A Unified Treatment for Knowledge Dynamics

Isabelle Bloch
Ecole Nationale Supérieure
des Télécommunications
Dept TSI - CNRS UMR 5141 LTCI
46 rue Barrault, Paris, France
Isabelle.Bloch@enst.fr

Ramón Pino-Pérez and **Carlos Uzcátegui**
Departamento de Matemáticas
Facultad de Ciencias
Universidad de Los Andes
Mérida, Venezuela
pino@ula.ve, uzca@ula.ve

Abstract

Using morpho-logics we show how to find explanations of observations, how to perform revision, contraction, fusion, in an unified way. In the framework of abduction, we show how to deal with observations inconsistent with the background theory and introduce methods to treat multiple observations. Based on these ideas we introduce a dynamics for transforming the background theory in function of observations.

Keywords: Explanations, revision, fusion, distance, morpho-logic.

Introduction

The tools of mathematical morphology (Serra 1982; 1988) can be applied to some areas of Artificial Intelligence where logical representations of knowledge are used. This is possible due to the duality between syntax and semantics of logical formulas. Actually we can identify a formula with the set of its models. Moreover, it is quite natural to consider distances in the space of models (for instance the Hamming distance). Having a distance is all we need to define in a straightforward way the basic operators of mathematical morphology: erosion and dilation (in fact it is enough to have a graph on the set of models).

In (Bloch & Lang 2000; 2002) morpho-logics were introduced: mathematical morphology for logical formulas. It is shown there how to use these operators to deal with revision and fusion. The idea to perform revision is to dilate enough the old belief set to meet the new information. To perform fusion, the idea is to dilate each source of information until the intersection of all of them is nonempty. This common part is the result of merging these sources of information.

In (Bloch, Pino-Pérez, & Uzcátegui 2001) it is shown how to construct explanatory relations using the erosion operator. The main idea is to find the *most central part* of a theory by successive erosions. Two explanatory relations were constructed and their behavior with respect to the postulates of rationality introduced in (Pino-Pérez & Uzcátegui 1999) analyzed.

In this work we present an unified view of these processes. This unified view allows to consider observations which are inconsistent with the background theory in a very natural way. In order to treat this case there are three basic situations considered here: the observation is unreliable and the background theory is reliable; the observation is reliable and the background theory is unreliable; the observation and the background theory are unreliable.

In particular the second case leads to consider changes in the background theory. This dynamics introduces a kind of learning process when successive observations are received or performed.

Another advantage of the treatment presented here is the introduction of explanatory relations for multiple observations. This is performed by combining fusion and abduction.

Our aim when presenting these methods which are all built upon the erosion or dilation operator of morpho-logics is to give a powerful, natural and uniform tool to treat many important aspects of knowledge dynamics based on logical representations. The general idea, unifying abduction, revision and fusion, is to “expand” or “shrink” a formula or the background theory until (or while) some properties like consistency are verified. Mathematical morphology operators provide formal tools for implementing these ideas.

We claim that these methods are paradigmatic, so studying their structural properties will give insights about the properties that general operators have to satisfy. We have given some structural properties of some of our operators. Nevertheless, we have not given a complete characterization of our operators.

Preliminaries

Let us recall here the basic principles of morpho-logics. Let PS be a finite set of propositional symbols. The language is generated by PS and the usual connectives. Well-formed formulas will be denoted by Greek letters φ, ψ... Worlds will be denoted by ω, ω'... and the set of all worlds by Ω. $Mod(\varphi) = \{\omega \in \Omega \mid \omega \models \varphi\}$ is the set of all worlds where φ is satisfied. Dilation and erosion (the two fundamental operations of mathematical morphology (Serra 1982)) of a formula φ by a structuring element B have been defined in (Bloch & Lang 2000) as follows:

$$Mod(D_B(\varphi)) = \{\omega \in \Omega \mid B(\omega) \cap Mod(\varphi) \neq \emptyset\}, \quad (1)$$

$$Mod(E_B(\varphi)) = \{\omega \in \Omega \mid B(\omega) \models \varphi\}, \quad (2)$$

where $B(\omega)$ is a set of worlds and $B(\omega) \models \varphi$ means $\forall \omega' \in B(\omega), \omega' \models \varphi$. It is usually called the structuring element and it contains the worlds that are in relation with ω. An important example is given by $B(\omega) = \{\omega' : \ d(\omega, \omega') \leq 1)\}$ where d is a distance function on Ω. This will be the only type of structuring element used in this paper.

The condition in Equation 1 expresses that the set of worlds in relation to ω should be consistent with φ, i.e.: $\exists \omega' \in B(\omega), \ \omega' \models \varphi$. The condition in Equation 2 is stronger and expresses that φ should be satisfied in all worlds which stand in relation to ω .

