Axiomatizing the Cyclic Interval Calculus

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Abstract

A model for qualitative reasoning about intervals on a cyclic time has been recently proposed by Balbiani and Osmani (Balbiani & Osmani 2000). In this formalism, the basic entities are intervals on a circle, and using considerations similar to Allen's calculus, sixteen basic relations are obtained, which form a jointly disjunctive and pairwise distinct (JEPD) set of relations. The purpose of this paper is to give an axiomatic description of the calculus, based on the properties of the *meets* relation, from which all other fifteen relations can be deduced. We show how the corresponding theory is related to cyclic orderings, and use the results to prove that any countable model of this theory is isomorphic to the cyclic interval structure based on the rational numbers. Our approach is similar to Ladkin's axiomatization of Allen's calculus, although the cyclic structures introduce specific difficulties.

Keywords: qualitative temporal reasoning, cyclic interval calculus, cyclic orderings, completeness, $\aleph_0$–categorical theories

Introduction

In the domain of qualitative temporal reasoning, a great deal of attention has been devoted to the study of temporal formalisms based on a dense and unbounded linear model of time. Most prominently, this is the case of Allen's calculus, where the basic entities are intervals of the real time line, and the 13 basic relations (Allen's relations) correspond to the possible configurations of the endpoints of two intervals (Allen 1981). Other calculi such as the cardinal direction calculus (Ligozat 1998a; 1998b), the n-point calculus (Balbiani & Condotta 2002), the rectangle calculus (Balbiani, Condotta, & Fariñas del Cerro 1999), the n-block calculus (Balbiani, Condotta, & Farinas del Cerro 2002) are also based on products of the real line equipped with its usual ordering relation, hence on products of dense and unbounded linear orderings.

However, many situations call for considering orderings which are *cyclic* rather than linear. In particular, the set of directions around a given point of reference has such a cyclic structure. This fact has motivated several formalisms in this direction: Isli and Cohn (Isli & Cohn 2000) and Balbiani *et al.* (Balbiani, Condotta, & Ligozat 2002) consider a calculus about points on a circle, based on qualitative ternary relations between the points. Schlieder's work on the concepts of orientation and panorama (Schlieder 1993; 1995) is also concerned with cyclic situations. Our work is more closely related to Balbiani and Osmani's proposal (Balbiani & Osmani 2000) which we will refer to as the *cyclic interval calculus*. This calculus is similar in spirit to Allen's calculus: in the same way as the latter, which views intervals on the line as ordered pairs of points (the starting and ending point of the interval), the cyclic interval calculus considers intervals on a circle as pairs of distinct points: two points on a circle define the interval obtained when starting at the first, going (say counterclockwise) around the circle until the second point is reached. The consideration of all possible configurations between the endpoints of two intervals defined in that way leads to sixteen basic relations, each one of which is characterized by a particular qualitative configuration. For instance, the relation *meets* corresponds to the case where the last point of the first interval coincides with the first point of the other, and the two intervals have no other point in common. Another interesting relation, which has no analog in the linear case, is the *mmi* relation[1], where the last point of each interval is the first point of the other (as is the case with two serpents, head to tail, each one of them devouring the other).

This paper is concerned with giving suitable axioms for the *meets* relation in the cyclic case. This single relation can be used to define all other 15 relations of the formalism (there is a similar fact about the *meets* relation in Allen's calculus). We give a detailed description of the way in which the axiomatization of cyclic orderings – using a ternary relation described in (Balbiani, Condotta, & Ligozat 2002)– relates to the axiomatization of cyclic intervals based on the binary relation *meets*. Our approach is very similar to the approach followed by Ladkin in his PhD thesis (Ladkin 1987) where he shows how the axiomatization of linear dense and unbounded linear orderings relates to the axiomatization proposed by Allen and Hayes for the interval calculus, in terms of the relation *meets*.

The core of the paper, apart from the choice of an appropriate set of axioms, rests on two constructions:

[1]The notation is mnemonic for *meets* and *meets inverse*.

- Starting from a cyclic ordering, that is a set of points equipped with a ternary order structure satisfying suitable axioms , the first construction defines a set of cyclic intervals equipped with a binary *meets* relation; and conversely.
- Starting from a set of cyclic intervals equipped with a *meets* relation, the second construction yields a set of points (the intuition is that two intervals which meet define a point, their meeting point) together with a ternary relation which has precisely the properties necessary to define a cyclic ordering.

The next step involves studying how the two constructions interact. In the linear case, a result of Ladkin's can be expressed in the language of category theory by saying that the two constructions define an equivalence of categories. Using Cantor's theorem, this implies that the corresponding theories are $\aleph_0$ categorical. In the cyclic case, we prove an analogous result: here again, the two constructions define an equivalence of categories. On the other hand, as shown in (Balbiani, Condotta, & Ligozat 2002), all countable cyclic orderings are isomorphic. As a consequence, the same fact is true of the cyclic interval structures which satisfy the axioms we give for the relation *meets*. This is the main result of the paper. We further examine the connections of these results to the domain of constraint-based reasoning in the context of the cyclic interval calculus, and we conclude by pointing to possible extensions of this work.

Building cyclic interval structures from cyclic orderings

This section is devoted to a construction of the cyclic interval structures we will consider in this paper, starting from cyclic orderings. In the next section, we will propose a set of axioms for these structures. Intuitively, each model can be visualized in terms of a set of oriented arcs (intervals) on a circle (an interval is identified by a starting point and an ending point on the circle), together with a binary *meets* relation on the set of intervals. Specifically, two cyclic intervals (m, n) and (m', n') are such that (m, n) *meets* (m', n') if $n = m'$ and n' is not between m and n, see Figure 1 (as a consequence, $n = m'$ is the only point that the two intervals have in common).

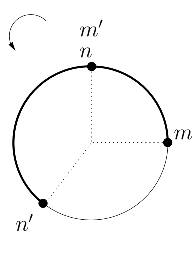

Figure 1: Two cyclic intervals (m, n) and (m', n') satisfying the *meets* relation.

In order to build interval structures, we start from cyclic orderings[2](Balbiani, Condotta, & Ligozat 2002). Intuitively, the cyclic ordering on a circle is similar to the usual ordering on the real line. In formal terms, a cyclic ordering is a pair $(\mathcal{P}, \prec)$ where $\mathcal{P}$ is a nonempty set of points, and $\prec$ is a ternary relation on $\mathcal{P}$ such that the following conditions are met, for all $x, y, z, t \in \mathcal{P}$:

P1. $\neg \prec (x, y, y)$;

P2. $\prec (x, y, z) \wedge \prec (x, z, t) \rightarrow \prec (x, y, t)$;

P3. $x \neq y \wedge x \neq z \rightarrow y = z \vee \prec (x, y, z) \vee \prec (x, z, y)$;

P4. $\prec (x, y, z) \leftrightarrow \prec (y, z, x) \leftrightarrow \prec (z, x, y)$;

P5. $x \neq y \rightarrow (\exists z \prec (x, z, y)) \wedge (\exists z \prec (x, y, z))$;

P6. $\exists x, y \;\; x \neq y$.

Definition 1 (The cyclic interval structure associated to a cyclic ordering) *Let* $(\mathcal{P}, \prec)$ *be a cyclic ordering. The cyclic interval structure* $\mathsf{CycInt}((\mathcal{P}, \prec))$ *associated to* $(\mathcal{P}, \prec\;)$ *is the pair* $(\mathcal{I}, meets)$ *where:*

- $\mathcal{I} = \{(x, y) \in \mathcal{P} \times \mathcal{P} : \exists z \in \mathcal{P} \; with \prec (x, y, z)\}$*. The elements of* $\mathcal{I}$ *are called (cyclic) intervals.*
- *meets is the binary relation defined by* $meets = \{((x, y), (x', y')) : y = x' \; and \; \prec (x, y, y')\}$.

As an example, consider the set $\mathcal{C}$ of all rational numbers contained in the interval $[0, 2\pi[$. Each rational number in that range represents a point in the unit circle centered at the origin in the Euclidean plane: $n \in [0, 2\pi[$ corresponds to the point with polar coordinates $(1, n)$. Let $\prec_{\mathcal{C}}$ the binary relation $[0, 2\pi[$ as follows: $\prec_{\mathcal{C}} (x, y, z)$ if and only if either $x < y < z$ or $y < z < x$ or $z < x < y$, where $x, y, z \in [0, 2\pi[$. We can easily check that the structure $(\mathcal{C}, \prec_{\mathcal{C}})$ we get is a cyclic ordering. Hence $\mathsf{C}ycInt((\mathcal{C}, \prec_{\mathcal{C}}))$ is a cyclic interval structure $(\mathcal{I}, meets)$. Each element $u = (x, y)$ of $\mathcal{I}$ can be viewed as the oriented arc containing all points between the points represented by x and y (we will refer to these two points as to the endpoints of the cyclic interval u and denote by u^- and u^+, respectively, the points associated to x and y). For instance, the cyclic intervals $(0, \pi/2)$, $(\pi/2, 0)$ and $(3\pi/2, \pi/2)$ are shown in Figure 2. Notice that no cyclic interval contains only one point (there are no punctual intervals), and that no interval covers the whole circle. Intuitively, two cyclic intervals are in the relation *meets* if

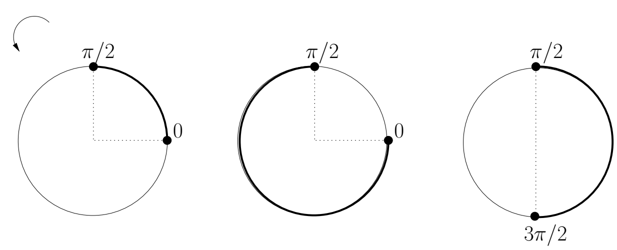

Figure 2: Three cyclic intervals.

