A Polynomial-time Algorithm for Constructing \( k \)-Maintainable Policies

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Abstract

In this paper we present a polynomial time algorithm for constructing \( k \)-maintainable policies (Nakamura, Baral, & Bjareland 2000). Our algorithm, in polynomial time, constructs a \( k \)-maintainable control policy, if one exists, or tells that no such policy is possible. Our algorithm is based on SAT Solving, and employs a suitable formulation of the existence of \( k \)-maintainable control in a fragment of SAT which is tractable. We then give a logic programming implementation of our algorithm and use it to give a standard procedural algorithm. We then present several complexity results about constructing \( k \)-maintainable controls, under different assumptions such as \( k = 1 \), and compact representation.

Introduction and Motivation

Consider an agent who is assigned the goal of "maintaining a room clean". There are various possible interpretations of this goal. A strict interpretation would be that the room should be always clean. This can be expressed in linear temporal logic as \( \Box \diamond \text{clean} \). A less strict interpretation of it would be to allow the room to get unclean (say while it is being used) but with a guarantee that it will be eventually clean. This can be expressed in linear temporal logic as \( \Box \neg \diamond \text{clean} \). There are two issues with this representation. First, it does not give a bound on how soon \text{clean} should be true after \( \neg \text{clean} \) becomes true. For the second issue, consider the case when the agent is not allowed to clean the room while it is being used and the room is being continuously used. In that case we can not blame the agent for the status of the room. But we can seek a different kind of guarantee. We can demand that the agent give a guarantee that as long as it is not interfered with (i.e., is allowed to clean) for \( k \) steps (or \( k \) units of time) it will have the room clean after that. This is formulated as \( k \)-maintainability in (Nakamura, Baral, & Bjareland 2000). When \( k = 1 \) it is referred to simply as "maintainability". This notion was earlier discovered (Dijkstra 1974).

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The inadequacy of the expression \( \Box \diamond f \) in capturing our intuition about "maintaining \( f \)" is because \( \Box \diamond f \) is defined on trajectories which do not distinguish between transitions due to agent and environment actions. Thus we can not distinguish the cases

(i) where the agent does its best to maintain \( f \) (and is sometimes thwarted by the environment), and can indeed make \( f \) true in some (say, \( k \)) steps if not interfered by the environment during them; and

(ii) where the agent really does not even try.

The main contributions of this paper can be summarized as follows.
1. We provide polynomial time algorithms that can construct $k$-maintainable control policies, if one exists. (In the rest of the paper we will refer to ‘control policy’ simply by ‘control’.) Our algorithm is based on SAT Solving, and employs a suitable formulation of the existence of $k$-maintainable control in a tractable fragment of SAT. We then proceed to give a logic programming implementation of this method, and finally distill from it a standard procedural algorithm.

2. We analyze the computational complexity of constructing $k$-maintainable controls, under different settings of the environment and the windows of opportunity open to the agent, as well as under different forms of representation. We show that the problem is complete for PTIME in the standard setting, where the possible states are enumerated, and complete for EXPTIME in a STRIPS-style setting where states are given by value assignments to fluents.

**Background: Actions, states, control policies and $k$-maintainability**

In this section we present a slightly revised versions of the definitions for $k$-maintainability presented in (Nakamura, Baral, & Bjarelard 2000). We start with defining the notion of a system that is used in the discrete event dynamic systems community (Ozveren, Willsky, & Antsaklis 1991).

**Definition 1 (System)** A system is a quadruple $A = (S, A, \Phi, poss)$, where

- $S$ is the set of system states;
- $A$ is the set of actions, which is the union of the set of agents actions, $A_{ag}$, and the set of environmental actions, $A_{env}$;
- $\Phi : S \times A \rightarrow 2^S$ is a non-deterministic transition function that specifies how the state of the world changes in response to actions; and
- $poss : S \rightarrow 2^A$ is a function that describes which actions are possible to take in which states.