Properties

The properties of these basic operations and of other derived operations are detailed in (Bloch & Lang 2000). The fundamental properties of erosion and dilation, that will be used intensively in the following, can be summarized as:

- Independence of the syntax (follows directly from the definition through the models).
- Monotonicity: erosion and dilation are increasing with respect to φ, i.e.

  $$\varphi \vdash \psi \ \Rightarrow \ E_B(\varphi) \vdash E_B(\psi), \quad (3)$$

  for any structuring element B, and a similar equation for dilation. Erosion is decreasing with respect to the structuring element, i.e. suppose that $\forall \omega \in \Omega, B(\omega) \subseteq B'(\omega)$ (for instance this happens when the distances d and d' used to define B and B' satisfy $d \leq d'$), then

  $$E_{B'}(\varphi) \vdash E_B(\varphi). \quad (4)$$

  Dilation is increasing with respect to the structuring element.
- If B is derived from a reflexive relation, i.e. such that $\forall \omega \in \Omega, \omega \in B(\omega)$, then erosion and dilation satisfy the following properties:

  $$\varphi \vdash D_B(\varphi), \quad (5)$$

  $$E_B(\varphi) \vdash \varphi. \quad (6)$$

  (This is called anti-extensivity and extensivity in set theoretical mathematical morphology). This condition obviously holds when B is defined by a distance, as it will be case in this paper. We will also consider symmetrical relations, i.e. $\forall (\omega, \omega') \in \Omega^2, \omega \in B(\omega') \Leftrightarrow \omega' \in B(\omega)$.
- Iteration: erosion and dilation satisfy an iteration property, which is expressed for symmetrical structuring elements as:

  $$E_B[E_{B'}(\varphi)] = E_{D_B(B')}(\varphi), \quad (7)$$

  $$D_B[D_{B'}(\varphi)] = D_{D_B(B')}(\varphi). \quad (8)$$

  For instance if $B = B'$, and if we denote by E^n the erosion of size n, i.e. by B dilated $(n-1)$ times by itself (this is typically the case for distance based operations where the structuring element is a ball of distance, as will be seen in the next subsection), we have:

  $$E^{n+n'}(\varphi) = E^{n'}[E^n(\varphi)] = E^n[E^{n'}(\varphi)], \quad (9)$$

  where n, n' denote the size of the erosion (i.e. the "radius" of the structuring element).
- Commutativity of erosion with conjunction and of dilation with disjunction:

  $$E_B(\wedge_{i=1}^m \varphi_i) = \wedge_{i=1}^m E_B(\varphi_i), \quad (10)$$

  $$D_B(\vee_{i=1}^m \varphi_i) = \vee_{i=1}^m D_B(\varphi_i). \quad (11)$$
- Erosion of a disjunction: erosion and disjunction do not commute, but we have a partial relation:

  $$E_B(\varphi) \vee E_B(\psi) \vdash E_B(\varphi \vee \psi). \quad (12)$$

  Similarly, dilation and conjunction do not commute, and only a partial relation holds.
- Dilation and erosion are dual operators with respect to the negation:

  $$D_B(\varphi) = \neg E_B(\neg \varphi). \quad (13)$$

Illustrative example

In all what follows, we will consider as an illustrative example the case where the structuring element is defined as a ball of the Hamming distance between worlds d_H, where $d_H(\omega, \omega')$ is the number of propositional symbols that are instantiated differently in both worlds. Then dilation and erosion

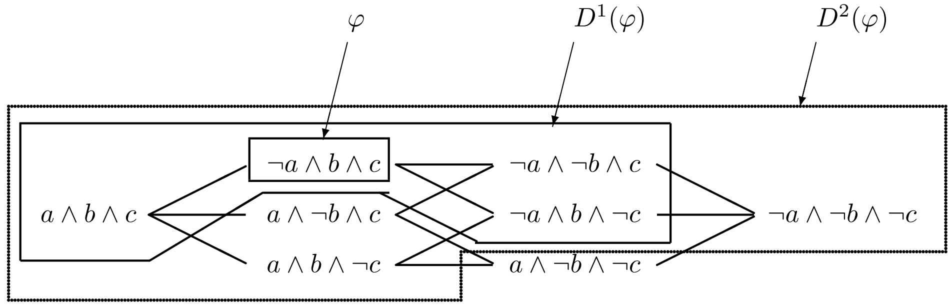

Figure 1: Graph representation of possible worlds with 3 symbols and an example of φ and two successive dilations. An arc between two nodes means that the corresponding nodes are at a distance to each other equal to 1.

of size n are defined from Equations 1 and 2 by using the distance balls of radius n as structuring elements:

$$Mod(D^n(\varphi)) = \{\omega \in \Omega \mid \exists \omega' \in \Omega, \omega' \models \varphi \; and \; d_H(\omega, \omega') \leq n\}, \quad (14)$$

$$Mod(E^n(\varphi)) = \{\omega \in \Omega \mid \forall \omega' \in \Omega, d_H(\omega, \omega') \leq n \Rightarrow \omega' \models \varphi\}. \quad (15)$$

We make use of a graph representation of worlds, where each node represents a world and a link represents an elementary connection between two worlds, i.e. being at distance 1 from each other. A ball of radius 1 centered at ω is constituted by ω and the extremities of the arcs originating in ω. This allows for an easy visualization of the effects of transformations.

Let us consider an example with three propositional symbols a, b, c. The possible worlds are represented in Figure 1.

Let us consider $\varphi = \neg a \wedge b \wedge c$. Then we have:

$$D^1(\varphi) = (\neg a \wedge b) \vee (\neg a \wedge c) \vee (b \wedge c),$$

$$D^2(\varphi) = \neg a \vee b \vee c = \neg(a \wedge \neg b \wedge \neg c).$$

These results are illustrated in Figure 1. Note that in this kind of figures the formula defined by a border is the disjunction of the formulas in the interior of the border.

Erosion can be computed very easily from any conjunctive normal form. Indeed, if φ is a disjunction of literals, i.e., $\varphi = l_1 \vee l_2 \vee ... \vee l_n$, then we have:

$$E^1(\varphi) = \wedge_{j=1}^n (\vee_{i \neq j} l_i). \quad (16)$$

This property, along with the commutativity of erosion with conjunction, allows to compute easily the erosion of any formula expressed as a CNF (see also (Lafage & Lang 2000)). By duality, dilation of any formula expressed as a disjunctive normal form is easy to compute.