[2]Actually, we use what are called "standard cyclic orderings" in (Balbiani, Condotta, & Ligozat 2002). We use the shorter term "cyclic ordering" in this paper.

and only if the ending point of the first coincides with the starting point of the other, and the intervals have no other point in common. For instance, $((3\pi/2, \pi/2), (\pi/2, \pi)) \in meets$, while $((3\pi/2, \pi/2), (\pi/2, 5\pi/3)) \notin meets$.

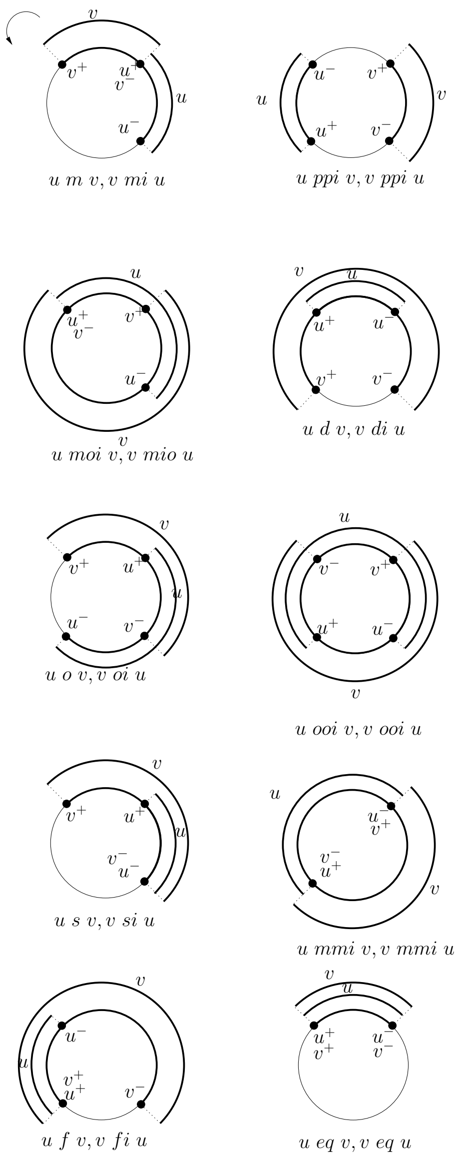

Figure 3: The 16 basic relations of the cyclic interval calculus.

Let $(\mathcal{I}, meets)$ be a cyclic interval structure. We now show how the other fifteen basic relations of the cyclic interval calculus defined by Balbiani and Osmani (Balbiani & Osmani 2000) can be defined using the *meets* relation. The 16 relations are denoted by the set of symbols $\{m, mi, ppi, mmi, d, di, f, fi, o, oi, s, si, ooi, moi, mio, eq\}$ (where m is the *meets* relation). Figure 3 shows examples of these relations. More formally, the relations other than *meets* are defined as follows[3]:

- $u\ \mathbf{ppi}\ v \overset{def}{\equiv} \exists w, x \quad u\ \mathbf{m}\ w\ \mathbf{m}\ v\ \mathbf{m}\ x\ \mathbf{m}\ u$,
- $u\ \mathbf{mmi}\ v \overset{def}{\equiv} \exists w, x, y, z \quad w\ \mathbf{m}\ x\ \mathbf{m}\ y\ \mathbf{m}\ z\ \mathbf{m}\ w \wedge z\ \mathbf{m}\ u\ \mathbf{m}\ y \wedge x\ \mathbf{m}\ v\ \mathbf{m}\ w$,
- $u\ \mathbf{d}\ v \overset{def}{\equiv} \exists w, x, y \quad w\ \mathbf{m}\ x\ \mathbf{m}\ u\ \mathbf{m}\ y\ \mathbf{m}\ w \wedge v\ \mathbf{mmi}\ w$,
- $u\ \mathbf{f}\ v \overset{def}{\equiv} \exists w, x \quad w\ \mathbf{m}\ x\ \mathbf{m}\ u\ \mathbf{m}\ w \wedge v\ \mathbf{mmi}\ w$,
- $u\ \mathbf{o}\ v \overset{def}{\equiv} \exists w, x, y, z \quad u\ \mathbf{m}\ v\ \mathbf{m}\ x\ \mathbf{m}\ u \wedge v\ \mathbf{m}\ x\ \mathbf{m}\ y\ \mathbf{m}\ v \wedge y\ \mathbf{m}\ z\ \mathbf{m}\ w$,
- $u\ \mathbf{s}\ v \overset{def}{\equiv} \exists w, x, y \quad w\ \mathbf{m}\ x\ \mathbf{m}\ v\ \mathbf{m}\ w \wedge x\ \mathbf{m}\ u\ \mathbf{m}\ y\ \mathbf{m}\ w$,
- $u\ \mathbf{ooi}\ v \overset{def}{\equiv} \exists w, x \quad w\ \mathbf{f}\ u \wedge w\ \mathbf{s}\ v \wedge x\ \mathbf{s}\ u \wedge x\ \mathbf{f}\ v$,
- $u\ \mathbf{moi}\ v \overset{def}{\equiv} \exists w, x, y \quad w\ \mathbf{m}\ x\ \mathbf{m}\ y\ \mathbf{m}\ w \wedge y\ \mathbf{ppi}\ u \wedge x\ \mathbf{ppi}\ v$,
- $u\ \mathbf{mio}\ v \overset{def}{\equiv} \exists w, x, y \quad w\ \mathbf{m}\ x\ \mathbf{m}\ y\ \mathbf{m}\ w \wedge x\ \mathbf{ppi}\ u \wedge y\ \mathbf{ppi}\ v$,
- $u\ \mathbf{eq}\ v \overset{def}{\equiv} \exists w, x \quad w\ \mathbf{m}\ u\ \mathbf{m}\ x \wedge w\ \mathbf{m}\ v\ \mathbf{m}\ x$.

The relations mi, di, fi, oi, si are the converse relations of m, d, f, o, s, respectively.

Axioms for cyclic interval structures: The CycInt theory

In this section, we give a set of axioms allowing to characterise the relation *meets* of cyclic intervals. Several axioms are motivated by intuitive properties owned by models of cyclic intervals. Other axioms are axioms of the relation *meets* of the intervals of the line (Ladkin 1987; Allen & Hayes 1985) adapted to the cyclic case.

In the sequel $u, v, w, \ldots$ will denote variables representing cyclic intervals. The symbol $|$ corresponds to the relation *meets*. The expression $v_1|v_2|\ldots|v_n$ with $v_1, v_2, \ldots, v_n$ n variables ($n > 2$) is an abbreviation for the conjunction $\bigwedge_{i=1}^{n-1} v_i|v_{i+1}$. Note that the expression $v_1|v_2|\ldots|v_n|v_1$ is equivalent to $v_2|\ldots|v_n|v_1|v_2$.

Another abbreviation used in the sequel is $\mathsf{X}(u, v, w, x)$. It is defined by the expression $u|v \wedge w|x \wedge (u|x \vee w|v)$. Intuitively, the satisfaction of $\mathsf{X}(u, v, w, x)$ expresses the fact that the cyclic interval u meets (is in relation *meets* with) the cyclic interval v, the cyclic interval w meets (is in relation *meets* with) the cyclic interval x and the two meeting points are the same points. In Figure 4 are represented the three possible cases for which $\mathsf{X}(u, v, w, x)$ is satisfied by cyclic intervals onto an oriented circle :

(a) $u|v$, $w|x$, $u|x$, $w|v$ are satisfied,

(b) $u|v$, $w|x$, $w|v$ are satisfied and $u|x$ is not satisfied,

(c) $u|v$, $w|x$, $u|x$ are satisfied and $w|v$ is not satisfied.

Now, it is possible for us to give the $\mathcal{C}yc\mathcal{I}nt$ axioms defined to axiomatize the relation *meets* of the cyclic interval models. After each axiom is given an intuitive idea of what

[3] Here we use the notation $v_1\ m\ v_2\ m\ \ldots\ m\ v_n$ where $v_1, v_2, \ldots, v_n$ are n variables ($n > 2$) as a shorthand for the conjunction $\bigwedge_{i=1}^{n-1} v_i\ m\ v_{i+1}$.

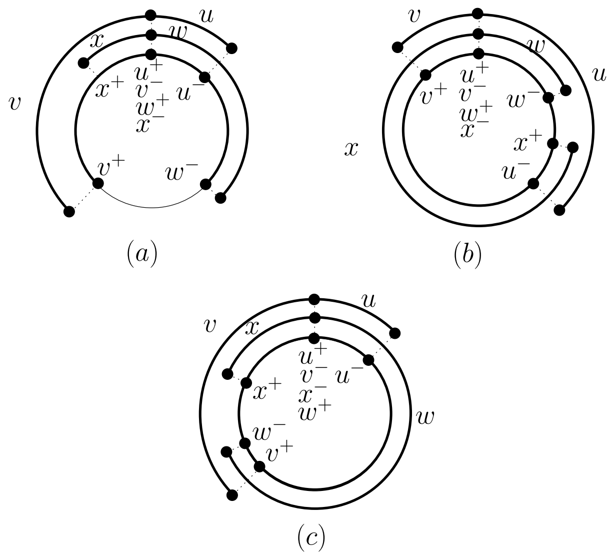

Figure 4: Satisfaction of $\mathsf{X}(u, v, w, x)$.

it expresses.