In practice, the functions $\Phi$ and $poss$ are required to be effectively (and efficiently) computable, and they may often be specified in a representation language such as in (Gelfond & Lifschitz 1992; Fikes & Nilson 1971). The possibility of an action has different meaning depending on whether it is an agent’s action or whether it is an environmental action. In case of an agent’s action, it is never dictated by the control policy followed by the agent. For environmental actions, it encodes the various possibilities that are being accounted for in the model. We tacitly assume here that possible actions lead always to some successor state, i.e., the axiom that $\Phi(s, a) \neq \emptyset$ whenever $a \in poss(s)$ holds for any state $s$ and action $a$, is satisfied by any system.

An example of a system $A = (S, A, \Phi, poss)$, where $S = \{b, c, d, f, g, h\}$, $A = \{a, a’, e\}$, and the transition function $\Phi$ is shown in Figure 1, where $s' \in \Phi(s, a)$ iff an arc $s \rightarrow s'$ labeled with $a$ is present and $poss(s)$ are all actions that label arcs leaving $s$. Notice that in this case, $\Phi(s, a)$ is deterministic, i.e., $\Phi(s, a)$ is a singleton if nonempty.

![Figure 1: Transition diagram of system A](image)

We now formally define the notion of a control and a super-control policy.

**Definition 2 (Control and super-control policy)** Given a system $A = (S, A, \Phi, poss)$ and a set $A_{ag} \subseteq A$ of agent actions,

- a control policy for $A$ w.r.t. $A_{ag}$ is a partial function $K : S \rightarrow A_{ag}$, such that $K(s) \in poss(s)$ whenever $K(s)$ is defined.
- a super-control policy for $A$ w.r.t. $A_{ag}$ is a partial function $K : S \rightarrow 2^{A_{ag}}$ such that $K(s) \subseteq poss(s)$ and $K(s) \neq \emptyset$ whenever $K(s)$ is defined.

As we mentioned earlier, the main intuition behind the notion of maintainability is that maintenance becomes possible only if there is a window of non-interference from the environment during which maintenance is performed by the agent. In other words, an agent $k$-maintains a condition $c$ if its control (or its reaction) is such that if we allow it to make the controlling actions without interference from the environment for at least $k$ steps, then it gets to a state that satisfies $c$. The definition of maintainability has the following parameters: (i) a set of initial states $S$ that the system may be initially in, (ii) a set of desired states $E$ that we want to maintain, (iii) (i) a set of initial states $S$ that the system may be initially in, (ii) a set of desired states $E$ that we want to maintain, (iii) a set $A_{ag} \subseteq A$ of agent actions, (iv) a function $exo : S \rightarrow 2^{A_{env}}$ detailing exogenous actions, such that $exo(s) \subseteq poss(s)$, and (vi) a control $K$ (mapping a relevant part of $S$ to $A_{ag}$) such that $K(s) \in poss(s)$.

The next step is to formulate when the control $K$ maintains $E$ assuming that the system is initially in one of the states in $S$. The exogenous actions are accounted for by defining the notion of a closure of $S$ with respect to the system $A_{K,exo} = (S, A, \Phi, poss_{K,exo})$, denoted by $\text{Closure}(S, A_{K,exo})$; where $poss_{K,exo}(s)$ is the set $\{K(s)\} \cup exo(s)$. This closure is the set of states that the system may get into starting from $S$ because of $K$ and/or $exo$. Maintainability is then defined by requiring the control to be such that if the system is in any state in the closure and is given a window of non-interference from exogenous actions, then it gets into a desired state. The importance of using the notion of closure is that one can focus only on a possibly smaller state of states, rather than all the states, thus limiting the possibility of an exponential blow-up - as warned in (Ginsberg 1989) - of the number of control rules. We now formally define the notions of closure and maintainability.
Definition 3 Given a system $A = (S, A, \Phi, \text{poss})$ and a state $s$, $R(A, s) \subseteq S$ is the smallest set of states that satisfies the following conditions: (i) $s \in R(A, s)$, and (ii) if $s' \in R(A, s)$, and $a \in \text{poss}(s')$, then $\Phi(s', a) \subseteq R(A, s)$.

