Majority Logic

Eric Pacuit and **Samer Salame**
Department of Computer Science
Graduate Center, City University of New York
365 5th Avenue, New York 10016
epacuit@cs.gc.cuny.edu, ssalame@gc.cuny.edu

Abstract

We extend graded modal logic (**GML**) to a logic that captures the concept of majority. We provide an axiomatization for majority logic, **MJL**, and sketch soundness and completeness proofs. Along the way, we must answer the question what is a majority of an infinite set? Majority spaces are introduced as a solution to this question.

Introduction

The language of modal logic has long been used to model intensional notions such as knowledge, belief and obligation. In this extended abstract we present a new modal logic which models an agent's ability to reason about majorities. The concept of majority plays an important role when an agent is faced with a decision in a social situation. For example, think of dinner with a group of friends. Chances are that many of the decisions, such as choice of restaurant, appetizers or wine, were based on the will of the majority. An extended example which illustrates this point is found in the next section. Of course, the concept of majority is integral to many voting systems. With these intuitions in mind, we propose a logic, **MJL**, in which the concept of majority is axiomatized.

Given a formula α, the language of normal modal logics can express "α is true in *all* accessible worlds" ($\Box\alpha$), and "α is true in *at least one* accessible world" ($\Diamond\alpha$). But suppose that we want to express that α is true in at least *three* accessible worlds or that α is true in a *majority (more than half)* of the accessible worlds. The language of normal modal logic cannot express such statements. The logic **MJL** presented in this paper will use modal operators that can specify exactly how many accessible worlds are of interest.

To start with, we add the graded modalities first discussed in (Fine 1972; Goble 1970). For any $n \in \mathbb{N}$, the formula $\Diamond_n\alpha$ is intended to mean α is true in strictly more than n accessible world, and so its dual $\Box_n\alpha$ is intended to mean $\neg\alpha$ is true in less than or equal to n accessible worlds. We may call $\Diamond_n\alpha$ an *at least* formula, since $\Diamond_n\alpha$ will be true precisely when α is true in *at least* $n+1$ accessible worlds. Similarly we may call $\Box_n\alpha$ *all but* formulas, since $\Box_n\alpha$ will be true precisely when α is false in *all but* n accessible worlds. For simplicity we write $\Diamond\alpha$ instead of $\Diamond_0\alpha$ and $\Box\alpha$ instead of $\Box_0\alpha$. For instance, if the formula $\Box_k\bot$ is true at some world w, then w has at most k accessible worlds.

We then extend the graded modal logic (**GML**) as described in (Fattorosi-Barnaba & Cerrato. 1988; Caro 1988) with a modal operator W, where $W\alpha$ is intended to mean α is true in more than or equal to half of the accessible worlds. Hence its dual, $M\alpha$ will mean α is true in more than half of the accessible worlds. Here M represents strict Majority and W represents Weak majority. In what follows, when we use "majority", we mean weak majority (i.e. more than or equal to 50%).

Before proceeding we should check that we are in fact gaining expressive power with the new modal operators. To see this note that **MJL** does not obey bisimulation. We can easily find two bisimular Kripke models where in one of them we have $W\alpha$ is true at some state s and in the other $W\alpha$ may not be true at a bisimular state. It follows that the operator W cannot be defined from the standard modal operators ($\Box$ and $\Diamond$). A similar argument shows that $\Diamond_n$ cannot be defined from the standard modal operators. For an extended discussion of this fact refer to (Fattorosi-Barnaba & Cerrato. 1988). Furthermore, a similar argument shows that the modal operator M cannot be expressed with the graded modal operators[1]

Given the intended interpretation of $W\alpha$, defining truth in a Kripke model is straightforward provided there are only finitely many accessible worlds. However, there are situations, such as in the canonical model, in which one cannot assume that the number of accessible worlds is finite. This leads us to the question of what is the majority of an infinite set? The standard definition, i.e. more than half, no longer makes sense. Should we consider the even numbers a weak majority of the natural numbers, and if so what about the

[1] In (de Rijke 2000) de Rijke develops a notion of bisimulation (g-bisimulation) for graded modal logic. He then uses this notion to prove some model theoretic results such as the finite model property. So, we need to show that there are two models that are g-bisimular but can be distinguished using the M operator. Of course, if the number of accessible worlds is fixed to be n then $M\alpha$ can be defined to be $\Diamond_{\lfloor n/2 \rfloor}\alpha$; however if $M\alpha$ can be defined using graded modal operators, then this definition must hold regardless of the number of accessible worlds.

set that contains all the even numbers plus the set $\{1, 3\}$? Mark Fey in (Fey 2002) proposes some interesting answers to this question. However, Fey's solutions are not appropriate for our framework and so we need another solution. We propose *majority spaces*, which generalize the concept of an ultrafilter, as a solution to the problem of defining a majority of an infinite set.

This extended abstract is organized as follows. The next section reviews graded modal logic. We then describe the language of majority logic and offer an axiomatization. After introducing majority spaces, we provide a Kripke style semantics and sketch the completeness proof.

Graded Modal Logic

In this section, we provide a brief overview of graded modal logic. Graded modal logic was first introduced in (Fine 1972; Goble 1970). It was then studied in (Fattorosi-Barnaba & Cerrato. 1988; H. J. Ohlbach & Hustadt 1995; Caro 1988; de Rijke 2000; Tobies 2001) in which issues of axiomatization, completeness, decidability and translations into predicate logic are discussed. We briefly discuss the language of graded modal logic and state some of the main results found in the literature. All results and proofs can be found in (Fattorosi-Barnaba & Cerrato. 1988) and (Caro 1988).

Definition 1 *Given a countable set of atomic propositions $\mathbb{P} = \{p_0, p_1, \ldots\}$, a formula α of **GML** can have the following syntactic form:*

$$\alpha := p \mid \neg\alpha \mid \alpha \vee \alpha \mid \Diamond_n \alpha$$

where $p \in \mathbb{P}$ and $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, we define $\Box_n \alpha := \neg\Diamond_n\neg\alpha$, and $\Diamond!_n\alpha := \Diamond_{n-1}\alpha \wedge \neg\Diamond_n\alpha$ $(n \neq 0)$ where $\Diamond!_0\alpha := \neg\Diamond_0\alpha$. So $\Diamond!_n\alpha$ will have the intended meaning that α is true in *exactly* n accessible worlds. Let $\mathcal{L}_{\mathbf{GML}}$ be the set of all well-formed formulas of **GML**.