Definition 2 (The $\mathcal{C}yc\mathcal{I}nt$ axioms)

A1. $\forall u, v, w, x, y, z \quad \mathsf{X}(u, v, w, x) \wedge \mathsf{X}(y, z, w, x) \rightarrow \mathsf{X}(u, v, y, z)$

Given three pairs of meeting cyclic intervals, if the meeting point defined by the first pair is the same as the one defined by the second pair and the meeting point defined by the second pair is the same as the one defined by the third pair then, the first pair and the second pair of meeting cyclic intervals define the same meeting point.

A2. $\forall u, v, w, x, y, z \quad \mathsf{X}(u, v, w, x) \wedge \mathsf{X}(y, u, x, z) \rightarrow \neg u|x \wedge \neg x|u$

Two cyclic intervals with the same endpoints do not satisfy the relation meets.

A3. $\forall u, v, w, x, y, z \quad u|v \wedge w|x \wedge y|z \wedge \neg u|x \wedge \neg w|v \wedge \neg u|z \wedge \neg y|v \wedge \neg w|z \wedge \neg y|x \rightarrow \exists r, s, t \quad r|s|t|r \wedge \mathsf{X}(u, v, r, s) \wedge (\mathsf{X}(w, x, s, t) \wedge \mathsf{X}(y, z, t, r)) \vee (\mathsf{X}(w, x, t, r) \wedge \mathsf{X}(y, z, s, t))$

Three distinct meeting points can be defined by three cyclic intervals satisfying the relation meets so that these three meeting cyclic intervals cover the circle in its entirety.

A4. $\forall u, v, w, x, \quad u|v \wedge w|x \wedge \neg u|x \wedge \neg w|v \rightarrow (\exists y, z, t, \quad y|z|t|y \wedge \mathsf{X}(y, z, w, x) \wedge \mathsf{X}(t, y, u, v)) \wedge (\exists y, z, t, \quad y|z|t|y \wedge \mathsf{X}(y, z, u, v) \wedge \mathsf{X}(t, y, w, x))$

Two meeting points are the endpoints of two cyclic intervals. Each one can be defined by two other cyclic intervals.

A5. $\forall u, v \quad (\exists w, x \quad u|w|x|v|u) \rightarrow (\exists y \quad u|y|v|u)$

Two meeting cyclic intervals define another cyclic interval corresponding to the union of these cyclic intervals.

A6. $\exists u \quad u = u$ *and* $\forall u \exists v, w \quad u|v|w|u$

There exists a cyclic interval and for every cyclic intervals there exist two other cyclic intervals such that they satisfy the relation meets in a cyclic manner (they satisfy the relation meets so that they cover the circle in its entirety).

A7. $\forall u, v \quad (\exists w, x \quad w|u|x \wedge w|v|x) \leftrightarrow u = v$

There does not exist two distinct cyclic intervals with the same endpoints.

A8. $\forall u, v, w \qquad u|v|w \rightarrow \neg u|w$

Two cyclic intervals separated by a third one cannot satisfy the relation meets.

From these axioms we can deduce several theorems which will be used in the sequel.

Proposition 1 *Every structure $(\mathcal{I}, |)$ satisfying the $\mathcal{C}yc\mathcal{I}nt$ axioms satisfies the following formulas:*

B1. $\forall u, v \quad u|v \rightarrow \neg v|u$

B2. $\forall u, v, w, x, y, z \quad \mathsf{X}(u, v, w, x) \wedge \mathsf{X}(y, u, x, z) \rightarrow w|v \wedge y|z$

B3. $\forall u, v \quad (\exists w \quad u|w|v|u) \rightarrow (\exists x, y \quad u|x|y|v|u)$

Proof

- $(B1)$ Let u, v be two cyclic intervals satisfying $u|v$. Suppose that $v|u$ is satisfied. It follows that $\mathsf{X}(u, v, u, v)$ and $\mathsf{X}(v, u, v, u)$ are satisfied. From Axiom **A2** follows that $u|v$ and $v|u$ cannot be satisfied. There is a contradiction.
- $(B2)$ Let u,v,w,x,y,z be cyclic intervals satisfying $\mathsf{X}(u, v, w, x)$ and $\mathsf{X}(y, u, x, z)$. From Axiom **A2** we can deduce that $u|x$ and $x|u$ are not satisfied. As $\mathsf{X}(u, v, w, x)$ and $\mathsf{X}(y, u, x, z)$ are satisfied, we can assert that $y|z$ and $w|v$ are satisfied.
- $(B3)$ Let u, v, w be cyclic intervals satisfying $u|w|v|u$. We have $u|w$, $w|v$ and $v|u$ which are satisfied. Moreover, since $v|u$ is satisfied, from **B1** we can deduce that $u|v$ and $w|w$ cannot be satisfied. From Axiom **A4** follows that there exists cyclic intervals x, y, z satisfying $x|y|z|x$, $\mathsf{X}(x, y, u, w)$ and $\mathsf{X}(z, x, w, v)$. From Axiom **A2** we can assert that $x|w$ and $w|x$ are not satisfied. From it and the satisfaction of $\mathsf{X}(x, y, u, w) \wedge \mathsf{X}(z, x, w, v)$, we can assert that $u|y$ and $z|v$ are satisfied. We can conclude that u, v, y, z satisfy $u|y|z|v|u$.

$\square$

From cyclic interval structures back to cyclic orderings

In this section, we show how to define a cyclic ordering $\prec$ onto a set of points from a set of cyclic intervals and a relation *meets* onto these cyclic intervals satisfying the $\mathcal{C}yc\mathcal{I}nt$ axioms. The line of reasoning used is similar to the one used by Ladkin (Ladkin 1987) in the linear case. Indeed, intuitively, a set of pairs of meeting cyclic intervals satisfying the relation *meets* at a same place will represent a cyclic point. Hence, a cyclic point will correspond to a meeting place. Three cyclic points l, m, n defined in this way will be in relation $\prec$ if, and only if, there exist three cyclic intervals satisfying the relation *meets* in a cyclic manner (so that they cover the circle in its entirety) so that their meeting points are successively l, m and n. Now, let us give more formally the definition of this cyclic ordering.

Let $(\mathcal{I}, |)$ be a pair defined by a set $\mathcal{I}$ and a binary relation $|$ onto $\mathcal{I}$ satisfying the $\mathcal{C}yc\mathcal{I}nt$ axioms. Let $\mathcal{J}$ be the subset of $\mathcal{I} \times \mathcal{I}$ defined by $\mathcal{J} = \{(u, v) \in \mathcal{I} \times \mathcal{I} : u|v\}$.

Definition 3 *Let $\doteq$ be the binary relation onto $\mathcal{J}$ defined by $(u, v) \doteq (w, x)$ iff $u|x$ or $w|v$.*

Note that since $u|v$ and $w|x$ are satisfied, we have $(u, v) \doteq (w, x)$ iff $\mathsf{X}(u, v, w, x)$ for all $u, v, w, x \in \mathcal{I}$.

Proposition 2 *$\doteq$ is a relation of equivalence.*

Proof From the definition of the relation $\doteq$ we can easily establish the properties of reflexivity and symmetry. Axiom **A1** allows us to assert that the relation $\doteq$ is a transitive relation. □

Given an element $(u, v) \in \mathcal{J}$, $\overline{uv}$ will denote the equivalence class corresponding to (u, v) with respect to the relation $\doteq$. Let $\mathcal{P}$ be the set of all equivalence classes of $\doteq$. We define the ternary relation $\prec$ onto $\mathcal{P}$ in the following way.

Definition 4 *Let $\overline{uv}$, $\overline{wx}$, $\overline{yz} \in \mathcal{P}$. $\prec(\overline{uv}, \overline{wx}, \overline{yz})$ iff $\exists r, s, t \in \mathcal{I}$ with $r|s|t|r$, $\overline{rs} = \overline{uv}$, $\overline{st} = \overline{wx}$ and $\overline{tr} = \overline{yz}$.*

See Figure 5 for an illustration of this definition. The

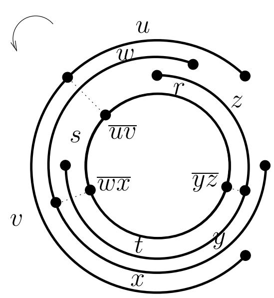

Figure 5: Satisfaction of $\prec(\overline{uv}, \overline{wx}, \overline{yz})$.

structure $(\mathcal{P}, \prec)$ obtained from $(\mathcal{I}, |)$ will be denoted by $\mathsf{CycPoint}((\mathcal{I}, |))$ in the sequel.

Theorem 1 *The structure $(\mathcal{P}, \prec)$ is a cyclic ordering.*

Proof We give the proof for Axioms $P1$ and $P2$ only. The proof for the other axioms is in the annex.