Definition 4 (Closure) Let $A = (S, A, \Phi, \text{poss})$ be a system and let $S' \subseteq S$ be a set of states. Then the closure of $A$ w.r.t. $S'$, denoted by $\text{Closure}(S, A)$, is defined by

$$\text{Closure}(S, A) = \bigcup_{s \in S} R(A, s).$$

Example 1 In the system $A$ in Figure 1, we have that $R(A, d) = \{d, h\}$ and $R(A, f) = \{f, g, h\}$, and therefore $\text{Closure}\{(d, f), A\} = \{d, f, g, h\}$.

Next we define the notion of unfolding a control.

Definition 5 (Unfold$_k(s, A, K)$) Let $A=(S, A, \Phi, \text{poss})$ be a system, let $s \in S$, and let $K$ be a control for $A$. Then $\text{Unfold}_k(s, A, K)$ is the set of all sequences $\sigma = s_0, s_1, \ldots, s_l$ where $l \leq k$ and $s_0 = s$ such that $K(s_j)$ is defined for all $j < l$, $s_{j+1} \in \Phi(s_j, K(s_j))$, and if $l < k$, then $K(s_k)$ is undefined.

Intuitively, an element of $\text{Unfold}_k(s, A, K)$ is a sequence of states of length at most $k + 1$ that the system may go through if it follows the control $K$ starting from the state $s$. The above definition of $\text{Unfold}_k(s, A, K)$ is easily extended to the case when $K$ is a super-control, meaning $K(s)$ is a set of actions instead of a single action. In that case, we overload $\Phi$ and for any set of actions $a^*$, define $\Phi(s, a^*) = \bigcup_{a \in a^*} \Phi(s, a)$.

We now define the notion of $k$-maintainability. This definition can be used to verify the correctness of a control.

Definition 6 ($k$-Maintainability) Given a system $A = (S, A, \Phi, \text{poss})$, a set of actions $A_{ag} \subseteq A$, and a specification of exogenous action occurrence $exo$, we say that a control $K$ for $A$ w.r.t. $A_{ag}$ $k$-maintains $S \subseteq S$ with respect to $E \subseteq S$, where $k \geq 0$, if it holds for each state $s \in \text{Closure}(S, A_{K,exo})$ and each sequence $\sigma = s_0, s_1, \ldots, s_l$ in $\text{Unfold}_k(s, A, K)$ with $s_0 = s$ that $(s_0, s_1, \ldots, s_l) \cap E = \emptyset$.

We say that a set of states $S \subseteq S$ (resp. $A$, if $S = S$) is $k$-maintainable, $k \geq 0$, with respect to a set of states $E \subseteq S$, if there exists a control $K$ which $k$-maintains $S$ w.r.t. $E$. Furthermore, $S$ (resp. $A$) is called maintainable w.r.t. $E$, if $S$ (resp. $A$) is $k$-maintainable w.r.t. $E$ for some $k \geq 0$.

Example 2 Reconsider the system $A$ in Figure 1. Let us assume that $A_{ag} = \{a, a'\}$, that $exo(s) = \{e\}$ iff $s = f$ and that $exo(s) = \emptyset$ otherwise. Suppose now that we want a 3-maintainable control policy for $S \subseteq \{b, c, d, h\}$ w.r.t. $E = \{h\}$. Clearly, such a control policy $K$ is to take $a$ in $b, c,$ and $d$. Indeed, $\text{Closure}\{b\}, A_{K,exo} = \{b, c, d, h\}$ and $\text{Unfold}_3(b, A, K) = \{(b, c, d, h)\}$. Suppose now that $\Phi(c, a) = \{d, f\}$ instead of $\{d\}$ (i.e., nondeterminism in c). Then, no $k$-maintainable control policy for $S \subseteq \{b\}$ w.r.t. $E = \{h\}$ exists for any $k \geq 0$. Indeed, the agent can always end up in the dead-end $g$. If, however, in addition $\Phi(g, a') = \{f, h\}$ and $a' \in \text{poss}(g)$, a 3-maintainable control policy $K$ is $K(s) = a$ for $s \in \{b, c, d, f\}$ and $K(g) = a'$.