Explanatory relations

In the logic-based approach to abduction, the background theory is given by a consistent set of formulas Σ. The notion of a *possible explanation* is defined by saying that a formula γ (consistent with Σ) is an explanation of α if $\Sigma \cup \{\gamma\} \vdash \alpha$. An explanatory relation is a binary relation $\rhd$ where the intended meaning of $\alpha \rhd \gamma$ is "γ is a *preferred explanation* of α". We prefer to write $\alpha \rhd \gamma$ instead of $\gamma \rhd \alpha$ because in explanatory reasoning, the input is an observation and the output is a preferred explanation of that observation.

In (Pino-Pérez & Uzcátegui 1999; 2003), a set of postulates that should be satisfied by preferred explanatory relations is proposed and discussed. The notation $\gamma \vdash_\Sigma \alpha$ will be used as an abbreviation of $\Sigma \cup \{\gamma\} \vdash \alpha$. For the sake of completeness we list some of them in Table 1.

Some justifications and predecessors of these rules were given in (Pino-Pérez & Uzcátegui 1999; 2000) (see also (Flach 1996; 2000a; 2000b)). An explanatory relation is called *Rational* if it satisfies LLE$_\Sigma$, E-CM, E-C-Cut, E-R-Cut and RS. All rational explanatory relations are of the following form (Pino-Pérez & Uzcátegui 1999): given a total pre-order $\preceq$ over Ω, we define the explanatory relation $\rhd_\preceq$ associated to $\preceq$ by

$$\alpha \rhd_\preceq \gamma \overset{def}{\Leftrightarrow} \text{mod}(\Sigma \cup \{\gamma\}) \subseteq \min(\text{mod}(\alpha), \preceq) \quad (17)$$

where $\min(\text{mod}(\alpha), \preceq)$ is the $\preceq$-minimal models of α.

Recall that an explicit assumption made in the definition of an explanatory relation is that an explanation together with the background theory has to logically imply the observation. In this paper we will allow a more permissive notion of explanation, which could be stated shortly as saying that an explanation has to *morphologically imply the observa-*

LLE$_\Sigma$	Left syntax independance	If $\vdash_\Sigma \alpha \leftrightarrow \alpha'$ and $\alpha \rhd \gamma$, then $\alpha \rhd \gamma'$
RLE$_\Sigma$	Right syntax independance	If $\vdash_\Sigma \gamma \leftrightarrow \gamma'$ and $\alpha \rhd \gamma$, then $\alpha \rhd \gamma'$
E-CM	Cautious Monotony	If $\alpha \rhd \gamma$ and $\gamma \vdash_\Sigma \beta$, then $(\alpha \wedge \beta) \rhd \gamma$
E-C-Cut	Cautious Cut	If $(\alpha \wedge \beta) \rhd \gamma$ and for all δ $[\alpha \rhd \delta \;\Rightarrow\; \delta \vdash_\Sigma \beta\,]$, then $\alpha \rhd \gamma$
E-R-Cut	Rational Cut	If $(\alpha \wedge \beta) \rhd \gamma$ and there is δ such that $\delta \vdash_\Sigma \beta$ and $\alpha \rhd \delta$, then $\alpha \rhd \gamma$
LOR	Disjunction on the left	If $\alpha \rhd \gamma$ and $\beta \rhd \gamma$, then $(\alpha \vee \beta) \rhd \gamma$
E-DR	Disjunctive rationality	If $\alpha \rhd \gamma$ and $\beta \rhd \delta$, then $(\alpha \vee \beta) \rhd \gamma$ or $(\alpha \vee \beta) \rhd \delta$
ROR	Disjunction on the right	If $\alpha \rhd \gamma$ and $\alpha \rhd \delta$, then $\alpha \rhd (\gamma \vee \delta)$
RS	Right Strengthening	If $\alpha \rhd \gamma$, $\gamma' \vdash_\Sigma \gamma$ and $\gamma' \not\vdash_\Sigma \bot$, then $\alpha \rhd \gamma'$

Table 1: Postulated that should be satisfied by preferred explanatory relations (Pino-Pérez & Uzcátegui 2003).

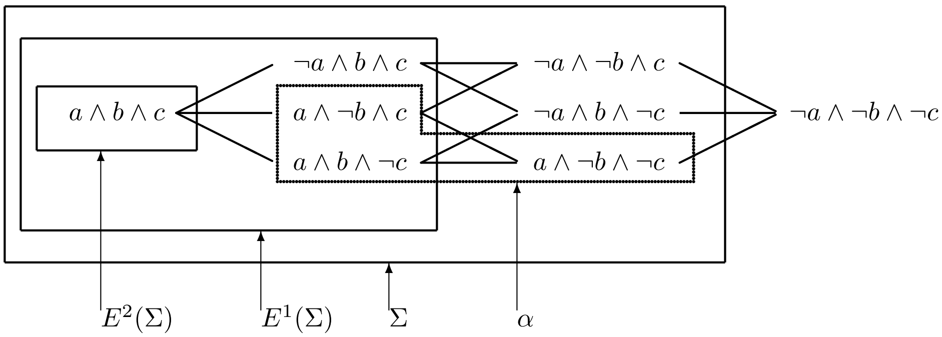

Figure 2: An example of last consistent erosion.

tion, that is, the set of models of the explanation has to be "morphologically closed" to the set of models of the observation. This will be made more precise in what follows.