- $\forall \overline{uv}, \overline{wx} \in \mathcal{P}, \quad \neg \prec(\overline{uv}, \overline{wx}, \overline{wx}) \quad (P1)$

Let $\overline{uv}, \overline{wx} \in \mathcal{P}$. Suppose that $\prec(\overline{uv}, \overline{wx}, \overline{wx})$ is satisfied. From the definition of $\prec$, there exist $y, z, t \in \mathcal{I}$ satisfying $y|z|t|y$ and such that $(y, z) \doteq (u, v)$, $(z, t) \doteq (w, x)$, $(t, y) \doteq (w, x)$. $\doteq$ owns the properties of transitivity and symmetry, in consequence, we can assert that $(z, t) \doteq (t, y)$. From it and from the definition of $\doteq$, we have $z|y$ or $t|t$ which are satisfied. As $|$ is an irreflexive relation, we can assert that $z|y$ is satisfied. Moreover, $y|z$ is also satisfied. There is a contradiction since the relation $|$ is an asymmetric relation.

- $\forall \overline{uv}, \overline{wx}, \overline{yz}, \overline{st} \in \mathcal{P}, \prec(\overline{uv}, \overline{wx}, \overline{yz}) \wedge \prec(\overline{uv}, \overline{yz}, \overline{st}) \rightarrow \prec(\overline{uv}, \overline{wx}, \overline{st}) \quad (P2)$

Let $\overline{uv}, \overline{wx}, \overline{yz}, \overline{st} \in \mathcal{P}$ which satisfy $\prec(\overline{uv}, \overline{wx}, \overline{yz})$ and $\prec(\overline{uv}, \overline{yz}, \overline{st})$. From the definition of $\prec$ we can deduce that there exist $m, n, o \in \mathcal{I}$ satisfying $m|n|o|m$, $\overline{mn} = \overline{uv}$, $\overline{no} = \overline{wx}$, $\overline{om} = \overline{yz}$. On the other hand, we can assert that there exist $p, q, r \in \mathcal{I}$ satisfying $p|q|r|p$, $\overline{pq} = \overline{uv}$, $\overline{qr} = \overline{yz}$ and $\overline{rp} = \overline{st}$. From the property of transitivity of the relation $\doteq$ and the equalities $\overline{mn} = \overline{uv}$, $\overline{pq} = \overline{uv}$, $\overline{om} = \overline{yz}$, $\overline{qr} = \overline{yz}$, we obtain the equalities $\overline{mn} = \overline{pq}$ and $\overline{om} = \overline{qr}$. Hence, from the definition of $\doteq$, we can assert that $\mathsf{X}(m, n, p, q)$ and $\mathsf{X}(o, m, q, r)$ are satisfied. From Theorem **B2**, it follows that $p|n$ and $o|r$ are also satisfied. From all this, we can deduce that $n|o|r|p|n$ is satisfied. From Axiom **A5**, we can assert that there exists l satisfying $n|l|p|n$. By rotation, we deduce that $p|n|l|p$ is satisfied. $n|l$ and $n|o$ are satisfied, in consequence, we have $\overline{nl} = \overline{no}$. From this equality, the transitivity of the relation $\doteq$ and the equality $\overline{no} = \overline{wx}$, we can assert that $\overline{nl} = \overline{wx}$. As $l|p$ and $r|p$ are satisfied, we have the equality $\overline{lp} = \overline{rp}$. From this equality, the transitivity of the relation $\doteq$ and the equality $\overline{rp} = \overline{st}$, we can deduce that $\overline{lp} = \overline{st}$. Consequently, $p|n|l|p$, $\overline{pn} = \overline{uv}$, $\overline{nz} = \overline{wx}$ and $\overline{zp} = \overline{st}$ are satisfied. Hence, from the definition of $\prec$, we can conclude that $\prec(\overline{uv}, \overline{wx}, \overline{st})$ is satisfied. □

Cyclic orderings yield models of CycInt

In this section, we prove that every structure of cyclic intervals defined from a cyclic ordering is a model of $\mathcal{C}yc\mathcal{I}nt$.

Theorem 2 *Let $(\mathcal{P}, \prec)$ be a cyclic ordering. $(\mathcal{I}, |) = \mathsf{CycInt}((\mathcal{P}, \prec))$ is a model of the $\mathcal{C}yc\mathcal{I}nt$ axioms.*

Proof In the sequel, given an element $u = (m, n) \in \mathcal{I}$, u^- (resp. u^+) will correspond to m (resp. to n). Let us prove that the axioms of $\mathcal{C}yc\mathcal{I}nt$ are satisfied by $(\mathcal{I}, |)$.

- **(A1)** Let $u, v, w, x, y, z \in \mathcal{I}$ satisfying $\mathsf{X}(u, v, w, x)$ and $\mathsf{X}(y, z, w, x)$. From the definition of X we can assert that $u|v$ and $y|z$ are satisfied. Hence the equalities $u^+ = v^-$, $w^+ = x^-$ and $y^+ = z^-$. Moreover, from the definition of

X, it follows that $u|x$ or $w|v$ and $y|x$ or $w|z$ are satisfied. Let us consider all the possible situations exhaustively:

- $u|x$ and $y|x$ are satisfied. It follows that $u^+ = x^-$ and $y^+ = x^-$ are satisfied. Hence, we have $u^+ = v^- = w^+ = x^- = y^+ = z^-$.
- $u|x$ and $w|z$ are satisfied. It follows that $u^+ = x^-$ and $w^+ = z^-$ are satisfied. Consequently, $u^+ = v^- = w^+ = x^- = y^+ = z^-$ is satisfied.
- $w|v$ and $y|x$ are satisfied. It follows that $w^+ = v^-$ and $y^+ = x^-$ are satisfied. Therefore, $u^+ = v^- = w^+ = x^- = y^+ = z^-$ is satisfied.
- $w|v$ and $w|z$ are satisfied. It follows that $w^+ = v^-$ and $w^+ = z^-$ are satisfied. Hence, $u^+ = v^- = w^+ = x^- = y^+ = z^-$ is satisfied.

Let us denote by l the identical points $u^+, v^-, w^+, x^-, y^+, z^-$. Suppose that $\mathsf{X}(u, v, y, z)$ is falsified. By using the fact that $u|v$ and $y|z$ are satisfied, we deduce that $u|z$ and $y|v$ are not satisfied. Since $u^+ = z^-$ and $y^+ = v^-$, $\prec (u^-, l, z^+)$ and $\prec (y^-, l, v^+)$ are not satisfied. From $\mathbf{P5}$, we get the satisfaction of $\prec (u^-, z^+, l)$ and the one of $\prec (y^-, v^+, l)$. As $u|v$ and $y|z$ are satisfied, $\prec (u^-, l, v^+)$ and $\prec (y^-, l, z^+)$ are also satisfied. Hence, by using $\mathbf{P4}$, we can assert that $\prec (l, y^-, v^+)$ and $\prec (l, v^+, u^-)$ are satisfied. From $\mathbf{P2}$, it follows that $\prec (l, y^-, u^-)$ is also satisfied. From the satisfaction of $\prec (u^-, z^+, l)$ and the one of $\mathbf{P4}$, it follows that $\prec (l, u^-, z^+)$ is satisfied. By using $\mathbf{P2}$, it results that $\prec (l, y^-, z^+)$ is satisfied. Recall that $\prec (y^-, l, z^+)$ is satisfied. From $\mathbf{P4}$ and $\mathbf{P2}$, it results that $\prec (y^-, z^+, z^+)$ is satisfied. From $\mathbf{P1}$, a contradiction follows. Consequently, we can conclude that $\mathsf{X}(u, v, y, z)$ is satisfied.