The following axiomatization was presented in (Caro 1988).

G0 All tautologies in the language of **GML**

G1 $\Diamond_{n+1}\alpha \to \Diamond_n\alpha \quad (n \in \mathbb{N})$

G2 $\Box_0(\alpha \to \beta) \to \Diamond_n\alpha \to \Diamond_n\beta \quad (n \in \mathbb{N})$

G3 $\Diamond!_0(\alpha \wedge \beta) \to ((\Diamond!_{n_1}\alpha \wedge \Diamond!_{n_2}\beta) \to \Diamond!_{n_1+n_2}(\alpha \vee \beta)) \quad (n_1, n_2 \in \mathbb{N})$

GML is closed under modus ponens (MP) and necessitation (N), i.e., from $\vdash \alpha$ infer $\vdash \Box\alpha$. We write $\vdash_{\mathbf{GML}} \alpha$ if α can be deduced from $G0 - G1$ using the rules MP and N.

GML formulas are interpreted in the usual Kripke structures. Let $\mathcal{M} = \langle S, R, V\rangle$ be a Kripke model, where S is a set of worlds, R is a binary relation over S and $V : \mathbb{P} \to 2^S$ is a valuation function. The boolean connectives and propositional variables are evaluated as usual. We will only show how the formula $\Diamond_n\alpha$ is evaluated at a world $s \in S$:

$$\mathcal{M}, s \models \Diamond_n\alpha \text{ iff } |\{t : sRt \text{ and } \mathcal{M}, t \models \alpha\}| > n$$

We say α is valid in $\mathcal{M}$ iff $\forall s \in S$, $\mathcal{M}, s \models \alpha$, and write $\mathcal{M} \models \alpha$. We write $\models \alpha$ if α is valid in all models (based on some class of frames[2]) We also make use of the following notation throughout this paper: $R(s) = \{t \mid sRt\}$ and for any formula α (of **MJL** or **GML**), $R_\alpha(s) = \{t \mid sRt \text{ and } t \models \alpha\}$. So, the above definition can be rewritten as

$$\mathcal{M}, s \models \Diamond_n\alpha \text{ iff } |R_\alpha(s)| > n$$

GML is shown to be sound and complete with respect to the class of all frames in (Fattorosi-Barnaba & Cerrato. 1988). Let $\mathfrak{F}$ be the class of all frames. It is easily verified that the axioms $G0 - G1$ are valid in any model based on $\mathfrak{F}$ and MP and N preserve validity. We state the completeness theorem below, but postpone discussion until later in this paper.

Theorem 1 (Completeness of GML) *Let $\mathfrak{F}$ be the class of all frames. Then for any formula α of **GML**, $\models \alpha$ iff $\vdash_{\mathbf{GML}} \alpha$.*

In (Caro 1988) **GML** is shown to be decidable by showing that **GML** has the finite model property. Maarten de Rijke (de Rijke 2000) arrives at the same conclusion using an extended notion of bisimulation appropriate for a modal language with graded modalities. de Rijke also establishes invariance and definability results. Finally in (Tobies 2001), Tobbies shows that the decidability problem for **GML** is in $PSPACE$.

Majority Logic: Syntax

We extend the graded modal language with a new modal operator W where W is interpreted as "weak majority."

Definition 1 *Given a countable set of atomic propositions $\mathbb{P} = \{p_0, p_1, \ldots\}$, a formula α of **MJL** can have the following syntactic form:*

$$\alpha := p \mid \neg\alpha \mid \alpha \vee \alpha \mid \Diamond_n\alpha \mid W\alpha$$

where $p \in \mathbb{P}$ and $n \in \mathbb{N}$.

Let $\mathcal{L}_{\mathbf{MJL}}$ be the set of all well-formed formulas of the majority modal logic. Define $M\alpha := \neg W\neg\alpha$. So, **MJL** takes the language of **GML** and closes under the operator W. Notice in particular that there are an infinite number of modal operators, one for each natural number plus the majority operators.

Axiomatization

We propose the following axiomatization of **MJL**. Since **MJL** extends graded modal logic, we will include the axiom schemes $G1, G2$ and $G3$. These axioms captures our intuitions when we can count accessible worlds. But what axioms shall we adopt to reason about "majority"? The following discussion will motivate the proposed axiomatization which can be found at the end of the discussion.

[2] Unless otherwise stated we will assume that we are working with models based on the class of all frames. Refer to (Blackburn, de Rijke, & Venema 2001) for more information on frames.

Suppose a group of friends are trying to decide where to go for dinner. As is common in most social situations, the goal is to keep as many people happy as possible. If more than half of the people want Indian for dinner and more than half want Italian for dinner, then there must be someone who wants both Italian and Indian. This is easy to see if we consider a specific example. If there are 10 friends deciding on dinner and 6 people want Indian and 6 people want Italian, then obviously at least someone wants both Indian and Italian[3]. This reasoning is captured by the following axiom scheme

$$M\alpha \wedge M\beta \to \Diamond(\alpha \wedge \beta)$$

Now, suppose that more than half of the friends want Italian for dinner. Also, suppose that every time the group eats Italian where wine is always served with Italian food. We can conclude that a majority of the friends want wine with dinner. And so, we will included the following axiom scheme

$$M\alpha \wedge \Box(\alpha \to \beta) \to M\beta$$

Suppose that you are put in charge of making dinner reservations for the group of 10 people. Given that 5 people prefer Italian and 5 people prefer Indian, what can you conclude if you are given additional information that more that 3 people do not like Italian and do not like Indian. The natural conclusion to draw is that more than 3 people like Indian and Italian food. Otherwise, say you conclude that only two people like both Indian and Italian food. This would mean that 3 people like Italian but not Indian, 3 people like Indian but not Italian and (more than) 3 people like neither Indian nor Italian. Since these sets are disjoint, the total sum of people is 11 or more, and so it must be the case that more than 3 people like Indian and Italian. This line of reasoning is captured by the following axiom scheme

$$W\alpha \wedge W\beta \wedge \Diamond_n(\neg\alpha \wedge \neg\beta) \to \Diamond_n(\alpha \wedge \beta) \quad (n \in \mathbb{N})$$

The final situation is similar to the above situation. except suppose that a majority of the people prefer Italian.

$$W\alpha \wedge M\beta \wedge \Diamond_n(\neg\alpha \wedge \neg\beta) \to \Diamond_{n+1}(\alpha \wedge \beta) \quad (n \in \mathbb{N})$$

The preceding discussion is summarized by the following list of axioms and rules.

