, the update operator.

Acknowledgements

This work was supported (in part) by a grant (No. PSCREG-38-749) from The City University of New York PSC-CUNY Research Award Program.

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