- **(A2)** Let $u, v, w, x, y, z \in \mathcal{I}$ satisfy $\mathsf{X}(u, v, w, x)$ and $\mathsf{X}(y, u, x, z)$. The following equalities are satisfied: $u+ = x-$ and $x^+ = u^-$. By using $\mathbf{P4}$ and $\mathbf{P1}$, we can assert that $\prec (u^-, u^+, x^+)$ and $\prec (x^-, x^+, u^+)$ cannot be satisfied. Hence, $u|x$ and $x|u$ are not satisfied.
- **(A3)** Let us prove the satisfaction of Axiom **A3**. Let $u, v, w, x, y, z \in \mathcal{I}$ satisfying $u|v$, $w|x$, $y|z$, $\neg u|x$, $\neg w|v$, $\neg u|z$, $\neg y|v$, $\neg w|z$, $\neg y|x$. From the satisfaction of $u|v$ (resp. $w|x$ and $y|z$), it follows that $u^+ = v^-$ (resp. $w^+ = x^-$ and $y^+ = z^-$). Let l (resp. m and n) the point defined by $l = u^+ = v^-$ (resp. $m = w^+ = x^-$ and $n = y^+ = z^-$). Suppose that $l = m$. the equality $u^+ = v^- = w^+ = x^-$ is satisfied. Since $w|x$ is true, we can deduce that $\prec (w^-, x^-, x^+)$ is also satisfied. Consequently, w^- and x^+ are distinct points. Let us consider the three points u^-, w^-, x^+. From $\mathbf{P3}$, we can assert that only four cases are possible: $u^- = w^-$ is satisfied, $u^- = x^+$ is satisfied, $\prec (w^-, x^+, u^-)$ is satisfied, or $\prec (w^-, u^-, x^+)$ is satisfied. By using, $\mathbf{P2}$, $\mathbf{P3}$ and $\mathbf{P4}$, we obtain for every case a contradiction:
  - $u^- = w^-$ is satisfied. As $w|x$ is satisified, $\prec (u^-, x^-, x^+)$ is also satisfied. Recall that $u^+ = x^-$. It follows that $u|x$ is satisfied. There is a contradiction.
  - $u^- = x^+$ is satisfied. As $u|v$ and $w|x$ are satisfied, we can assert that $\prec (u^+, v^-, v^+)$ and $\prec (w^-, x^-, x^+)$ are satisfied. Hence, $\prec (x^+, v^-, v^+)$ and $\prec (w^-, v^-, x^+)$ are also satisfied. By using $\mathbf{P4}$, we can deduce that $\prec (v^-, v^+, x^+)$ and $\prec (v^-, x^+, w^-)$ are satisfied. From $\mathbf{P2}$ it follows that $\prec (v^-, v^+, w^-)$ is also satisfied. From $\mathbf{P4}$ follows the satisfaction of $\prec (w^-, v^-, v^+)$. Moreover, we have the equality $w^+ = v^-$. Consequently, $w|v$ is satisfied. There is a contradiction.
  - $\prec (w^-, x^+, u^-)$ is satisfied. From $\mathbf{P4}$, we obtain the satisfaction of $\prec (x^+, u^-, w^-)$. As $w|x$ is satisfied, we deduce that $\prec (w^-, x^-, x^+)$ is satisfied. Hence, $\prec (x^+, w^-, x^-)$ is also satisfied ($\mathbf{P4}$). From $\mathbf{P2}$, we can assert that $\prec (x^+, u^-, x^-)$ is satisfied. From $\mathbf{P4}$, $\prec (u^-, x^-, x^+)$ is satisfied. As $x^- = u^+$ is satisfied, we can assert that $u|x$ is satisfied. There is a contradiction.
  - $\prec (w^-, u^-, x^+)$ is satisfied. Hence, u^- and x^+ are distinct points. Moreover, we know that u^+ and x^+ are distinct points from the fact that x^- and u^+ are equal. From $\mathbf{P3}$, $\prec (u^-, x^+, x^-)$ or $\prec (u^-, x^-, x^+)$ is satisfied. Suppose that $\prec (u^-, x^-, x^+)$ is satisfied. Since we have the equality $u^+ = x^-$, $u|x$ is satisfied. There is a contradiction. It results that $\prec (u^-, x^+, x^-)$ must be satisfied. From the satisfaction of $\prec (w^-, u^-, x^+)$ and $\mathbf{P4}$, we deduce that $\prec (u^-, x^+, w^-)$ is satisfied. From the satisfaction of $w|x$ and from $\mathbf{P4}$, we can assert that $\prec (x^-, x^+, w^-)$ is satisfied. $\prec (u^-, x^+, x^-)$ is satisfied, hence, from $\mathbf{P4}$ we can deduce that $\prec (u^-, x^+, x^-)$ is satisfied. From $\mathbf{P4}$, we obtain the satisfaction of $\prec (x^-, u^-, x^+)$. From $\mathbf{P2}$, it results that $\prec (x^-, u^-, w^-)$ is satisfied. Hence, $\prec (u^+, u^-, w^-)$ is satisfied. From the satisfaction of $u|v$ and from $\mathbf{P4}$ it follows that $\prec (u^+, v^+, u^-)$ is satisfied. From $\mathbf{P2}$ we can assert that $\prec (u^+, v^+, w^-)$ is satisfied. In consequence, $\prec (w^+, v^+, w^-)$ is satisfied. Hence, from $\mathbf{P4}$, $\prec (w^-, w^+, v^+)$ is satisfied. It results that $w|v$ is satisfied. There is a contradiction.

Consequently, we can assert that $l \neq m$. In a similar way, we can prove that $l \neq n$ and $m \neq n$. Now, we know that l, m, n are distinct points. From $\mathbf{P3}$, we can just examine two cases:

- $\prec (l, m, n)$ is satisfied. Let $r = (n, l)$, $s = (l, m)$ and $t = (m, n)$. We have $r|s|t|r$ which is satisfied. Suppose that $u|s$ is falsified. It follows that $\prec (u^-, l, m)$ is also falsified. As l is different from u^- and m, we have $u^- = m$ or $\prec (u^-, m, l)$ which is satisfied.
  - ∗ Suppose that $u^- = m$ is satisfied. Since $u|v$ is satisfied, it follows that $\prec (u^-, u^+, v^+)$ is satisfied. Consequently, $\prec (m, l, v^+)$ is true. From $\mathbf{P4}$, it follows that $\prec (l, v^+, m)$ is satisfied. From all this, the satisfaction of $\prec (l, m, n)$ and $\mathbf{P2}$, we can assert that $\prec (l, v^+, n)$ is satisfied. From $\mathbf{P4}$, we deduce that $\prec (n, l, v^+)$ is satisfied. As $l = v^-$, $r|v$ is satisfied.
  - ∗ Suppose that $\prec (u^-, m, l)$ is satisfied. From $\mathbf{P4}$, it follows that $\prec (l, u^-, m)$ is satisfied. From all this, the satisfaction of $\prec (l, m, n)$ and $\mathbf{P2}$, we can assert that $\prec (l, u^-, n)$ is satisfied. As $u|v$ is satisfied, we can deduce that $\prec (u^-, u^+, v^+)$ is satisfied. Consequently, $\prec (u^-, l, v^+)$ is also satisfied. From $\mathbf{P4}$, it results that $\prec (l, v^+, u^-)$ is satisfied. From all this

and the satisfaction of $\prec (l, u^-, n)$, we can deduce that $\prec (l, v^+, n)$ is satisfied. By using $\mathbf{P4}$, we obtain the satisfaction of $\prec (n, l, v^+)$. As $l = v^-$, we deduce that $r|v$ is satisfied.

It results that $u|s$ or $r|v$ is satisfied. Hence, $\mathsf{X}(u, v, r, s)$ is satisfied. With a similar line of reasoning, we can prove that $\mathsf{X}(w, x, s, t)$ and $\mathsf{X}(y, z, t, r)$ are satisfied.

– $\prec (l, n, m)$ is satisfied. Let $r = (m, l)$, $s = (l, n)$ and $t = (n, m)$. We have $r|s|t|r$ which is satisfied. In a similar way, we can prove that $\mathsf{X}(u, v, r, s)$, $\mathsf{X}(y, z, s, t)$ and $\mathsf{X}(w, x, t, r)$ are satisfied.

- For Axioms **A4-A5-A6-A7-A8**, the proofs can be found in the annex.

□

Categoricity of $\mathcal{CycInt}$

In this section, we establish the fact that the countable models satisfying the $\mathcal{CycInt}$ axioms are isomorphic. In order to prove this property, let us show that for every cyclic interval there exist two unique "endpoints".

Proposition 3 *Let $\mathcal{M} = (\mathcal{I}, |)$ a model of $\mathcal{CycInt}$. Let $(\mathcal{P}, \prec)$ be the structure* $\mathsf{CycPoint}(\mathcal{M})$. *For every $u \in \mathcal{I}$ there exist $L_u, U_u \in \mathcal{P}$ such that :*

1. *$\exists v \in \mathcal{I}$ such that $(v, u) \in L_u$,*

2. *$\exists w \in \mathcal{I}$ such that $(u, w) \in U_u$,*

3. *L_u (resp. U_u) is the unique element of $\mathcal{P}$ satisfying (1.) (resp. (2.)),*

4. *$L_u \neq U_u$.*

Proof From Axiom **A6**, we can assert that there exist $v, w \in \mathcal{I}$ such that $u|w|v|u$ is satisfied. Consequently, $u|w$ and $v|u$ are satisfied. By defining L_u by $L_u = \overline{vu}$ and U_u by $U_u = \overline{uw}$, the properties (1) and (2) are satisfied. Now, let us prove that the property (3) is satisfied. Suppose that there exists L'_u such that there exists $x \in \mathcal{I}$ with $(x, u) \in L'_u$. We have $(v, u) \equiv (x, u)$. It follows that $L_u = L'_u$. Now, suppose that there exists U'_u such that there exists $y \in \mathcal{I}$ with $(u, y) \in U'_u$. We have $(u, w) \equiv (u, y)$. It follows that $U_u = U'_u$. Hence, we can assert that property (3) is true. Now, suppose that $L_u = U_u$. It follows that $(v, u) \equiv (u, w)$. As a result, $v|w$ or $u|u$ is satisfied. We know that $|$ is an irreflexive relation. Moreover, from Axiom **A8** we can assert that $v|w$ cannot be satisfied. It results that there is a contradiction. Hence, L_u and U_u are distinct elements. □

From an initial model of $\mathcal{CycInt}$, we have seen that we can define a cyclic ordering. Moreover, from this cyclic ordering we can generate a cyclic interval model. We are going to show that this generated cyclic interval model is isomorphic to the initial cyclic interval model.