Axiom 1 *Classical propositional tautologies*

Axiom 2 $\Diamond_{n+1}\alpha \to \Diamond_n\alpha \ \ (n \in \mathbb{N})$

Axiom 3 $\Box(\alpha \to \beta) \to (\Diamond_n\alpha \to \Diamond_n\beta) \quad (n \in \mathbb{N})$

Axiom 4 $\Diamond!_0(\alpha\wedge\beta) \to ((\Diamond!_{n_1}\alpha\wedge\Diamond!_{n_2}\beta) \to \Diamond!_{n_1+n_2}(\alpha\vee\beta)) \ \ (n_1, n_2 \in \mathbb{N})$

Axiom 5 $M\alpha \wedge M\beta \to \Diamond(\alpha \wedge \beta)$

Axiom 6 $M\alpha \wedge \Box(\alpha \to \beta) \to M\beta$

Axiom 7 $W\alpha\wedge W\beta\wedge\Diamond_n(\neg\alpha\wedge\neg\beta) \to \Diamond_n(\alpha\wedge\beta) \quad (n \in \mathbb{N})$

Axiom 8 $W\alpha\wedge M\beta\wedge\Diamond_n(\neg\alpha\wedge\neg\beta) \to \Diamond_{n+1}(\alpha\wedge\beta) \quad (n \in \mathbb{N})$

MP From α and $\alpha \to \beta$ derive β.

NEC From α derive $\Box\alpha$.

We write $\vdash_{\mathbf{MJL}} \alpha$ if α can be deduced from Axioms 1 - 8 using the rules MP and N. If it is clear from context, we may write $\vdash \alpha$ instead of $\vdash_{\mathbf{MJL}} \alpha$.

Properties of the Axioms

We now discuss some of the properties of the axioms proposed in the previous section. Some of these properties turn out to be useful in the completeness proof and others are natural properties of majorities and weak majorities.

This first lemma gives some consequences of the proposed axioms. The lemma also shows that our axiomatization captures many natural properties of "majority" and "weak majority". Part (i) shows M and W are both normal modal operators. Part (ii) is equivalent to saying that given any set X and any subset of X either it or its complement (or both) constitutes weak majority of X. (iii) - (viii) are obvious properties of majority and weak majority sets.

Lemma 2 *Suppose that α and β are arbitrary formulas of **MJL**. Then*

i. If $\vdash \alpha \to \beta$ then $\vdash M\alpha \to M\beta$ and $\vdash W\alpha \to W\beta$.
ii. $\vdash W\alpha \vee W\neg\alpha$
iii. $\vdash M\alpha \to W\alpha$ and $\vdash M\alpha \to \Diamond\alpha$
iv. $\vdash \neg M\bot$
v. $\vdash \Box\alpha \to W\alpha$
vi. $\vdash M\alpha \wedge W\beta \to \Diamond(\alpha \wedge \beta)$
vii. $\vdash W\alpha \wedge \Diamond(\neg\alpha \wedge \beta) \to M(\alpha \vee \beta)$
viii. $\vdash W\alpha \wedge W\beta \wedge \neg\Diamond(\alpha \wedge \beta) \to \neg\Diamond(\neg\alpha \wedge \neg\beta)$

Proof Suppose that α and β are any formulas of **MJL**.

i. Suppose that $\vdash \alpha \to \beta$. We will show $\vdash M\alpha \to M\beta$ and $\vdash W\alpha \to W\beta$. By the NEC, $\vdash \Box(\alpha \to \beta)$, and so by modus ponens and Axiom 6, $\vdash M\alpha \to M\beta$. $\vdash W\alpha \to W\beta$ follows easily using contraposition.
ii. $\vdash \neg\Diamond(\alpha \wedge \neg\alpha) \to \neg(M\neg\alpha \wedge M\alpha)$ is an instance of Axiom 5. Hence $\vdash \Box\top \to (W\alpha \vee W\neg\alpha)$. By NEC $\vdash \Box\top$. Therefore by MP, $\vdash W\alpha \vee W\neg\alpha$.
iii. $\vdash M\alpha \to W\alpha$ follows from (ii), and $\vdash M\alpha \to \Diamond\alpha$ is an instance of Axiom 5.
iv. Using (iii) we get $\vdash \neg\Diamond\bot \to \neg M\bot$, and hence $\vdash \neg M\bot$.
v. Using axiom 5 $\vdash \neg\Diamond(\neg\alpha \wedge \neg\alpha) \to \neg(M\neg\alpha \wedge M\neg\alpha)$ which is $\vdash \Box\alpha \to W\alpha$
vi. Note by axiom 6 we have $\vdash M\alpha \wedge \Box(\alpha \to \neg\beta) \to M\neg\beta$. So we have $\vdash M\alpha \wedge W\beta \to \neg\Box(\alpha \to \neg\beta)$ and so using propositional calculus $\vdash M\alpha \wedge W\beta \to \Diamond(\alpha \wedge \beta)$
vii. $W\alpha \wedge \Diamond(\neg\alpha \wedge \beta) \to W\alpha \wedge \Diamond(\neg\alpha \wedge (\alpha \vee \beta)) \wedge \Box\neg(\alpha \wedge \neg(\alpha \vee \beta))$ but we have $W\alpha \wedge \Diamond(\neg\alpha \wedge (\alpha \vee \beta)) \wedge \Box\neg(\alpha \wedge \neg(\alpha \vee \beta)) \to M(\alpha \vee \beta)$

[3] Of course two people could have said that they don't care where they eat. This is not the same as two people saying that they want *both* Italian and Indian, but for the purposes of this example we will assume that a "don't care" vote is a vote for both options

viii. By lemma 2-(vi) as a substitution instance, we get $\vdash W\alpha \wedge M(\beta \vee \neg\alpha) \rightarrow \Diamond(\alpha \wedge \beta)$ which is equivalent to $\vdash W\alpha \wedge \neg\Diamond(\alpha \wedge \beta) \rightarrow \neg M(\beta \vee \neg\alpha)$, but the above item of the lemma, $\vdash W\beta \wedge \Diamond(\neg\alpha \wedge \neg\beta) \rightarrow M(\beta \vee \neg\alpha)$. Thus we get $\vdash W\beta \wedge W\alpha \wedge \neg\Diamond(\alpha \wedge \beta) \rightarrow W\beta \wedge \neg M(\beta \vee \neg\alpha)$ and hence $\vdash W\beta \wedge \neg M(\beta \vee \neg\alpha) \rightarrow \neg\Diamond(\neg\alpha \wedge \neg\beta)$