Viewing abduction and revision as the same process

We will show in this section a unified treatment of abduction and revision. In particular, we will put in the same framework some of the results from (Bloch & Lang 2000; Bloch, Pino-Pérez, & Uzcátegui 2001).

In the following we fix a distance (for instance the Hamming distance) upon which we define a structuring element in order to define the operators of erosion and dilation.

An idea for finding explanations to an observation α in the setting of the background theory Σ consists in eroding Σ as much as possible but still under the constraint that it remains consistent with α (see (Bloch, Pino-Pérez, & Uzcátegui 2001)):

$$E_{\ell c}(\Sigma, \alpha) = E^n(\Sigma) \;\text{ for }\; n = \max\{k : E^k(\Sigma) \wedge \alpha \not\vdash \bot\}, \tag{18}$$

where the subscript ℓc stands for last consistent erosion.

Then, from this operator, the following explanatory relation was defined in (Bloch, Pino-Pérez, & Uzcátegui 2001) and shown to be a Rational explanatory relation:

$$\alpha \rhd^{\ell c} \gamma \;\;\overset{def}{\Leftrightarrow}\;\; \gamma \vdash E_{\ell c}(\Sigma, \alpha) \wedge \alpha, \tag{19}$$

Let us come back to the illustrative example, and take (see Figure 2): $\Sigma = a \vee b \vee c$, and $\alpha = (a \wedge \neg b \wedge c) \vee (a \wedge b \wedge \neg c) \vee (a \wedge \neg b \wedge \neg c)$. We have: $E^1(\Sigma) = (a \vee b) \wedge (a \vee c) \wedge (b \vee c)$, $E^2(\Sigma) = a \wedge b \wedge c$, and finally $E^3(\Sigma) \vdash \bot$. Therefore:

$$E^1(\Sigma) \wedge \alpha = (a \wedge \neg b \wedge c) \vee (a \wedge b \wedge \neg c)$$

and $E^2(\Sigma) \wedge \alpha \vdash \bot$. Therefore the value of n in Equation 18 is equal to 1. For Definition 19, we have $\alpha \rhd^{\ell c} (a \wedge \neg b \wedge c) \vee (a \wedge b \wedge \neg c)$, $\alpha \rhd^{\ell c} (a \wedge \neg b \wedge c)$ and $\alpha \rhd^{\ell c} (a \wedge b \wedge \neg c)$.

There is an alternative way of looking at $\rhd^{\ell c}$ which will be particularly useful in what follows. The iteration of the erosion operator provides a method of linearly pre-ordering the models of Σ. Consider the following relation among models:

$$\omega \leq_E \omega' \;\;\overset{def}{\Leftrightarrow}\;\; \forall k \;\; (\omega' \in E^k(\Sigma) \rightarrow \omega \in E^k(\Sigma)). \tag{20}$$

It is clear that $\leq_E$ is a total pre-order and it is not difficult to verify that the following holds:

$$\alpha \rhd^{\ell c} \gamma \iff \mathrm{mod}(\{\gamma\}) = \min(\mathrm{mod}(\Sigma \cup \{\alpha\}), \leq_E). \tag{21}$$

Notice that from the equivalence 21 and by the results in (Pino-Pérez & Uzcátegui 1999), it is clear that $\rhd^{\ell c}$ is, as we already said, a rational explanatory relation.

Now we will recall the presentation of revision made in (Bloch & Lang 2000) based on the dilation operator. The idea is to dilate Σ (which is not necessarily consistent with α) until it becomes consistent with α. More precisely, we define $*$ as:

$$\Sigma * \alpha = D^n(\Sigma) \wedge \alpha \tag{22}$$

where $n = \min\{k : D^k(\Sigma) \wedge \alpha \nvdash \bot\}$ (it should be noted that n depends on α).

As before, the iteration of the dilation operator provides a method of linearly pre-ordering the models. Consider the following relation among models:

$$\omega \leq_D \omega' \quad \overset{def}{\Leftrightarrow} \quad \forall k \ \ (\omega' \in D^k(\Sigma) \rightarrow \omega \in D^k(\Sigma)). \tag{23}$$

Indeed, it is clear that $\leq_D$ is a total pre-order; we will call it the *total preorder associated to* Σ *using successive dilations*. It is not difficult to verify that the following holds:

$$\text{mod}(\Sigma * \alpha) = \min(\text{mod}(\Sigma \cup \{\alpha\}), \leq_D). \tag{24}$$

By the well known representation theorem for revision operators (see for instance (Katsuno & Mendelzon 1991)), it follows from Equation 24 that $*$ is an AGM revision operator (Bloch & Lang 2000).

We observe a great resemblance between the definitions of pre-orders defined by Equations 20 and 23. Actually we can define a sequence associated to Σ in the following way: let m be such that $E^m(\Sigma) \nvdash \bot$ and $E^{m+1}(\Sigma) \vdash \bot$ and let n be such that $D^n(\Sigma) = \top$ (the theory containing the tautologies). Then we define the fundamental sequence (T_i) associated to Σ (relative to operators E and D) from $i = 0$ to $i = n + m$ as follows

$$T_i = \begin{cases} E^{m-i}(\Sigma) & \text{if } \ i \leq m \\ D^{i-m}(\Sigma) & \text{if } \ i > m \end{cases}$$

Now we can define the fundamental total preorder by the following

$$\omega \preceq_f \omega' \quad \overset{def}{\Leftrightarrow} \quad \forall k \ \ (\omega' \in T^k \rightarrow \omega \in T^k). \tag{25}$$

This pre-order will be called *the fundamental preorder associated to* Σ. This preorder is the main tool for what follows. It is then natural to associate to each observation α the collection

$$M(\alpha) = \min(\text{mod}(\alpha), \preceq_f).$$

Note that the criterion used to define $M(\alpha)$ is based on the morphology operators D and E. The interpretation we give to $M(\alpha)$ is that it contains those worlds that are (morphologically) more relevant given the observation α. Therefore for the task of revising Σ or explaining α we only look at $M(\alpha)$. This will be made precise in the result that follows. We will denote by $C(\alpha)$ the formula whose models are exactly $M(\alpha)$.