Proposition 4 *Let $\mathcal{M} = (\mathcal{I}, |)$ a model of the $\mathcal{CycInt}$ axioms. $\mathcal{M}$ is isomorphic to $(\mathcal{I}', |') =$* $\mathsf{CycInt}(\mathsf{CycPoint}(\mathcal{M}))$.

Proof Let f be the mapping from $\mathcal{I}$ onto $\mathcal{I}'$ defined by $f(u) = (L_u, U_u)$, *i.e.* $f(u) = (\overline{vu}, \overline{uw})$ for any $v, w \in \mathcal{I}$ satisfying $v|u$ and $u|w$. Let us show that f is a one-to-one mapping. Let $(\overline{uv}, \overline{wx}) \in \mathcal{I}'$. We have $u|v$ and $w|x$ which are satisfied and $u|x$ and $w|v$ which are falsified (in the contrary case we would have $\overline{uv} = \overline{wx}$). From $\mathbf{A4}$, it follows that there exist y, z, t satisfying $y|z|t|y$, $\mathsf{X}(y, z, w, x)$ and $\mathsf{X}(t, y, u, v)$. Note that $L_y = \overline{ty} = \overline{uv}$ and $U_y = \overline{yz} = \overline{wx}$. Consequently, there exists $y \in \mathcal{I}$ such that $f(y) = (\overline{uv}, \overline{wx})$. Now, suppose that there exist $u, v \in \mathcal{I}$ such that $f(u) = f(v)$. Suppose that $f(u) = (\overline{wu}, \overline{ux})$ and $f(v) = (\overline{yv}, \overline{vz})$. We have $\overline{wu} = \overline{yv}$ and $\overline{ux} = \overline{vz}$. It follows that $(w, u) \mathrel{\hat{=}} (y, v)$ and $(u, x) \mathrel{\hat{=}} (v, z)$. From all this, we have $w|u$, $y|v$, $u|x$ and $v|z$ which are satisfied. Four possible situations must be considered:

- $w|v$ and $u|z$ are satisfied. It follows that $w|v|z$ and $w|u|z$ are satisfied.
- $w|v$ and $v|x$ are satisfied. It follows that $w|v|x$ and $w|u|x$ are satisfied.
- $y|u$ and $u|z$ are satisfied. It follows that $y|v|z$ and $y|u|z$ are satisfied.
- $y|u$ and $v|x$ are satisfied. It follows that $y|v|z$ and $y|u|z$ are satisfied.

For each case, by using **A7**, we can deduce the equality $u = v$. Consequently, f is a one-to-one mapping.
Now, let us show that $u|v$ if, and only if, $f(u)|'f(v)$. We will denote $f(u)$ by $(\overline{wu}, \overline{ux})$ and $f(v)$ by $(\overline{yv}, \overline{vz})$. Suppose that $u|v$ is satisfied. It follows that $(u, x) \mathrel{\hat{=}} (y, v)$, hence, $\overline{ux} = \overline{yv}$. For this reason, $f(u)|'f(v)$ is satisfied. Now, suppose that $f(u)|'f(v)$ is satisfied. It follows that $\prec (\overline{wu}, \overline{ux}, \overline{vz})$ and $\overline{ux} = \overline{yv}$ are satisfied. Hence, there exist $r, s, t \in \mathcal{I}$ such that $r|s|t|r$, $\overline{rs} = \overline{wu}$, $\overline{st} = \overline{ux}$ and $\overline{tr} = \overline{vz}$ are satisfied. From the equalities $\overline{rs} = \overline{wu}$ and $\overline{st} = \overline{ux}$, we can assert that $u|x$, $s|t$, $r|s$ and $w|u$ are satisfied. Moreover, one of the following cases is satisfied:

- $r|u$ and $u|t$ are satisfied. It follows that $r|u|t$ and $r|s|t$ are satisfied.
- $r|u$ and $s|x$ are satisfied. It follows that $r|s|x$ and $r|u|x$ are satisfied.
- $w|s$ and $u|t$ are satisfied. It follows that $w|u|t$ and $w|s|t$ are satisfied.
- $w|s$ and $s|x$ are satisfied. It follows that $w|s|x$ and $w|u|x$ are satisfied.

For each case, from **A7**, we can deduce the equality $u = s$. From the equalities $\overline{st} = \overline{yv}$ and $\overline{tr} = \overline{vz}$, we can deduce that $s|t$, $y|v$, $t|r$ and $v|z$ are satisfied. Moreover, one of the following cases is satisfied:

- $s|v$ and $t|z$ are satisfied. It follows that $s|t|z$ and $s|v|z$ are satisfied.
- $s|v$ and $v|r$ are satisfied. It follows that $s|v|r$ and $s|t|r$ are satisfied.
- $y|t$ and $t|z$ are satisfied. It follows that $y|t|z$ and $y|v|z$ are satisfied.
- $y|t$ and $v|r$ are satisfied. It follows that $y|t|r$ and $y|v|r$ are satisfied.

For each case, from Axiom **A7**, we can deduce that $v = t$. Hence, we have the equalities $u = s$ and $v = t$. We can conclude that $u|v$ is satisfied. □

Now, let us show that two cyclic interval models generated by two countable cyclic orderings are isomorphic.

Proposition 5 *Let $(\mathcal{P}, \prec)$ and $(\mathcal{P}', \prec')$ be two cyclic orderings with $\mathcal{P}$ and $\mathcal{P}'$ two countable sets of points.* $\mathsf{CycInt}((\mathcal{P}, \prec))$ *and* $\mathsf{CycInt}((\mathcal{P}', \prec'))$ *are isomorphic.*

Proof Let $(\mathcal{I}, |)$ and $(\mathcal{I}', |')$ be defined by $\mathsf{CycInt}((\mathcal{P}, \prec))$ and $\mathsf{CycInt}((\mathcal{P}', \prec'))$. We know that $(\mathcal{P}, \prec)$ and $(\mathcal{P}', \prec')$ are isomorphic (Balbiani, Condotta, & Ligozat 2002). Let g be an isomorphism from $(\mathcal{P}, \prec)$ to $(\mathcal{P}', \prec')$. Let h be the mapping from $\mathcal{I}$ onto $\mathcal{I}'$ defined by $h((l, m)) = (g(l), g(m))$. First, let us show that $(g(l), g(m)) \in \mathcal{I}'$. As $(l, m) \in \mathcal{I}$, there exists $n \in \mathcal{P}$ satisfying $\prec (l, m, n)$. It follows that $\prec' (g(l), g(m), g(n))$ is satisfied. It results that $(g(l), g(m)) \in \mathcal{I}'$. Now, let us show that for every $(l, m) \in \mathcal{I}'$, there exists $(n, o) \in \mathcal{I}$ such that $h((n, o)) = (l, m)$. We can define n and o by $n = g^{-1}(l)$ and $o = g^{-1}(m)$. Indeed, $h(g^{-1}(l), g^{-1}(m)) = (g(g^{-1}(l)), g(g^{-1}(m))) = (l, m)$. Now, let $(l, m), (n, o) \in \mathcal{I}$ such that $h((l, m)) = h((n, o))$. It follows that $g(l) = g(n)$ and $g(m) = g(o)$. Therefore, we have $l = n$ and $m = o$. Hence, we obtain the equality $(l, m) = (n, o)$. Finally, let us show that for all $(l, m), (n, o) \in \mathcal{I}$, $(l, m)|(n, o)$ is satisfied iff $h((l, m))|'h((n, o))$ is satisfied. $(l, m)|(n, o)$ is satisfied iff $\prec (l, m, o)$ and $m = n$ are satisfied. Hence, $(l, m)|(n, o)$ is satisfied iff $\prec' (g(l), g(m), g(o))$ and $g(m) = g(n)$ are satisfied. For these reasons, we can assert that $(l, m)|(n, o)$ is satisfied iff $h((l, m))|'h((n, o))$ is satisfied. We can conclude that h is an isomorphism. □

In the sequel, $(\mathcal{Q}, \prec)$ will correspond to the cyclic ordering on the set of rational numbers $\mathcal{Q}$, defined by $\prec (x, y, z)$ iff $x < y < z$ or $y < z < x$ or $z < x < y$, with $x, y, z \in \mathcal{Q}$ and $<$ the usual linear order on $\mathcal{Q}$. It is time to

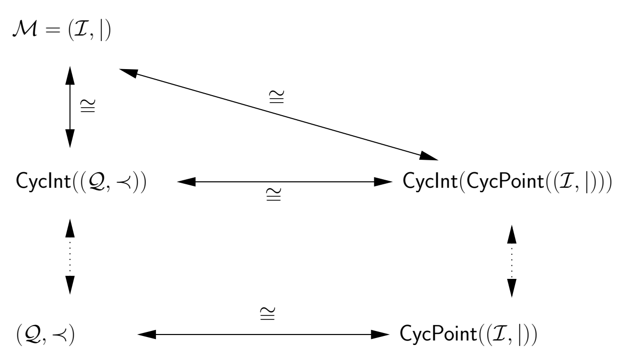

Figure 6: Every countable model of $\mathcal{CycInt}$ $(\mathcal{I}, |)$ is isomorphic to $\mathsf{CycInt}((\mathcal{Q}, \prec))$.

establish the main result of this section.

Theorem 3 *The theory axiomatized by $\mathcal{CycInt}$ is $\aleph_0$–categorical. Moreover, its countable models are isomorphic to* $\mathsf{CycInt}((\mathcal{Q}, \prec))$.