□

Using the language of graded modal logic, we can find a formula that expresses exactly how many worlds are accessible at any given state. For any $n \in \mathbb{N}$, the formula $\Diamond_n!\top$ will be true at some world w iff there are exactly n accessible worlds. Similarly, the formulas $\Box_n\bot$ will be true at some world w iff there are *at most* n accessible worlds. We will define $\mathbf{A_n} := \Diamond_n!\top$, $\mathbf{A_{\leq n}} := \Box_n\bot$ and $\mathbf{A_{>n}} := \Diamond_n\top$. So, $\mathbf{A_n}$ is true at a state s if there are exactly n accessible worlds. The following lemma show what happens when we know the exact number of accessible worlds.

Lemma 3 *Suppose that $\mathbf{A_n}$ is the formula defined above. Then*

i. $\vdash \mathbf{A_n} \rightarrow (\Box_{\lfloor n/2\rfloor}\alpha \vee \Box_{\lfloor n/2\rfloor}\neg\alpha)$ *For all* $n \in \mathbb{N}$

ii. $\vdash \mathbf{A_n} \rightarrow (\Box_{\lfloor n/2\rfloor-1}\alpha \rightarrow \Diamond_{\lfloor n/2\rfloor}\alpha)$ *For all* $n > 2$

iii. *For all* $n \in \mathbb{N}$, $\vdash \mathbf{A_n} \rightarrow (M\alpha \leftrightarrow \Diamond_{\lfloor n/2\rfloor}\alpha)$.

Proof Part (i) and (ii) are statements of graded modal logic, and given the completeness and soundness proofs in (Caro 1988; Fattorosi-Barnaba & Cerrato. 1988), follow easily from semantic arguments. We first note the following properties which are instances of Axioms 8 and 7 respectively (let $\beta = \alpha$).

1. $\vdash M\alpha \wedge \Diamond_n\neg\alpha \rightarrow \Diamond_{n+1}\alpha$
2. $\vdash W\alpha \wedge \Diamond_n\neg\alpha \rightarrow \Diamond_n\alpha$

We need only show property (iii).

- Suppose that $\vdash \mathbf{A_n} \wedge \Diamond_{\lfloor n/2\rfloor}\alpha$. By part (i) of this lemma we get $\vdash \mathbf{A_n} \wedge \Diamond_{\lfloor n/2\rfloor}\alpha \rightarrow \Box_{\lfloor n/2\rfloor}\alpha$ and by (2) we get $\vdash \mathbf{A_n} \wedge \Diamond_{\lfloor n/2\rfloor}\alpha \wedge \Box_{\lfloor n/2\rfloor}\alpha \rightarrow M\alpha$
- Suppose that $\vdash \mathbf{A_n} \wedge M\alpha$. If $n = 0$ or $n = 1$ then we have $M\alpha \rightarrow \Diamond\alpha$ so we get $\Diamond_{\lfloor n/2\rfloor}\alpha$. Assume $n > 2$ by (1) we have $M\alpha \rightarrow (\Diamond_{\lfloor n/2\rfloor}\alpha \vee \Box_{\lfloor n/2\rfloor-1}\alpha)$. Using part (ii) of this lemma we get $\vdash \mathbf{A_n} \wedge M\alpha \rightarrow \Diamond_{\lfloor n/2\rfloor}\alpha$

□

Majority Logic: Semantics

In this section we will present the semantics for **MJL**. As usual, formulas of **MJL** will be interpreted over Kripke models. The formula $W\alpha$ will be true provided that the set of all accessible worlds in which α is true is a majority of the set of all accessible worlds. The definition makes sense only if there are *finitely* many accessible worlds. But what constitutes a majority of an infinite set? The following section offers a solution to this question.

Recall that if S is any set of states and R a binary relation on S, then $R(s) = \{t \mid sRt\}$ and for any formula α (of **MJL** or **GML**), $R_\alpha(s) = \{t \mid sRt \text{ and } t \models \alpha\}$. This definition of course depends on the definition of truth in a model which is given below.

Majority Spaces

A very interesting situation arises when a Kripke model is not finitely generated, that is when $R(s)$ may be infinite for some state $s \in S$. While the semantics of a majority subset is very clear in the finite case, it is not clear what should constitute a majority when there are an infinite number of possibilities. We cannot for example stipulate that every infinite set is a (strict) majority. This would create the unsatisfactory situation where a set and it's complement could be a majority.

Another natural choice would be to call a set $X \subseteq R(s)$ a majority if X^C is finite, i.e take the majority sets to be the co-finite sets. However, suppose that $R(s) = X_1 \cup X_2 \cup X_3$, where X_1, X_2, and X_3 are nonempty pairwise disjoint sets Then one would expect that for some i and j where $i \neq j$, $X_i \cup X_j$ would be a majority. This is certainly true in the finite case, and so one would expect it to be true in the infinite case. However, it is easy to come up with an example where all of the X_i are infinite; and so, none of the $X_i \cup X_j$ would be a majority.

Instead of trying to define a majority set as some special subset of $R(s)$, we will let a model stipulate which sets are to be considered a majority.

Definition 2 *Let W be any set. We will call any set $\mathfrak{M} \subseteq 2^W$ a* **majority system** *if it satisfies the following properties.*

M1. *If $X \subseteq W$, then either $X \in \mathfrak{M}$ or $X^C \in \mathfrak{M}$.*

M2. *If $X \in \mathfrak{M}$, $Y \in \mathfrak{M}$ and $X \cap Y = \varnothing$, then $Y = X^C$.*

M3. *Suppose that $X \in \mathfrak{M}$ and $F \subseteq X$ is any finite set. If G is any set where $G \cap X = \varnothing$ and $|F| \leq |G|$, then $(X - F) \cup G \in \mathfrak{M}$.*

The pair $\langle W, \mathfrak{M}\rangle$ will be called a **weak majority space**. Given a set W, a set $X \subseteq W$ will be called a **strict majority** (with respect to $\mathfrak{M}$) if $X \in \mathfrak{M}$ and $X^C \notin \mathfrak{M}$. X will be called a **weak majority** if $X \in \mathfrak{M}$ and $X^C \in \mathfrak{M}$. We need to check that the above properties correspond to our intuitions about majority sets. We will call any set $X \in \mathfrak{M}$ a **majority set**.