### Polynomial time methods to construct $k$-maintainable control policies

Now that we have defined the notion of $k$-maintainability, our next step is to show how some $k$-maintainable control can be constructed in an automated way. We start with some historical background. There has been extensive use of situation control rules (Drummond 1989) and reactive control in the literature. But there have been far fewer efforts (Kabanza, Barbeau, & St-Denis 1997) to define correctness of such control rules, and to automatically construct correct control rules. In (Kaelbling & Rosenschein 1991), it is suggested that in a control rule of the form: “if condition $c$ is satisfied then do action $a'$”, the action $a$ is the action that leads to the goal from any state where the condition $c$ is satisfied. In (Baral & Son 1998) a formal meaning of “leads to” is given as: for all states $s$ that satisfy $c$, $a$ is the first action of a minimal cost plan from $s$ to the goal. Using this definition, an algorithm is presented in (Nakamura, Baral, & Bjarelend 2000) to construct $k$-maintainable controls. This algorithm is sound but not complete, in the sense that it generates correct controls only, but there is no guarantee that it will find always a control if one exists. The difficulty in developing a complete algorithm – also recognized in (Jensen, Veloso & Bryant 2003) – can be explained as follows. Suppose one were to do forward search from a state in $S$. Now suppose there are multiple actions from this state that ‘lead’ to $E$. Deciding on which of the actions or which subsets one needs to choose is a nondeterministic choice necessitating backtracking if one were to discover that a particular choice leads to a state (due to exogenous actions) from where $E$ can not be reached. Same happens in backward search too. In this paper we overcome the problems one faces in following the straightforward approaches and give a sound and complete algorithm for constructing $k$-maintainable control policies.

We provide it in two sets: First we consider the case when the transition function $\Phi$ is deterministic, and then we generalize to the case where $\Phi$ may be non-deterministic. In each case, we present different methods, which illustrates our discovery process and also eases a better grasp of the final algorithm. We first present an encoding of our problem as a propositional theory and appeal to propositional SAT solvers to construct the control. As it turns out, this encoding is in a tractable fragment of SAT, for which specialized solvers (in particular, Horn SAT solvers) can be used easily. Finally, we present a direct algorithm distilled from the previous methods.

The reasoning behind this line of presentation is the following:

(i) It illustrates the methodology of using SAT and Horn SAT encodings to solve problems;

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1 Here only $K(s)$ for $s \in \text{Closure}(S, A_{K,exo})$ is of relevance. For all other $s$, $K(s)$ can be arbitrary or undefined.
(ii) the encodings allow us to quickly implement and test algorithms;
(iii) the proof of correctness mimics the encodings; and
(iv) we can exploit known complexity results for Horn SAT to determine the complexity of our algorithm, and in particular to establish tractability.

As for ii), we can make use of Answer Set Solvers such as DLV (Eiter et al. 2000) or Smmodels (Niemelä & Simons 1997) which extend Horn logic programs by nonmonotonic negation. These solvers allow to compute efficiently the least model and some maximal model of a Horn theory, which can be exploited to construct robust or “small” controls, respectively.

The problem we want to solve, which we refer to as k-Maintain, has the following input and output:

**Input**: An input \( I \) is a system \( A = (S, A, \Phi, \text{poss}) \), sets of states \( E \subseteq S \) and \( S \subseteq S \), a set \( A_{\text{ag}} \subseteq A \), a function \( \text{exo} \), and an integer \( k \geq 0 \).

**Output**: A control \( K \) such that \( S \) is \( k \)-maintainable with respect to \( E \) (using the control \( K \)), if such a control exists. Otherwise the output is the answer that no such control exists.

We assume here that the functions \( \text{poss}(s) \) and \( \text{exo}(s) \) can be efficiently evaluated; e.g., if both functions are given by their graphs (i.e., in a table).

**Deterministic transition function \( \Phi(s, a) \)**

We start with the case of deterministic transitions, i.e., \( \Phi(s, a) \) is a singleton set \( \{s'\} \) whenever nonempty. In abuse of notation, we simply will write \( \Phi(s, a) = s' \) in this case.