Proof Let $\mathcal{M}$ a model of $\mathcal{CycInt}$. $\mathcal{M}$ is isomorphic to $\mathsf{CycInt}(\mathsf{CycPoint}(\mathcal{M}))$. $\mathsf{CycInt}(\mathsf{CycPoint}(\mathcal{M}))$ is isomorphic to $\mathsf{CycInt}((\mathcal{Q}, \prec))$. By composing the isomorphisms, we have $\mathsf{CycInt}((\mathcal{Q}, \prec))$ which is isomorphic to $\mathcal{M}$. □

As a direct consequence of this theorem we have that the set of the theorems of $\mathcal{CycInt}$ is syntactically complete and decidable.

Application to constraint networks

Balbiani and Osmani (Balbiani & Osmani 2000) use constraint networks to represent the qualitative information about cyclic intervals. A network is defined as a pair (V, C), where V is a set of variables representing cyclic intervals and C is a map which, to each pair of variables (V_i, Vj) associates a subset C_{ij} of the set of all sixteen basic relations. The main problem in this context is the consistency problem, which consists in determining whether the network has a so-called solution: a solution is a map m from the set of variables V_i to the set of cyclic intervals in $\mathcal{C}$ such that all constraints are satisfied. The constraint C_{ij} is satisfied if and only if, denoting by m_i and m_j the images of V_i and V_j respectively, the cyclic interval m_i is in one of the relations in the set C_{ij} with respect to m_j (the set C_{ij} is consequently given a disjunctive interpretation in terms of constraints).

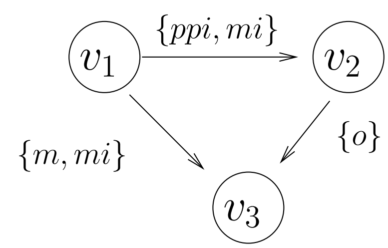

Figure 7: A constraint network on cyclic intervals.

A first interesting point is the fact that the axiomatization we have obtained allows us to check the consistency of a constraint network on cyclic intervals by using a theorem prover. Indeed, the procedure goes as follows: First, translate the network (V, C) into an *equivalent* logical formula Φ. Then, test the validity of the formula (or its validity in a specific model) by using the $\mathcal{CycInt}$ axiomatization.

As an example, consider the constraint network in Figure 7. The corresponding formula is $\Phi = (\exists v_1, v_2, v_3) \; ((v_1 \; ppi \; v_2 \vee v_1 \; mi \; v_2) \wedge (v_1 \; m \; v_3 \vee v_1 \; mi \; v_3) \wedge (v_2 \; o \; v_3))$.

In order to show that this network is consistent, we would have to prove that this formula is valid with respect to $\mathcal{CycInt}$, or satisfiable for a model such as $\mathcal{C}$. In order to show inconsistency, we have to consider the negation of Φ.

Usually a local constraint propagation method, called the path-consistency method, is used to solve this kind of constraint network. The method [4] consists in removing from

[4]In the case of cyclic interval networks, the path-consistency

each constraint C_{ij} all relations which are not compatible with the constraints in C_{ik} and C_{kj}, for all 3-tuples i, j, k. This is accomplished by using the composition table of the cyclic interval calculus which, for each pair (a, b) of basic relations, gives the composition of a with b, that is the set of all basic relations c such that there exists a configuration of three cyclic intervals u, v, w with $u\ a\ v$, $v\ b\ w$ and $u\ c\ w$. For instance, the composition of m with d consists in the relation ppi.The composition table of the cyclic interval calculus can be automatically computed by using our axiomatization. Indeed, in order to decide whether c belongs to the composition of a with b, it suffices to prove that the formula $(\exists u, v, w)\ \ (u\ a\ v \wedge v\ b\ w \wedge u\ c\ w)$ is valid. In order to prove that, conversely, c does not belong to this composition, one has to consider the negated formula $\neg(\exists u, v, w)\ \ (u\, a\, v \wedge v\, b\, w \wedge u\, c\, w)$.

Conclusions and further work

We have shown in in paper how the theory of cyclic orderings, on the one hand, and the theory of cyclic intervals, on the other hand, can be related. We proposed a set of axioms for cyclic intervals and showed that each countable model is isomorphic to the model based on cyclic intervals on the rational circle. Determining whether the first order theory of the *meets* relation between cyclic orderings admits the elimination of quantifiers is to our knowledge an open problem we are currently examining. Another question is whether the axioms of the $\mathcal{C}ycInt$ theory are independent. Still another interesting direction of research is the study of finite models of cyclic intervals. To this end, we will have to consider discrete cyclic orderings (which consequently do not satisfy axiom **P5**). This could lead to efficient methods for solving the consistency problem for cyclic interval networks: Since these involve only a finite number of variables, they should prove accessible to the use of finite models.

Annex

Proof (End of proof of Theorem 1)

- $\forall \overline{uv}, \overline{wx}, \overline{yz} \in \mathcal{P}, \overline{uv} \neq \overline{wx} \wedge \overline{wx} \neq \overline{yz} \wedge \overline{uv} \neq \overline{yz} \rightarrow\ \prec (\overline{uv}, \overline{wx}, \overline{yz}) \vee \prec (\overline{uv}, \overline{yz}, \overline{wx})\ \ (P3)$

  Let $\overline{uv}, \overline{wx}, \overline{yz} \in \mathcal{P}$ satisfying $\overline{uv} \neq \overline{wx}$, $\overline{wx} \neq \overline{yz}$ and $\overline{uv} \neq \overline{yz}$. From the definitions of $\mathcal{P}$ and $\stackrel{\circ}{=}$ we can assert that $u|v$, $w|x$, $y|z$, $\neg u|x$, $\neg w|v$, $\neg u|z$, $\neg y|v$, $\neg w|z$, $\neg y|x$ are satisfied. From Axiom **A3** we can deduce that there exist r, s, t satisfying $r|s|t|r$ and such that $\mathsf{X}(u, v, r, s)$, $\mathsf{X}(w, x, s, t)$, $\mathsf{X}(y, z, t, r)$ or $\mathsf{X}(u, v, r, s)$, $\mathsf{X}(w, x, t, r)$, $\mathsf{X}(y, z, s, t)$ are satisfied. From all this, we can conclude that $\prec (\overline{uv}, \overline{wx}, \overline{yz}) \vee \prec (\overline{uv}, \overline{yz}, \overline{wx})$ is satisfied.

- $\forall \overline{uv}, \overline{wx}, \overline{yz} \in \mathcal{P}, \prec (\overline{uv}, \overline{wx}, \overline{yz}) \leftrightarrow \prec (\overline{wx}, \overline{yz}, \overline{uv}) \leftrightarrow \prec (\overline{yz}, \overline{uv}, \overline{wx})\ \ (P4)$

  Let $\overline{uv}, \overline{wx}, \overline{yz} \in \mathcal{P}$ satisfying $\prec (\overline{uv}, \overline{wx}, \overline{yz})$. From the definition of $\prec$, we have $u|v$, $w|x$ and $y|z$ which are satisfied and there exist r, s, t satisfying $r|s|t|r$, $\overline{rs} = \overline{uv}$, $\overline{st} = \overline{wx}$ and $\overline{tr} = \overline{yz}$. By rotation, we can assert that $s|t|r|s$ is also satisfied. From this, we can deduce that $\prec (\overline{wx}, \overline{yz}, \overline{uv})$ is satisfied. In a similar way, we can prove that $\prec (\overline{wx}, \overline{yz}, \overline{uv}) \rightarrow \prec (\overline{yz}, \overline{uv}, \overline{wx})$ and $\prec (\overline{yz}, \overline{uv}, \overline{wx}) \rightarrow \prec (\overline{uv}, \overline{wx}, \overline{yz})$ are satisfied.

- $\forall \overline{uv}, \overline{wx} \in \mathcal{P}, \overline{uv} \neq \overline{wx} \rightarrow ((\exists \overline{yz} \in \mathcal{P}, \prec (\overline{uv}, \overline{wx}, \overline{yz})) \wedge (\exists \overline{rs} \in \mathcal{P}, \prec (\overline{uv}, \overline{rs}, \overline{wx})))\ \ (P5)$

  Let $\overline{uv}, \overline{wx} \in \mathcal{P}$ such that $\overline{uv} \neq \overline{wx}$. From the definition of $\mathcal{P}$ and the one of the relation $\stackrel{\circ}{=}$ we can assert that $u|v$, $w|x$, $\neg u|x$ and $\neg w|v$ are satisfied. From Axiom **A4** we deduce that there exist y, z, t such that $y|z|t|y \wedge \mathsf{X}(y, z, w, x) \wedge \mathsf{X}(t, y, u, v))$ is satisfied and that there exist q, r, s such that $q|r|s|q \wedge \mathsf{X}(q, r, u, v) \wedge \mathsf{X}(s, q, w, x))$ is satisfied. Consequently, there exists y, z, t such that $\prec (\overline{yz}, \overline{zt}, \overline{ty})$, $\overline{yz} = \overline{wx}$, $\overline{ty} = \overline{uv}$ are satisfied and there exist q, r, s such that $\prec (\overline{qr}, \overline{rs}, \overline{sq})$, $\overline{qr} = \overline{uv}$, $\overline{sq} = \overline{wx}$ are satisfied. Hence, there exists $\overline{zt} \in \mathcal{P}$ such that $\prec (\overline{wx}, \overline{zt}, \overline{uv})$ is satisfied , and there exists $\overline{rs} \in \mathcal{P}$ such that $\prec (\overline{uv}, \overline{rs}, \overline{wx})$ is satisfied. From **C3** we can conclude that there exists $\overline{zt} \in \mathcal{P}$ satisfying $\prec (\overline{uv}, \overline{wx}, \overline{zt})$, and that there exists $\overline{rs} \in \mathcal{P}$ satisfying $\prec (\overline{uv}, \overline{rs}, \overline{wx})$.