It is easy to see that majority spaces are closed under superset. We show that many of the intuitions we have about majority sets on a finite space remain in a majority space. For example, we show that given any majority set X, if we add something new to X, then this new formed set will be a strict majority. We also show that if a set W is infinite, then all majority sets must also be infinite.

Lemma 4 *If X is a weak majority and $F \neq \varnothing$ is a set such that $F \not\subseteq X$, then $X \cup F$ is a strict majority.*

Proof Suppose that X is a weak majority and $F \neq \varnothing$ is any set such that $F \not\subseteq X$. Notice first that since $X \in \mathfrak{M}$ and $X \subseteq X \cup F$, $X \cup F \in \mathfrak{M}$ by fact **??** (this is true for any set F). We need only show that $(X \cup F)^C \notin \mathfrak{M}$. Suppose that $(X \cup F)^C \in \mathfrak{M}$. By property M2, since $X \in \mathfrak{M}$, $(X \cup F)^C \in \mathfrak{M}$ and $X \cap (X \cup F)^C = \varnothing$, we must have $(X \cup F)^C = X^C$ which implies $F \subseteq X$. But this contradicts the assumption that $F \not\subseteq X$.

□

Lemma 5 *Suppose that $\langle W, \mathfrak{M} \rangle$ is a majority space and that W is infinite. If $X \in \mathfrak{M}$ then X is infinite.*

Proof Suppose that $\langle W, \mathfrak{M} \rangle$ is a majority space and W is infinite. Suppose that $X \subseteq W$ is finite and $X \in \mathfrak{M}$. Note that since X is finite, X^C is infinite. Take any finite set $G \subset X^C$, where $|X| \leq |G|$ (such a set must exist since W is infinite). Then by property M3, $(X - X) \cup G = G \in \mathfrak{M}$; and so, by property M2, $G = X^C$. But this is a contradiction since G is finite and X^C is infinite.

□

This last proposition demonstrates that our notion of an infinite majority is equivalent to the natural notion of a majority when we only have a finite number of elements. In other words, we will show that if W is a finite set, then the majority sets are the sets that have more than or equal to half of the elements. We will also show that when W is finite, the sets that can be called a majority (i.e. satisfy properties M1-M3) are the sets that have more than or equal to half of the elements.

Proposition 6 *Suppose that W is a finite set and that $\mathfrak{M}' = \{M \subseteq W \ : \ |M| \geq |W|/2\}$, Then*

$$\langle W, \mathfrak{M}' \rangle \text{ is a majority space}$$

Furthermore, if $\langle W, \mathfrak{M} \rangle$ is any other majority space then $\mathfrak{M} = \mathfrak{M}'$.

Proof Suppose that W is a finite set and $\mathfrak{M}'$ is as defined above. We must first show that $\langle W, \mathfrak{M}' \rangle$ is a majority space. For any set, $X \subseteq W$, since $|X| + |X^C| = |W|$, either $|X| \geq |W|/2$ or $|X^C| \geq |W|/2$ and so either $X \in \mathfrak{M}'$ or $X^C \in \mathfrak{M}'$. Hence property M1 is satisfied. For property M2, suppose that $X, Y \in \mathfrak{M}'$, and $X \cap Y = \varnothing$. Since $|X| \geq |W|/2$ and $|Y| \geq |W|/2$, $|X| + |Y| \geq |W|$. But since $X \cup Y \subseteq W$, $|X \cup Y| \leq |W|$ and so $|X \cup Y| = |W|$. Therefore, $X \cup Y = W$ (this follows since X and Y are assumed to be subsets of W). Since X and Y are disjoint and $X \cup Y = W$, then $Y = X^C$. Finally we need to show that property M3 is satisfied. Suppose that $X \in \mathfrak{M}'$. Then $|X| \geq |W|/2$. Suppose that $F \subseteq X$ and G is any finite set such that $|F| \leq |G|$ and $G \cap X = \varnothing$. Then

$$\begin{aligned} |(X-F) \cup G| &= |(X-F)| + |G| - |(X-F) \cap G| \\ &= |X - F| + |G| \\ &\geq |X - F| + |F| \\ &= |X \cup F| = |X| \geq |W|/2 \end{aligned}$$

so, $(X - F) \cup G \in \mathfrak{M}'$.

Let $\langle W, \mathfrak{M} \rangle$ be any majority space and let $X \in \mathfrak{M}$. We must now show that $|X| \geq |W|/2$. Suppose not, that is suppose that $|X| < |W|/2$. Therefore $|X^C| > |X|$. Let $Y \subseteq X^C$ and $|Y| = |X|$ (such a set must exist since $|X^C| > |X|$). Then by property M3, since $|X| \leq |Y|$ and $Y \cap X = \varnothing$, $(X - X) \cup Y = Y \in \mathfrak{M}$. But by property M2, $Y = X^C$. But this is a contradiction, since $|X| < |W|/2$ and $|Y| < |W|/2$. Hence, $|X| \geq |W|/2$.

□

Example of a majority space In this section we provide concrete example of a majority space. Furthermore, we show that this majority space is not an ultrafilter.

Given a set W, a filter $\mathcal{F}$ is any non-empty collection of subsets of W that is closed under intersection and superset. A filter is an ultrafilter if for all sets A either $A \in \mathcal{F}$ or $A^C \in \mathcal{F}$. Finally, $\mathcal{F}$ is principal if $\mathcal{F}$ contains a singleton, and $\mathcal{F}$ is non-principal if it is not principal (hence contains all cofinite sets). Given any infinite set, Zorn's lemma implies the existence of a non-principal ultrafilter.