Our first algorithm to solve k-Maintain will be based on a reduction to propositional SAT solving. Given an input \( I \) for k-Maintain, we construct a SAT instance \( \text{sat}(I) \) in propositional time such that \( \text{sat}(I) \) is satisfiable if and only if the input \( I \) allows for a \( k \)-maintainable control, and that the satisfying assignments for \( \text{sat}(I) \) encode possible such controls.

In our encoding, we shall use for each state \( s \in S \) propositional variables \( s_0, s_1, \ldots, s_k \). Intuitively, \( s_i \) will denote that (i) there is a path from state \( s \) to some state in \( E \) using only agent actions and at most \( i \) of them, to which we refer as “there is an a-path from \( s \) to \( E \) of length at most \( i \),” and that (ii) from each state \( s' \) reachable from \( s \), there is an a-path from \( s' \) to \( E \) of length at most \( k \).

The encoding \( \text{sat}(I) \) contains the following formulas:

1. For all \( s \in S \), and for all \( j, 0 \leq j < k \):
   \[ s_j \Rightarrow s_{j+1} \]

2. For all \( s \in E \cap S \):
   \[ s_0 \]

3. For any states \( s, s' \in S \) such that \( \Phi(a, s) = s' \) for some action \( a \in \text{exo}(s) \):
   \[ s_k \Rightarrow s'_k \]

4. For all \( s \in S \setminus E \) and all \( i, 1 \leq i \leq k \):
   \[ s_i \Leftrightarrow \bigvee_{s' \in \text{PS}(s)} s'_{i-1}, \text{ where } \text{PS}(s) = \{s' \in S : \exists a \in A_{\text{ag}} \cap \text{poss}(s) : s' = \Phi(a, s)\}; \]

5. For all \( s \in S \setminus E \):
   \[ \neg s_0 \]

The intuition behind the above encoding is as follows. For \( 0 \), (1), (4), and (5) it is ok to assume that the intuitive meaning of \( s_i \) is as given by \( i \). Thus, the clauses in (0) state that if there is an a-path from \( s \) to \( E \) of length at most \( j \) then, logically, there is also an a-path of length at most \( j+1 \). Next, the clauses in (1) say that for states \( s \in S \cap E \), there is an a-path of length 0 from \( s \) to \( E \), Next, (4) states that for any starting state \( s \) in \( S \) outside \( E \), there is an a-path from \( s \) to \( E \) of length at most \( k \), and finally (5) that for any state \( s \) outside \( E \), there is no a-path from \( s \) to \( E \) of length 0. Ignoring part (ii) of the meaning of \( s_i \), the clauses in (3) state that if, for any state \( s \), there is an a-path from \( s \) to \( E \) of length at most \( i \), then for some possible agent action \( a \) and successor state \( s' = \Phi(a, s) \), there is an a-path from \( s' \) to \( E \) of length at most \( i+1 \). For (2), let us consider the full intuitive meaning of \( s_i \). The “if” part of (2) expresses that there is an a-path from \( s \) to \( E \) of length at most \( k \), and for each state \( s'' \) reachable from \( s \), there is an a-path from \( s'' \) to \( E \) of length at most \( k \). The “then” part of (2) expresses that there is an a-path from \( s' \) to \( E \) of length at most \( k \), and for each state \( s'' \) reachable from \( s' \), there is an a-path from \( s'' \) to \( E \) of length at most \( k \). The link between the \( s \) in the “if” part and the \( s' \) in the “then” part is that an exogenous action takes one from \( s \) to \( s' \). Thus (2) follows from the intuitive meaning of \( s_i \).

Given any model of \( \text{sat}(I) \), we can extract a desired control \( K \) from it by defining \( K(s) = a \), if \( s \) is ok to be in the closure for \( k \)-maintenance (indicated by truth of \( s_k \) in \( M \)) but outside \( E \), and \( a \) is possible agent action in \( s \) such that \( s' = \Phi(s, a) \) and \( s' \) closer to \( E \) than \( s \) is. In case of multiple possible \( a \) and \( s' \), one \( a \) can be arbitrarily picked. Otherwise, \( K(s) \) is left undefined.