Theorem 1 *Let* Σ, α *and* γ *consistent formulas.*

1. *If* α *is consistent with* Σ, *then* $\alpha \rhd^{\ell c} \gamma$ *iff* $\gamma \vdash C(\alpha)$.
2. *If* α *is inconsistent with* Σ, *then* $\Sigma * \alpha = C(\alpha)$.

The previous result suggests the following definitions

$$\alpha \rhd_f \gamma \quad \overset{def}{\Leftrightarrow} \quad \gamma \vdash C(\alpha) \tag{26}$$

and

$$\Sigma *_f \alpha = C(\alpha) \tag{27}$$

where α and γ are consistent formulas.

Some comments about these definitions should be made. First of all, even when an observation is inconsistent with the background theory Σ there is a formula γ such that $\alpha \rhd_f \gamma$. That is to say, we can "explain" more observations with $\rhd_f$ than with $\rhd^{\ell c}$. The interpretation we give to this fact is that for explaining an observation it is allowed (if necessary) to "change" the background theory (this will be made more precise in the section about dynamics). Thus in the explanatory process described by $\rhd_f$ the observation is absolutely reliable. Notice also that $\rhd_f$ is not an explanatory relation as defined above, since some explanations might not be consistent with Σ.

The operator $*_f$ is not an AGM operator (for Σ), since when the observation α to be incorporated is consistent with Σ we have only $\Sigma *_f \alpha \vdash \Sigma \wedge \alpha$. The reason for this is that $*_f$ is based on explanations, so even when $\Sigma \vdash \alpha$ some explanation for α has to be found. Note that the previous remark says that $*_f$ does not satisfy the postulate K*4, which has been criticized by some authors in particular in (Ryan 1994). Unlike Ryan's operators, which are based on ordered theory presentations, K*4 is the only AGM postulate which is not satisfied by $*_f$.

Multiple observations

Suppose now that instead of an observation α we have n observations $\alpha_1, \ldots \alpha_n$ simultaneously. Note that two or more observations can be repeated in this sequence. Thus, it is quite natural to represent this by a multiset $\Phi = \{\alpha_1, \ldots \alpha_n\}$. The problem we address in this section can be stated as follows: how to explain Φ in the setting of the background theory Σ?

We propose a solution to this problem based on fusion where the principle of reliability of the *new observation* holds. The mechanism is roughly speaking the following:

1. Perform the fusion of Φ. Let $\Delta(\Phi)$ be this result.
2. Use an explanation relation $\rhd$ to find γ such that $\Delta(\Phi) \rhd \gamma$.

Now, we describe in detail the mechanism previously sketched. To fusion Φ we will use the method called majority merging (Konieczny & Pino-Pérez 1998; 2002)[1]. For this end, we first associate to each member of Φ a distance function on Ω which, as usual, is given by the rank of each world with respect to a total pre-order. Notice that Equation 23 defines a total pre-order for a given formula α_i (putting $\Sigma = \alpha_i$ in Equation 23). These relations will be denoted by $\preceq_{\alpha_i}$, for $i = 1, \cdots, n$. Let $r_{\alpha_i}(\omega)$ be the level of the world ω in the preorder $\preceq_{\alpha_i}$, *i.e.* the first k such that $\omega \in \mathrm{mod}(D^k(\alpha_i))$. For instance, if $\omega \in \mathrm{mod}(\alpha_i)$, then $r_{\alpha_i}(\omega) = 0$ and if $\omega \in \mathrm{mod}(D^1(\alpha_i)) \setminus \mathrm{mod}(\alpha_i)$, then $r_{\alpha_i}(\omega) = 1$.

Next we define a total pre-order $\preceq_\Phi$ in the following way: $\omega \preceq_\Phi \omega'$ iff $\Sigma_{\alpha_i \in \Phi} r_{\alpha_i}(\omega) \leq \Sigma_{\alpha_i \in \Phi} r_{\alpha_i}(\omega')$. Then the majority merging of Φ is given by:

$$\mathrm{mod}(\Delta(\Phi)) = \min(\Omega, \preceq_\Phi).$$

Now use the relation $\rhd_f$, given by Equation 26, to define the explanations of $\Delta(\Phi)$. Thus we finally get the following definition for the explanations of the multiset Φ:

$$\Phi \rhd_m \gamma \quad \stackrel{def}{\Leftrightarrow} \quad \Delta(\Phi) \rhd_f \gamma \tag{28}$$

The first thing to notice is that if Φ contains only one observation α, then $\Phi \rhd_m \gamma$ iff $\alpha \rhd_f \gamma$. The definition of $\rhd_m$ has something else interesting. The notion of explanation used is not the classical one. There is not a formal logical connection between Φ and its explanations. A feature of a merging process is that its output should satisfy the majority. Thus there are Φ such that $\Delta(\Phi)$ does not share any models with any of the members of Φ. So we can not say that an explanation of a group of observations will explain at least one of them. However, $\Delta(\Phi)$ can be regarded as a consensual observation and thus $\rhd_m$ provides its explanations.