Let X_1, X_2, X_3 be three disjoint sets such that each X_i is infinite. Let $\mathfrak{U}_i$ be a non-principal ultrafilter over X_i. Now Let $X = X_1 \cup X_2 \cup X_3$. Define

$$\mathfrak{M} = \{x | \exists i \neq j \text{ such that } x \cap X_i \in \mathfrak{U}_i \text{ and } x \cap X_j \in \mathfrak{U}_j\}$$

We claim that $(X, \mathfrak{M})$ is a majority space. Here is the proof:

M1 Take $x \subseteq X$. Assume $x \notin \mathfrak{M}$ then assume without loss of generality that $x \cap X_1 \notin \mathfrak{U}_1$ and $x \cap X_2 \notin \mathfrak{U}_2$. Then from the definition of the ultrafilter we have: $x^C \cap X_1 \in \mathfrak{U}_1$ and $x^C \cap X_2 \in \mathfrak{U}_2$. So $x^C \in \mathfrak{M}$

M2 Let $x \in \mathfrak{M}, y \in \mathfrak{M}$ and $x \cap y = \varnothing$ Since we only have three sets X_1, X_2, X_3 then there is i such that $x \cap X_i \in \mathfrak{U}_i$ and $y \cap X_i \in \mathfrak{U}_i$. So $x \cap y \in \mathfrak{U}_i$ and thus $\varnothing \in \mathfrak{U}_i$ which is a contradiction.

M3 $x \in \mathfrak{M}$ and Let $y = (x - f) \cup g$ where f in a finite subset of x and $|f| \leq |g|$. The proof goes easily using two facts:

Fact 1 if $x_i \in \mathfrak{U}_i$ then for any finite set $f \in X_i$ we have $x_i - f \in \mathfrak{U}_i$.
proof: suppose $(x_i - f) \notin \mathfrak{U}_i$ then $X_i - (x_i - f) \in \mathfrak{U}_i$. So $(X_i - (x_i - f)) \cap x_i \in \mathfrak{U}_i$ and that is $f \in \mathfrak{U}_i$ which a contradiction since $\mathfrak{U}_i$ is a non-principal ultrafilter.

Fact 2 if $x_i \in \mathfrak{U}_i$ then for any $g \in X_i$ we have $x_i \cup g \in \mathfrak{U}_i$

Hence $(X, \mathfrak{M})$ is a majority space. Notice that $X_1 \cup X_2 \in \mathfrak{M}$ and $X_2 \cup X_3 \in \mathfrak{M}$ but their intersection $X_2 \notin \mathfrak{M}$. So $\mathfrak{M}$ is not an ultrafilter over X. It should be clear that this example can be generalized to any odd number of disjoint sets.

Majority Models

In this section we will extend the definition of a Kripke model in order to define truth of a majority logic formula.

Definition 3 *A* **majority model** *is a tuple $\mathcal{M} = \langle S, R, V, m \rangle$. Where S is any set of states, R is an accessibility relation and V is the valuation function $V : \mathbb{P} \rightarrow 2^S$, and $m : S \rightarrow 2^{2^S}$ is a* **majority function** *such that for each $s \in S$, $\langle R(s), m(s) \rangle$ is a majority space.*

So, m assigns a majority space to each state. Let $s \in S$ be any state. We will define truth of a formula α at state s in model $\mathcal{M}$ as follows:

1. $\mathcal{M}, s \models p$ iff $s \in V(p)$, where $p \in \mathbb{P}$
2. $\mathcal{M}, s \models \neg\alpha$ iff $\mathcal{M}, s \not\models \alpha$
3. $\mathcal{M}, s \models \alpha \vee \beta$ iff $\mathcal{M}, s \models \alpha$ or $\mathcal{M}, s \models \beta$
4. $\mathcal{M}, s \models \Diamond_n \alpha$ iff $|R_\alpha(s)| > n \quad (n \in \mathbb{N})$
5. $\mathcal{M}, s \models W\alpha$ iff $R_\alpha(s) \in m(s)$

And so $\mathcal{M}, s \models M\alpha$ iff $R_{\neg\alpha}(s) \notin m(s)$. First notice that if $R(s)$ is finite for each $s \in S$, then by proposition 6, then $\mathcal{M}, s \models W\alpha$ iff $|R_\alpha(s)| \geq |R(s)|/2$. We will now show that the axioms of majority logic are valid in all majority models.

Theorem 7 ***MJL*** *is sound with respect to the class of all majority models.*

Proof Soundness was shown in (**?**) for axioms 1 - 4, MP, and Nec. Let $\mathcal{M} = \langle S, R, V, m\rangle$ be any majority model and $s \in S$. We will show Axiom 5 - 8 are true at state s. Since s is arbitrary, each axiom will be valid in $\mathcal{M}$; and hence, the axioms are sound. All of the proofs are straightforward and are left to the reader. As an example, we show the result holds for Axiom 7 and 8.

Axiom 7: Assume $s \models W\alpha \wedge W\beta \wedge \Diamond_n(\neg\alpha \wedge \neg\beta)$ so we have $R_\alpha(s) \in m(s), R_\beta(s) \in m(s)$ and $|R_{\neg\alpha\wedge\neg\beta}(s)| > n$ we need to prove that $|R_{\alpha\wedge\beta}(s)| > n$. Assume $|R_{\alpha\wedge\beta}(s)| \leq n$. Let $X \subset R_{\neg\alpha\wedge\neg\beta}(s)$ where $|X| = n$ (X is a proper subset). Let $Y = (R_\beta(s) - R_{\alpha\wedge\beta}(s)) \cup X$ according to M3 $Y \in m(s)$ and we have $Y \cap R_\alpha(s) = \varnothing$ and $Y \neq R^C_\alpha(s)$ which is a contradiction with M2. So $|R_{\alpha\wedge\beta}(s)| > n$ and thus $s \models \Diamond_n(\alpha \wedge \beta)$

Axiom 8: Assume $s \models W\alpha \wedge M\beta \wedge \Diamond_n(\neg\alpha \wedge \neg\beta)$ so we have $R_\alpha(s) \in m(s), R_{\neg\beta}(s) \notin m(s)$ and $R_{\neg\alpha\wedge\neg\beta}(s) > n$ we need to prove that $|R_{\alpha\wedge\beta}(s)| > n + 1$. Assume $|R_{\alpha\wedge\beta}(s)| \leq n + 1$. But $R_{\neg\beta}(s) = (R_\alpha(s) - R_{\alpha\wedge\beta}(s)) \cup R_{\neg\alpha\wedge\neg\beta}(s)$ and by M3 we get $R_{\neg\beta}(s) \in m(s)$ which is a contradiction.

□

Completeness

We adapt the proof of (Fattorosi-Barnaba & Cerrato. 1988) to show completeness for **MJL**. This section will sketch the completeness proof.

Given any consistent set of formulas of majority logic, Γ, using Lindenbaum's Lemma, we can construct a maximally consistent superset of Γ. As usual, the states of our canonical model will be maximally consistent sets. In what follows, Γ will always be assumed to be a maximally consistent set of formulas.