- $\exists \overline{uv}, \overline{wx} \in \mathcal{P},\ \ \overline{uv} \neq \overline{wx}.\ \ (P6)$

  From Axiom **A6** we can assert that there exist u, v, w satisfying $u|v|w|u$. Hence, there exist $\overline{uv}, \overline{vw}, \overline{wu} \in \mathcal{P}$ such that $\prec (\overline{uv}, \overline{vw}, \overline{wu})$ is satisfied. From **P1** we deduce that $\overline{uv}$ and $\overline{vw}$ are distinct classes.

□

Proof (End of proof of Theorem 2)

- **(A4)** Let $u, v, w, x \in \mathcal{I}$ satisfying $u|v$, $w|x$, $\neg u|x$, and $\neg w|v$. $\prec (u^-, u^+, v^+)$, $\prec (w^-, w^+, x^+)$ with $u^+ = v^-$ and $w^+ = x^-$ are satisified. Let l and m defined by $l = u^+ = v^-$ and $m = w^+ = x^-$. Suppose that $l = m$. As $\prec (u^-, u^+, v^+)$ and $\prec (w^-, w^+, x^+)$ are satisfied, we have $\prec (u^-, l, v^+)$ and $\prec (w^-, l, x^+)$ which are also satisfied. Hence, we have $u^- \neq l$ and $x^+ \neq l$. From **P3**, we can just consider three cases: $u^- = x^+$ is satisfied, $\prec (u^-, l, x^+)$ is satisfied, or $\prec (u^-, x^+, l)$ is satisfied. From **P2** and **P4**, we can deduce a contradiction for every case. We can assert that $l \neq m$. From **P5**, we can deduce there exist $n, o \in \mathcal{P}$ satisfying $\prec (l, m, n)$ and $\prec (l, o, n)$. Let us define three cyclic intervals y, z, t by $y = (l, m)$, $z = (m, n)$ and $t = (n, l)$. From the satisfaction of $\prec (l, m, n)$ and **P4**, we can deduce that $y|z|t|y$ is satisfied. Let us suppose that $y|x$ is not satisfied. As $y^+ = x^-$, it follows that $\prec (y^-, y^+, x^+)$ is not satisfied. We have $y^- \neq y^+$ and $y^+ \neq x^+$. From **P3**, it follows that $y^- = x^+$ or $\prec (y^-, x^+, y^+)$ is satisfied. Let us examine these two possible cases.

method is not complete even for atomic networks: path-consistency does not insure consistency.

– $y^- = x^+$ is satisfied. It follows that $x^+ = l = u^+ = v^-$. From the satisfaction of $w|x$, we have $\prec (w^-, w^+, x^+)$ which is satisfied, with $w^+ = x^-$. Since $\prec (l, m, n)$ is satisfied, $\prec (x^+, w^+, n)$ is also satisfied. From $\mathbf{P4}$, we can deduce that $\prec (w^+, x^+, w^-)$ and $\prec (w^+, n, x^+)$ are satisfied. From $\mathbf{P2}$ follows that $\prec (w^+, n, w^-)$ is satisfied. Hence, from $\mathbf{P4}$, we obtain the satisfaction of $\prec (w^-, w^+, n)$. As $w^+ = m$, $w|z$ is satisfied.

– $\prec (y^-, x^+, y^+)$ is satisfied. Hence, $\prec (l, x^+, w^+)$ is satisfied. As $\prec (l, m, n)$ is satisfied, $\prec (l, w^+, n)$ is also satisfied. From $\mathbf{P4}$, it follows that $\prec (w^+, n, l)$ and $\prec (w^+, l, x^+)$ are satisfied. From $\mathbf{P2}$, we can deduce that $\prec (w^+, n, x^+)$ is satisfied. As $w|x$ is satisfied, $\prec (w^-, w^+, x^+)$ is satisfied, with $w^+ = x^-$. From $\mathbf{P4}$, we have $\prec (w^+, x^+, w^-)$ which is satisfied. From $\mathbf{P2}$, we deduce that $\prec (w^+, n, w^-)$ is satisfied. From $\mathbf{P4}$, it follows that $\prec (w^-, w^+, n)$ is satisfied. We have $w^+ = m$. It results that $w|z$ is satisfied.

Hence, $\mathsf{X}(y, z, w, x)$ is satisfied. In a similar way, we can prove that $\mathsf{X}(t, y, u, v)$ is satisfied. By defining y, z, t by $y = (m, l)$, $z = (l, o)$ and $t = (o, m)$, we can also prove that $\mathsf{X}(y, z, u, v)$ and $\mathsf{X}(t, y, w, x)$ are satisfied.

- **(A5)** Let $u, v, w, x \in \mathcal{I}$ satisfying $u|w|x|v|u$. We have the following equalities: $u^+ = w^-$, $w^+ = x^-$, $x^+ = v^-$ and $v^+ = u^-$. Let us define l_1 (resp. l_2, l_3 and l_4) by $l_1 = u^+ = w^-$ (resp. $l_2 = w^+ = x^-$, $l_3 = x^+ = v^-$ and $l_4 = v^+ = u^-$). Consider the pair $y = (l_1, l_3)$. As $w|x$ is satisfied, we can deduce the satisfaction of $\prec (l_1, l_2, l_3)$. Hence, we can assert that $l_1 \neq l_3$. From $\mathbf{P5}$, it follows that there exists l satisfying $\prec (l_1, l_3, l)$. It results that $y = (l_1, l_3)$ belongs to $\mathcal{I}$.
  Suppose that $u|y$ is not satisfied. Since $u^+ = l_1$, $\prec (u^-, l_1, l_3)$ is not satisfied. u^- and l_1 are distinct points and, l_1 and l_3 are also distinct points. From the satisfaction of $v|u$, we can deduce that $\prec (l_3, u^-, u^+)$ is satisfied. It follows that $l_3 \neq u^-$. Consequently, Axiom $\mathbf{P3}$ and the non satisfaction of $u|y$ allow us to assert that $\prec (u^-, l_3, l_1)$ is satisfied. As $v|u$ is satisfied, $\prec (l_3, u^-, l_1)$ is also satisfied. From $\mathbf{P4}$ and from $\mathbf{P2}$, it follows that $\prec (l_3, u^-, u^-)$ is satisfied. From Axiom $\mathbf{P1}$, it results a contradiction. In consequence, $u|y$ is satisfied. With a similar line of reasoning, by supposing that $y|v$ is not satisfied, we obtain a contradiction. Hence, $u|y|v|u$ is satisfied.

- **(A6)** From $\mathbf{P6}$, we can deduce that there exist $l, m \in \mathcal{P}$ such that $l \neq m$. From $\mathbf{P5}$, it follows that there exists n satisfying $\prec (l, m, n)$. Let $u = (l, m)$, we have $u \in \mathcal{I}$ and $u = u$. Now, let us prove the second part of the axiom. Let $u = (l, m) \in \mathcal{I}$. By definition of $\mathcal{I}$, there exists $n \in \mathcal{P}$ such that $\prec (l, m, n)$. Let $v = (m, n)$ and $w = (n, l)$. From **P4**, $\prec (m, n, l)$ and $\prec (n, l, m)$ are satisfied. From all this, we deduce that $u|v$, $v|w$ and $w|u$ are satisfied.

- **(A7)** Let $u, v, w, x \in \mathcal{I}$ satisfying $w|u|x$ and $w|v|x$. The following equalities are satisfied: $w^+ = u^-$, $u^+ = x^-$, $w^+ = v^-$, $v^+ = x^-$. It follows that $(u^-, u^+) = (v^-, v^+)$. Consequently, we can assert that $u = v$. Let $u, v \in \mathcal{I}$ such that $u = v$. We know that $u^- \neq u^+$. From $\mathbf{P5}$, it follows that there exists $l \in \mathcal{P}$ satisfying $\prec (u^-, u^+, l)$. Let $w = (l, u^-)$ and $x = (u^+, l)$. From $\mathbf{P4}$, we deduce that $\prec (l, u^-, u^+)$ is satisfied. From all this, we can assert that $w, x \in \mathcal{I}$ and that $w|u$ and $u|x$ are satisfied. Since $(u^-, u^+) = (v^-, v^+)$, we can assert that $w|v|x$ is satisfied.

- **(A8)** Let $u, v, w \in \mathcal{I}$ satisfying $u|v|w$. It follows that $u^+ = v^-$ and $v^+ = w^-$. Moreover, as $\prec (u^-, v^-, v^+)$ is satisfied, we have $v^- \neq v^+$. In consequence, $u^+ \neq w^-$. Hence, we can assert that $u|w$ is not satisfied.

□

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