In particular, for \( k = 0 \), only the clauses from (1), (2), (4) and (5) do exist. As easily seen, \( \text{sat}(I) \) is satisfiable in this case if and only if \( S \subseteq E \) and no exogenous action leads outside \( E \), i.e., the closure of \( S \) under exogenous actions is contained in \( E \). This means that no actions of the agent are required at any point in time, and we thus obtain the trivial 0-control \( K \) which is undefined on all states, as desired.

The next result states that the SAT encoding works properly in general.

**Proposition 1** Let \( I \) consist of a system \( A = (S, A, \Phi, \text{poss}) \) where \( \Phi \) is deterministic, a set \( A_{\text{ag}} \subseteq A \), sets of states \( E \subseteq S \) and \( S \subseteq S \), an exogenous function \( \text{exo} \), and an integer \( k \). For any model \( M \) of \( \text{sat}(I) \), let \( C_M = \{s \in S : M \models s_k\} \), and for any state \( s \in C_M \) let \( \ell_M(s) \) denote the smallest index \( j \) such that \( M \models s_j \) (i.e., \( s_0, s_1, \ldots, s_{j-1} \) are \( false \) and \( s_j \) is \( true \)), which we call the level of \( s \) w.r.t. \( M \). Then,

(i) \( S \) is \( k \)-maintainable w.r.t. \( E \) iff \( \text{sat}(I) \) is satisfiable.

(ii) Given any model \( M \) of \( \text{sat}(I) \), the partial function \( K_M^+ : S \rightarrow 2^{A_{\text{ag}}} \) defined on \( C_M \setminus E \) such that

\[
K_M^+(s) = \{a \in A_{\text{ag}} \cap \text{poss}(s) : \Phi(s, a) = s', s' \in C_M, \ell_M(s') < \ell_M(s)\},
\]
is a valid super-control for \( A \) w.r.t. \( A_{ag} \):

(iii) any control \( K \) which refines \( K^+_M \) for some model \( M \) of \( \text{sat}(I) \) \( k \)-maintains \( S \) w.r.t. \( E \). □

Horn SAT encoding While \( \text{sat}(I) \) is constructible in polynomial time from \( I \), we can not automatically infer that solving \( k \)-\text{MAINTAIN} is polynomial, since SAT is a canonical NP-hard problem. However, a closer look at the structure of the clauses in \( \text{sat}(I) \) reveals that this instance is solvable in polynomial time. Indeed, it is a reverse Horn theory; i.e., by reversing the propositions, we obtain a Horn theory. Let us use propositions \( \overline{s_i} \) whose intuitive meaning is converse of the meaning of \( s_i \). Then the Horn theory corresponding to \( \text{sat}(I) \), denoted \( \text{sat}(I) \), is as follows:

(0) For all \( s \in S \) and \( j \), \( 0 \leq j < k \):
\[
\overline{s_j} \to \overline{s_{j+1}}
\]
(1) For all \( s \in E \cap S \):
\[
\overline{s_0} \to 1.
\]
(2) For any states \( s, s' \in S \) such that \( s' = \Phi(a, s) \) for some action \( a \in \text{exo}(s) \):
\[
\overline{s'_k} \to \overline{s_k}.
\]
(3) For any state in \( S \setminus E \), and for all \( i, 1 \leq i \leq k \):
\[
\left( \bigwedge_{a' \in \text{exo}(s)} \overline{s_{i-1}} \right) \to \overline{s_i},
\]
where
\[
PS(s) = \{ s' \in S | \exists a \in A_{ag} \cap \text{poss}(s); s' = \Phi(a, s) \}.
\]
(4) For all \( s \in S \setminus E \):
\[
\overline{s_k} \to 1.
\]
(5) For all \( s \in S \setminus E \):
\[
\overline{s_0}.
\]

Here, \( \perp \) denotes falsity. We then obtain a result similar to Proposition 1, and the models \( M \) of \( \text{sat}(I) \) lead to \( k \)-maintainable controls, which we can construct similarly; just replace in part (ii) \( C_M \) with \( \overline{C}_M = \{ s \in S | M \not\models \overline{s_k} \} \).