When constructing a canonical model for a graded modal logic, it is necessary to control the number of worlds accessible from any given state. Given any state, i.e. maximally consistent set, Γ, our goal is to construct $R(\Gamma)$ such that

$$\Diamond_n\alpha \in \Gamma \text{ iff } |\{\Gamma' \in R(\Gamma) \mid \alpha \in \Gamma'\}| > n$$

We will construct a satisfying family for each Γ, denoted by $SF(\Gamma)$ so that we may define $R(\Gamma) = SF(\Gamma)$ and then R will satisfy the above property. To this end we will present the following definitions and lemmas from (Caro 1988). Recall that ω is the first countable ordinal, and that $\omega{+}1 = \omega \cup \{\omega\}$. Let Φ be the set of all maximally consistent sets.

Definition 4 *The function* $\mu : \Phi \times \Phi \to \omega + 1$ *is defined as follows: for every* $\Gamma_1, \Gamma_2 \in \Phi$

$$\mu(\Gamma_1, \Gamma_2) = \omega \textit{ if for any } \alpha \in \Gamma_2, \Diamond_n\alpha \in \Gamma_1 \textit{ for all } n \in \mathbb{N}$$
$$\mu(\Gamma_1, \Gamma_2) = \min\{n \in \mathbb{N} : \Diamond!_n\alpha \in \Gamma \textit{ and } \alpha \in \Gamma_2\} \textit{ o.w.}$$

The function μ is well defined (refer to (Fattorosi-Barnaba & Cerrato. 1988; Caro 1988) for more about this function). The following lemma is an easy consequence of definition 4

The main idea is that μ will tell us how many accessible worlds are needed. Given two maximally consistent sets, Γ_1, Γ_2, $\mu(\Gamma_1, \Gamma_2)$ tells us the minimum number of copies of Γ_2 that are needed to be accessible from Γ_1.

We are now ready to define the **satisfying family** of a maximally consistent set Γ_0.

Definition 5 *Let* $\Gamma_0 \in \Phi$. *The set*

$$SF(\Gamma_0) = \bigcup\{\{\Gamma\} \times \mu(\Gamma_0, \Gamma) : \Gamma \in \Phi\}$$

will be called the satisfying family of Γ_0.

An element of $SF(\Gamma_0)$ is of the form $\langle\Gamma, n\rangle$ where $n < \mu(\Gamma_0, \Gamma)$, therefore we shall think of $SF(\Gamma_0)$ as made up of $\mu(\Gamma_0, \Gamma)$ ordered copies of Γ, for any $\Gamma \in \Phi$.

The following theorem is the main theorem from (Caro 1988).

Theorem 8 *For any* α *and any* $n \in \mathbb{N}$,

$$\Diamond_n\alpha \in \Gamma_0 \textit{ iff } |\{\Gamma \in SF(\Gamma_0) : \alpha \in \Gamma\}| > n$$

where to simplify notations, we identify a couple $\langle\Gamma, n\rangle$ *(*$n < \mu(\Gamma_0, \Gamma)$*) with its first component.*

We can now define the canonical for majority logic. We define the canonical model $\mathcal{M}^\Lambda = \langle S^\Lambda, R^\Lambda, V^\Lambda, m^\Lambda\rangle$ for **MJL** as follows, where Λ is a consistent set of majority logic formulas. First of all, let

$$\mu(\Gamma) = \sup\{\mu(\Gamma', \Gamma) \mid \Gamma' \in \Phi\}$$

So $\mu(\Gamma)$ gives the maximum number of copies of Γ that needed in the canonical model. Define

$$S^\Lambda = \bigcup\{\{\Gamma\} \times \mu(\Gamma) \mid \Gamma \in \Phi\} \cup \{\langle\Gamma, 0\rangle \mid \mu(\Gamma) = 0\}$$

So we may think of S^Λ as made up of $\mu(\Gamma)$ copies of Γ if $\mu(\Gamma) \neq 0$, and by one copy of Γ if $\mu(\Gamma) = 0$, for any maximally consistent set Γ.

For each $\langle\Gamma, i\rangle \in S^\Lambda$ define,

$$R^\Lambda(\langle\Gamma, i\rangle) = SF(\Gamma)$$

and for every proposition p and every $\langle\Gamma, i\rangle \in S^\Lambda$ we set:

$$V^\Lambda(p) = \{\langle\Gamma, i\rangle \mid p \in \Gamma\}$$

We need only define a majority function $m^\Lambda : S^\Lambda \to 2^{2^{S^\Lambda}}$. In what follows we will write Γ instead of $\langle\Gamma, i\rangle \in S^\Lambda$. This abuse of notation should not cause any confusion and so will be used to simplify the presentation.

Let $R^\Lambda_\alpha(\Gamma) = SF_\alpha(\Gamma) = \{\Gamma' : \Gamma' \in SF(\Gamma) \text{ and } \alpha \in \Gamma'\}$. We are ready to define $m^\Lambda(\Gamma)$ so that $\langle R^\Lambda(\Gamma), m^\Lambda(\Gamma)\rangle$ is a majority space.

Given any maximally consistent set Γ, it is easy to see that exactly one of the following cases must be true:

1. $\Diamond!_n\top \in \Gamma$ for some $n \in \mathbb{N}$
2. $\Diamond_n\top \in \Gamma\ \forall n \in \mathbb{N}$

If we are in Case 1, then $|SF(\Gamma)| = n$, and so we can define

$$m^{\Lambda}(\Gamma) = \{X \ : \ X \subseteq SF(\Gamma) \text{ and } |X| \geq \lceil |SF(\Gamma)|/2 \rceil\}$$

By Proposition 6, $\langle R(\Gamma), m^{\Lambda}(\Gamma)\rangle$ is a majority space.

So suppose that we are in case 2, that is for all $n \in \mathbb{N}$, $\Diamond_n\top \in \Gamma$. We need some definitions before we proceed.

Definition 6 *Let Y be any set and $X \subseteq 2^Y$. Then define*

$$X^f = \{A \mid \exists B \in X \textit{ such that } A = (B - F) \cup G$$
$$\textit{where } F \textit{ is finite, } |F| \leq |G| \textit{ and } X \cap G = \varnothing\}$$

So, X^f is X **closed under finite perturbations**. It is easy to see that $X \subseteq X^f$ (take F and G both to be empty).