On the other hand, since we are dealing with multisets (on the left hand side) the relation $\rhd_m$ is far from being an explanatory relation as defined above. However, it is interesting that we have the following fact.

Proposition 1 *The relation* $\rhd_m$ *satisfies* RLE, RS *and* ROR.

[1]Therein some other methods, for instance the Max method or the Gmax method, can be found.

The postulates involving the left part of the relation remain to be defined and studied. One of the main difficulties is how to explain $\Phi \sqcup \{\alpha\}$ knowing the explanations of α and Φ (here $\sqcup$ denote the union of multisets where the multiplicity is taken into account). To say it in an equivalent form, which is the postulate corresponding to LOR in the setting of multiple observations? Let us consider the following two candidates:

- $\Phi_1 \rhd \gamma$ and $\Phi_2 \rhd \gamma \Rightarrow \Phi_1 \sqcup \Phi_2 \rhd \gamma$
- $\Phi_1 \rhd \gamma$ and $\Phi_2 \rhd \gamma \Rightarrow \{\Delta(\Phi_1) \vee \Delta(\Phi_2)\} \rhd \gamma$

The second one, that is more constrained, would seem more adequate to our framework. It is very likely that these two postulates are not easy to be satisfied.

Dynamic observations and approximation process

Let us consider the explanation process as described above, but now we are going to update the background theory according to observation Φ.

The background theory Σ induces a preorder by successive dilations as in Equation 23 which will be denoted by $\preceq_\Sigma$. Thus we can see the information of the background theory like a more complex structure. Actually it is a pair $(\Sigma, \preceq_\Sigma)$. At the time that a complex observation Φ arrives, we find its explanations as in the previous Section and we transform the pair of base $(\Sigma, \preceq_\Sigma)$ in a new pair $(\Sigma', \preceq'_{\Sigma'})$ taking into account Φ. So we have to describe the following process:

$$(\Sigma, \preceq_\Sigma) \stackrel{\Phi}{\longmapsto} (\Sigma', \preceq'_{\Sigma'}) \tag{29}$$

To simplify the notation we will write $\preceq$ and $\preceq'$ instead of, respectively $\preceq_\Sigma$ and $\preceq'_{\Sigma'}$.

Let $\preceq'$ be $\preceq_{lex(\preceq_\Phi, \preceq)}$ and Σ' be the theory whose models are $\min(\Omega, \preceq')$, where $\preceq_{lex(\preceq_\Phi, \preceq)}$ is the lexicographical preorder associated to the preorders $\preceq_\Phi$ and $\preceq$. More precisely for any total preorders $\preceq_1$ and $\preceq_2$, the lexicographical preorder is defined as:

$$\omega \preceq_{lex(\preceq_1, \preceq_2)} \omega' \quad \stackrel{def}{\Leftrightarrow} \quad (\omega \prec_1 \omega') \quad \text{or} \quad (\omega \approx_1 \omega' \quad \text{and} \quad \omega \preceq_2 \omega').$$

An interpretation of $\preceq'$ is as follows. We give higher priority to the members of Φ and also to their preferences, but we try to accommodate as much as possible the initial preferences given by $(\Sigma, \preceq)$. The new background theory Σ' is the result of a compromise between the original pair $(\Sigma, \preceq)$ and the new incoming information.

This process can be iterated after receiving more observations. It would be then desirable that if the same multiset of observations Φ is received, then our state should not change. This is the content of the following result:

Proposition 2 *If* $(\Sigma, \preceq) \overset{\Phi}{\longmapsto} (\Sigma', \preceq')$ *then* $(\Sigma', \preceq')$ *is a stable state for* Φ*,* i.e.

$$(\Sigma', \preceq') \overset{\Phi}{\longmapsto} (\Sigma', \preceq')$$

.

Let us observe that in the transition of Equation 29, the pre-order $\preceq'$ is a refinement of $\preceq_\Phi$. Thus, it is easy to prove the following proposition:

Proposition 3 *Let* Σ *be any theory. Define* $(\Sigma_0, \preceq_0) = (\Sigma, \preceq_\Sigma)$*. Then, there are* $\Phi_1, \ldots \Phi_k$ *such that* $(\Sigma_{i-1}, \preceq_{i-1}) \overset{\Phi_i}{\longmapsto} (\Sigma_i, \preceq_i)$ *for* $i = 1, \ldots, k$ *and* $\preceq$ *is a linear* order *and* Σ_k *is complete.*

This can be interpreted as a learning process in which there are some sequences leading to a complete knowledge. In particular the theories produced starting from Σ_k will be also complete.

The whole process was modeled as an explanatory process. In fact, we can define dynamic explanations of multiple observations in the light of a pair $(\Sigma, \preceq)$ as follows[2]. Let $\preceq'$ be such that

$$(\Sigma, \preceq) \overset{\Phi}{\longmapsto} (\Sigma', \preceq')$$

Using the same idea behind the definition of the explanatory relation given in Equation 17, we define the relation $\rhd_{\preceq'}$ associated to $\preceq'$ as:

$$\alpha \rhd_{\preceq'} \gamma \quad \overset{def}{\Leftrightarrow} \quad \mathrm{mod}(\gamma) \subseteq \min(\mathrm{mod}(\alpha), \preceq') \quad (30)$$

and then we define

$$\Phi \rhd_d \gamma \quad \overset{def}{\Leftrightarrow} \quad \Delta(\Phi) \rhd_{\preceq'} \gamma$$

Thus Proposition 2 says that when Φ is again observed, then the explanations of Φ with respect to $(\Sigma, \preceq)$ and $(\Sigma', \preceq')$ are the same.