Notice that \( \overline{C}_M \) coincides with the set of states \( C_M \) for the model \( \overline{M} \) of \( \text{sat}(I) \) such that \( \overline{M} \models p \iff M \not\models p \), for each atom \( p \).

**Example 3** For the first (deterministic) instance \( I \) in Example 2, the encoding \( \text{sat}(I) \) yields the least model
\[
M = \{ f_3, f_2, f_1, f_0, f_3, f_2, f_1, f_0, b_2, b_1, b_0, c_1, c_0, c_0 \};
\]
hence, \( \overline{C}_M = \{ b, c, d, h \} \), which gives rise to the super-control \( K^+_M \) such that \( K^+_M(s) = \{ a \} \) for \( s \in \{ b, c, d \} \) and \( K^+_M(s) \) is undefined for \( s \in \{ f, g, h \} \). In this case, there is a single control \( K \) refining \( K^+_M \), which is given by \( K(s) = a \) for \( s \in \{ b, c, d \} \) discussed above. □

As computing a model of a Horn theory is a well-known polynomial problem (Dowling & Gallier 1984), we thus obtain the following result.

**Theorem 2** Under deterministic state transitions, problem \( k \)-\text{MAINTAIN} is solvable in polynomial time. □

An interesting aspect of the above is that, as well-known, each satisfiable Horn theory \( T \) has the least model, \( M_T \), which is given by the intersection of all its models. Moreover, the least model is computable in linear time, cf. (Dowling & Gallier 1984). This model not only leads to a \( k \)-maintainable control, but also leads to a maximal control, in the sense that the control is defined on a greatest set of states outside \( E \) among all possible \( k \)-maintainable controls for \( S' \) w.r.t. \( E \) such that \( S' \subseteq S' \). This gives a clear picture of which other states may be added to \( S \) while \( k \)-maintainability is preserved; namely, any states in \( C_{M_T} \).

Furthermore, any control \( K \) computed from \( M_T \) applying the method in Proposition 1 (using \( \overline{C}_{M_T} \)) works for such an extension of \( S \) as well.

On the other hand, intuitively a \( k \)-maintainable control constructed from some maximal model of \( \text{sat}(I) \) with respect to the propositions \( \overline{s_k} \) is undefined to a largest extent, and works merely for a smallest extension. We may generate, starting from \( M_T \), such a maximal model of \( T \) by trying to flip first, step by step all propositions \( \overline{s_k} \) which are false to true, as well as other propositions entailed. In this way, we can generate a maximal model of \( T \) on \( \{ \overline{s_k} | s \in S \setminus E \} \) in polynomial time, from which a “lean” control can also be computed in polynomial time.

**Non-deterministic transition function** \( \Phi(s, a) \)

We now generalize our method for constructing \( k \)-maintainable controls to the case in which transitions due to \( \Phi \) may be non-deterministic. As before, we first present a general propositional SAT encoding, and then rewrite to a propositional Horn SAT encoding. To explain some of the notations, we need the following definition, which generalizes the notion of an \( a \)-path to the non-deterministic setting.

**Definition 7** (a-path) We say that there exists an \( a \)-path of length at most \( k \geq 0 \) from a state \( s \) to a set of states \( S' \), if either \( s \in S' \), or \( s \notin S' \), \( k > 0 \) and there is some action \( a \in A_{ag} \cap \text{poss}(s) \) such that for every \( s' \in \Phi(s, a) \) there exists an \( a \)-path of length at most \( k - 1 \) from \( s' \) to \( S' \).