Definition 7 *Let Y be any set and $X \subseteq 2^Y$, then define*

$$\overline{X} = \{A \subseteq Y \ : \ A \notin X \textit{ and } A^C \in X\}$$

Note that $A \in X \cup \overline{X}$ iff $A^C \in X \cup \overline{X}$.

Let Γ be any maximally consistent set. We will now construct $m^{\Lambda}(\Gamma)$:

1. Define $\mathfrak{M}_0(\Gamma) = \{SF_{\alpha}(\Gamma) \mid W\alpha \in \Gamma\}$
2. Define $\mathfrak{M}_1(\Gamma) = (\mathfrak{M}_0(\Gamma))^f$. That is take $\mathfrak{M}_0(\Gamma)$ and close off under finite perturbations.
3. Let $\mathcal{O} = SF(\Gamma) - (\mathfrak{M}_1(\Gamma) \cup \overline{\mathfrak{M}_1(\Gamma)})$. The set $\mathcal{O}$ contains the "other" sets. That is the sets X such that neither X nor X^C have made it into $\mathfrak{M}_1(\Gamma)$. In order to satisfy $M1$, we must pick one of X or X^C to be elements of the majority space. These choices must be made in a way that is consistent with the properties $M1 - M3$. Let $\mathcal{U}$ be any non-principal ultrafilter over $SF(\Gamma)$. Define

$$m^{\Lambda}(\Gamma) = \mathfrak{M}_1(\Gamma) \cup (\mathcal{O} \cap \mathcal{U})$$

The following lemma shows that the construction above gives us what we want, namely a way to construct a majority space at each state in the canonical model.

Lemma 9 *Given any maximally consistent set Γ, $\langle R^{\Lambda}(\Gamma), m^{\Lambda}(\Gamma)\rangle$ is a majority space.*

We now have enough to prove the main truth lemma. The proof of the main truth lemma will be as usual, with the only interesting case being the following lemma.

Lemma 10 *For any maximally consistent set Γ and any formula α of **MJL**,*

$$R^{\Lambda}_{\alpha}(\Gamma) \in m^{\Lambda}(\Gamma) \textit{ iff } W\alpha \in \Gamma$$

Given the previous lemmas, the truth lemma follows easily:

Lemma 11 (Truth Lemma) *For any formula α and any $\Gamma \in S^{\Lambda}$ we have*

$$\mathcal{M}^{\Lambda}, \Gamma \models \alpha \textit{ iff } \alpha \in \Gamma$$

Given the truth lemma for **GML** and **MJL**, the completeness theorem follows using a standard argument.

Theorem 12 (Canonical Model Theorem for MJL) *if Λ is a consistent majority logic then:*

$$\alpha \in \Lambda \textit{ iff } \alpha \textit{ is valid in } \mathcal{M}_{\Lambda} \textit{ (for any } \alpha)$$

Conclusion and Future Work

We have extended graded modal logic with an operator W that can express the concept of weak majority. In order to interpret W in a Kripke structure, we defined a majority space. A majority space extends the well-defined concept of a majority of a finite set to an infinite set. A axiom system was presented and the proof of soundness and completeness was sketched.

Along the way, we looked at how to define the majority of an infinite set. Instead of trying to find a naturally occurring definition, we define a majority space which gives a lot of room in the definition of a majority subset of an infinite set. Thus if asked if the even numbers ($\mathbb{E}$) are a strict majority or a weak majority of the natural numbers ($\mathbb{N}$), we would answer that it depends on what is being modelled. On the one hand, it seems clear that $\mathbb{E}$ is a weak majority of $\mathbb{N}$. However, consider the following sequence of sets: $\{0, 2, 1\}, \{0, 2, 4, 1, 3\}, \{0, 2, 4, 6, 1, 3, 5\}, \ldots$. The first set has a strict majority of even numbers, and since each new set adds only one even number and one odd number, every element of this sequence has a strict majority of even numbers. The limit of this sequence is $\mathbb{N}$; and so if we think of $\mathbb{N}$ as being "constructed" by this sequence of sets, one would expect that $\mathbb{E}$ is a *strict* majority.

The main technical question is the decidability of **MJL**. Since it was shown in (Caro 1988) the graded modal logic has the finite model property, we expect that **MJL** will share this property.

We also point out that we cannot express the statement "among the worlds in which α is true, β is a majority " in our language. Such statements are often used when reasoning about candidates in an election. For example, among the Democratic registered voters, Kerry has the majority of their votes. We would like to extend the language of majority logic with an operator that can express such statements. A step in this direction would be to introduce a binary modality $\leq$, in which the intended meaning of $\alpha \leq \beta$ is α is true in "less" states than β.

Finally, we point to some possible applications of our logic. Although, the primary interest of this paper is technical, we feel that our framework can be used to reason about social software (see (Parikh 2002) for more information) such as voting systems (Brams & Fishburn 2002). This line of research will be pursued in a different paper.

Acknowledgments

Both authors would like to thank Rohit Parikh for many useful comments and suggesting that this paper be written. We also thank the Knowledge, Games and Beliefs group at CUNY for allowing us to present this work at various stages of its development. Finally, we thank Joel Hamkins, Noson Yanofsky and anonymous referees for very useful comments and suggestions.

References

Blackburn, P.; de Rijke, M.; and Venema, Y. 2001. *Modal Logic*. Cambridge University.

Brams, S., and Fishburn, P. 2002. voting procedures. *Handbook of social science and welfare* 1.

Caro, F. D. 1988. Graded modalities, ii (canonical models). *Studia logica* 47:1–10.

de Rijke, M. 2000. A note on graded modal logic. *Studia Logica* 64:271–283.

Fattorosi-Barnaba, M., and Cerrato., C. 1988. Graded modalities i. *Studia Logica* 47:99–110.

Fey, M. 2002. Mey's theorem with an infinite population. Research report, University of Rochester.

Fine, K. 1972. In so many possible worlds. *Notre Dame Journal of formal logic* 13(4):516–520.

Goble, L. 1970. Grades of modality. *Logique et Analyse* 13:323–334.

H. J. Ohlbach, R. A. S., and Hustadt, U. 1995. Proof theory and modal logic. 245–285.

Parikh, R. 2002. Social software. *Synthese* 132:187–211.

Tobies, S. 2001. Pspace reasoning for graded modal logics. *Journal of Logic and Computation* 11(1):85–106.