Suppose $(\Sigma, \preceq)$ is the initial state and Φ is observed. Our next result says that if Φ is coherent with the initial state, then the initial explanatory relation does not need to be modified. This is the case, for instance, when the set of formulas Φ is consistent with Σ.

Proposition 4 *If mod*$(\Sigma') \subseteq$ *mod*(Σ)*, then* $\Phi \rhd_d \gamma \;\Leftrightarrow\; \Delta(\Phi) \rhd_{\preceq} \gamma$*.*

[2]This process is analogous to revision of complex epistemic states by complex epistemic states (see (Konieczny & Pino-Pérez 2000; Benferhat *et al.* 2000)) in two aspects: the complex representation and the use of strong priority of observations which is traduced by the definition of $\preceq'$ using the lexicographical preorder resulting from two preorders.

Now if an inconsistent observation is repeatedly observed, then the tendency will be to iteratively update the background theory so that it becomes closer and closer to Φ. This can be interpreted as an approximation process. We start with $(\Sigma, \preceq)$. At first iteration, α is observed. If it happens to be inconsistent with Σ, then we will explain α and it will become consistent with Σ'. Of course, the situation is much more complex if instead of a single observation we receive several of them.

Unreliability

A natural question is how to explain an observation which is inconsistent with the background theory. We have already made some comments about this and we will explore further this issue in this section. Inconsistency of the observation with the theory may come from the fact that the observation is unreliable. Instead of dilating the theory to achieve consistency, we can then keep the theory and extend the observation until it becomes consistent with Σ. This extension can be again performed by a dilation (but of α this time), and can be interpreted as a way to introduce explicitly imprecision in the observation. Indeed, under a probabilistic (or possibilistic) interpretation, it is usual that reliability and precision are antagonist: a very precise observation has a low reliability. Conversely, an observation with high probability is usually imprecise (since it should include many possible cases).

This idea amounts to reverse the revision operator based on dilation introduced above, and consider $\alpha * \Sigma$, with the same $*$:

$$\alpha * \Sigma = \Sigma \wedge D^n(\alpha) \qquad (31)$$

for $n = \min\{k,\ \Sigma \wedge D^k(\alpha) \not\vdash_\Sigma \bot\}$. Now we can define a notion of an explanation as follows:

$$\alpha \rhd_u \gamma \quad \overset{def}{\Leftrightarrow} \quad \gamma \vdash_\Sigma \alpha * \Sigma \qquad (32)$$

Note that this does not define an explanation in the strict sense, since $\gamma \not\vdash \alpha$ if α is inconsistent with Σ.

Proposition 5 *The relation* $\rhd_u$ *defined by Equation 32 satisfies the postulates* RLE_Σ, LLE_Σ, RS, ROR, LOR *and* $\mathsf{E\text{-}DR}$. *The postulates* $\mathsf{E\text{-}CM}$, $\mathsf{E\text{-}C\text{-}Cut}$ *and* $\mathsf{E\text{-}R\text{-}Cut}$ *are not satisfied.*

Explanation as a fusion process

When the observation is not consistent with the theory, instead of changing the theory, we will try to find some compromise between the theory and the observation. This can be considered as a fusion or a negotiation process. The simplest idea to achieve this is to select the formulas which are "midway" between the theory and the observation. This can be formalized using morphological dilations as follows:

$$\gamma \rhd_{fus} \alpha \quad \stackrel{def}{\Leftrightarrow} \quad \gamma \vdash D^n(\Sigma) \wedge D^n(\alpha) \qquad (33)$$

where n is the smallest size of dilation such that the dilations of Σ and α are consistent (as was suggested in (Bloch & Lang 2002)):

$$n = \min\{k : \ D^k(\Sigma) \wedge D^k(\alpha) \not\vdash \bot\}. \qquad (34)$$

Again this does not define an explanation in the usual sense since $\gamma \not\vdash \alpha$. It can be interpreted as a recommendation, a decision, according to the interpretation in terms of fusion.

Proposition 6 *The relation* $\rhd_{fus}$ *satisfies* RLE$_\Sigma$, LLE$_\Sigma$, LOR, ROR, E-DR *and* RS.

In Equation 33, Σ and α are dilated by the same structuring element, and play therefore a similar role in the fusion. However, we can consider that we have more confidence in the theory than in the observation (or the reverse) and dilate less Σ than α. Dilation allows to perform such an asymmetrical fusion in a quite straightforward way.

For instance we can dilate α "m times faster" than Σ (m being a positive integer). Then, let

$$n = \min\{k : \ D^k(\Sigma) \wedge D^{mk}(\alpha) \not\vdash \bot\}. \qquad (35)$$

We can define a preferred explanation as:

$$\gamma \rhd_{wfus} \alpha \quad \stackrel{def}{\Leftrightarrow} \quad \gamma \vdash D^n(\Sigma) \wedge D^{mn}(\alpha). \qquad (36)$$

Conclusion

We have shown how to use the two basic operators of morpho-logics, erosion and dilation, to perform many processes of dynamical knowledge, in particular, revision and fusion. The tasks concerning the search of explanations have been incorporated in the same dynamic setting due to the resemblance in the way to produce explanations and the way to produce a revised theory. Thus, with the help of fundamental sequence we can express abduction and revision in the same way. This is done by Equations 26 and 27.

This uniform method suggests a generalization of explanatory relations. Thus we can treat observations which are either consistent or inconsistent with the background theory. This can be done, in at least three ways: first, considering the observation as very reliable. This is the case of $\rhd_f$. Second, considering the observation unreliable. This is the case of $\rhd_u$. The third way consists in considering the background and the observation equally unreliable or weightedly unreliable, this is the case of $\rhd_{fus}$ and $\rhd_{wfus}$ respectively.

We have introduced a new method to treat multiple (and simultaneous) observations; this is the relation $\rhd_m$ defined in Equation 28. This method has as a parameter the merging operator Δ used to perform the pretreatment of the multiple observations Φ. We have shown a few structural properties. However, the postulates concerning the manipulations of Φ (the left rules) remain to be discovered.

Finally we have introduced complex *background knowledge* and dynamic explanatory relations at the same time that we defined an update of the background knowledge. These are the relations $\rhd_d$ and the transitions $\stackrel{\Phi}{\longmapsto}$ between complex (background) knowledge. We noted that the iterations of this process eventually give a kind of complete information. Again here, the structural properties characterizing this kind of process remain to be discovered.

Acknowledgements

We would like to thank FONACIT (Program of Postgrados Integrados de Matemáticas, Venezuela) and División de Postgrado de la Facultad de Ciencias de la Universidad de Los Andes for the partial support to the first author for visiting The Universidad de Los Andes. The second author thanks the Ecole Nationale Supérieure des Télécommunications de Paris for the support for visiting to TSI Department of this institution.

References

Benferhat, S.; Konieczny, S.; Papini, O.; and Pérez, R. P. 2000. Iterated Revision by Epistemic States: Axioms, Semantics and Syntax. In *Proceedings of the Fourteenth European Conference on Artificial Intelligence, ECAI 2000*, 13–17.

Bloch, I., and Lang, J. 2000. Towards Mathematical Morpho-Logics. In *8th International Conference on Information Processing and Management of Uncertainty in Knowledge based Systems IPMU 2000*, volume III, 1405–1412.

Bloch, I., and Lang, J. 2002. Towards Mathematical Morpho-Logics. In Bouchon-Meunier, B.; Gutierrez-Rios, J.; Magdalena, L.; and Yager, R., eds., *Technologies for Constructing Intelligent Systems*. Springer. 367–380.

Bloch, I.; Pino-Pérez, R.; and Uzcátegui, C. 2001. Explanatory Relations based on Mathematical Morphology. In *ECSQARU 2001*, 736–747.

Flach, P. A. 1996. Rationality Postulates for Induction. In Shoham, Y., ed., *Sixth Conference of*

Theoretical Aspects of Rationality and Knowledge (TARK96), 267–281.

Flach, P. 2000a. Logical Characterisations of Inductive Learning. In Gabbay, D. M., and Kruse, R., eds., *Abductive Reasoning and Learning*. Kluwer Academic Publishers. 155–196.

Flach, P. 2000b. On the Logic of Hypothesis Generation. In Flach, P., and Kakas, A., eds., *Abduction and Induction*. Kluwer Academic Publishers. 89–106.

Katsuno, H., and Mendelzon, A. O. 1991. Propositional Knowledge Base Revision and Minimal Change. *Artificial Intelligence* 52:263–294.

Konieczny, S., and Pino-Pérez, R. 1998. On the Logic of Merging. In *6th International Conference on Principles of Knowledge Representation and Reasoning*, 488–498.

Konieczny, S., and Pino-Pérez, R. 2000. A Framework for Iterated Revision. *Journal of Applied and Non Classical Logics* 10(3-4):339–367.

Konieczny, S., and Pino-Pérez, R. 2002. Merging Information: A Qualitative Framework. *Journal of Logic and Computation* 12(5):773–808.

Lafage, C., and Lang, J. 2000. Logical Representation of Preferences for Group Decision Making. In Cohn, A. G.; Giunchiglia, F.; and Selman, B., eds., *7th International Conference on Principles of Knowledge Representation and Reasoning KR 2000*, 457–468. Breckenridge, CO: Morgan Kaufmann, San Francisco, CA.

Pino-Pérez, R., and Uzcátegui, C. 1999. Jumping to Explanations versus jumping to Conclusions. *Artificial Intelligence* 111:131–169.

Pino-Pérez, R., and Uzcátegui, C. 2000. Ordering Explanations and the Structural Rules for Abduction. In Cohn, A. G.; Giunchiglia, F.; and Selman, B., eds., *7th International Conference on Principles of Knowledge Representation and Reasoning KR 2000*, 637–646. Breckenridge, CO: Morgan Kaufmann, San Francisco, CA.

Pino-Pérez, R., and Uzcátegui, C. 2003. Preferences and Explanations. *Artificial Intelligence* 149(1):1–30.

Ryan, M. D. 1996. Belief Revision and Ordered Theory Presentations. In Fuhrmann, A., and Rott, H., eds., *Logic, Action and Information.* Walter De Gruyter. Berlin. 129–151.

Serra, J. 1982. *Image Analysis and Mathematical Morphology*. London: Academic Press.

Serra, J. 1988. *Image Analysis and Mathematical Morphology, Part II: Theoretical Advances*. London: Academic Press (J. Serra Ed.).