In the following encoding of an instance \( I \) of problem \( k \)-\text{MAINTAIN} to SAT, referred to as \( \text{sat}(I) \), \( s_i \) will again intuitively denote that (i) there is an \( a \)-path from \( s \) to \( E \) of length at most \( i \), and (ii) from each state \( s' \) reachable from \( s \), there is an \( a \)-path from \( s' \) to \( E \) of length at most \( k \). The proposition \( s_{a_i}, i > 0 \), will denote that for such \( s \) there is an \( a \)-path from \( s \) to \( E \) of length at most \( i \) starting with action \( a \) (in \( \text{poss}(s) \)). The encoding \( \text{sat}(I) \) has again groups (0)–(5) of clauses as follows:

(0) (1), (4) and (5) are the same as in \( \text{sat}(I) \).
(2) For any state \( s \in S \) and \( s' \) such that \( s' \in \Phi(a, s) \) for some action \( a \in \text{exo}(s) \):
\[
s_k \Rightarrow s'_k
\]
(3) For every state \( s \in S \setminus E \) and for all \( i, 1 \leq i \leq k \):
\[
s_i \Rightarrow v_{a \in A_{ag} \cap \text{poss}(s)} s_{a_i};
\]
(3.1) \( s_i \Rightarrow v_{a \in A_{ag} \cap \text{poss}(s)} s_{a_i}; \)
(3.2) for every \( a \in A_{ag} \cap \text{poss}(s) \) and \( s' \in \Phi(s, a) \):
\[
s_{a_i} \Rightarrow s'_{i-1};
\]
(3.3) for every $a \in A_{ag} \cap poss(s)$, if $i < k$:
\[ s_i \cdot a \Rightarrow s_{i+1}. \]

Group (2) above is very similar to group (2) of $sat(I)$ in the previous subsection. The only change is that we now have $s' \in \Phi(a, s)$ instead of $s' = \Phi(a, s)$. The main difference is in group (3). We now explain those clauses, but while doing it ignore the aspect (ii) of the meaning of $s_i$. The clauses in (3.1) state that if there is an $a$-path from $s$ to $E$ of length at most $i$, then there is some possible action $a$ for the agent, such that for each state $s'$ that potentially results by taking $a$ in $s$, there must be an $a$-path from $s'$ to $E$ of length at most $i-1$ (expressed by 3.2). The clauses $s_i \cdot a \Rightarrow s_{i+1}$ in (3.3) say that on a longer $a$-path from $s$ the agent must be able to pick $a$ also. Notice that there are no formulas in $sat(I)$ which forbid to pick different actions $a$ and $a'$ in the same state $s$, and thus we have a super-control; however, we can always refine it easily to a control.

**Proposition 3** Let $I$ consist of a system $A = (S, A, \Phi, poss)$, a set $A_{ag} \subseteq A$, sets of states $E, S \subseteq S$, an exogenous function $exo$, and an integer $k$. For any model $M$ of $sat'(I)$, let $C_M = \{ s \in S | M \models s_k \}$, and for any state $s \in C_M \setminus E$ let $\ell_M(s)$ denote the smallest index $j$ such that $M \models s_{a,j}$ for some action $a \in A_{ag} \cap poss(s)$, which we call the a-level of $s$ w.r.t. $M$. Then,

(i) $S$ is k-maintainable w.r.t. $E$ iff $sat'(I)$ is satisfiable;

(ii) given any model $M$ of $sat'(I)$, the partial function $K_M^{'}: S \rightarrow 2^{A_{ag}}$ which is defined on $C_M \setminus E$ by
\[ K_M^{'}(s) = \{ a | M \models s_{a, \ell_M(s)} \} \]

is a valid super-control; and

(iii) any control $K$ which refines $K_M^{'}$ for some model $M$ of $sat'(I)$ k-maintains $S$ w.r.t. $E$. \hfill \Box

One advantage of the encoding $sat'(I)$ over the encoding $sat(I)$ for deterministic transition function $\Phi$ above is that it directly gives us the possibility to read off a suitable control from the $s_i \cdot a$ propositions, $a \in poss(s)$, which are true in any model $M$ that we have looking at the transition function $\Phi(s, a)$ again. On the other hand, the encoding is more involved, and uses a larger set of propositions. Nonetheless, the structure of the formulas in $sat'(I)$ is benign for computation and allows us to compute a model, and from it a k-maintainable control in polynomial time.

**Horn SAT encoding (general case)** The encoding $sat'(I)$ is, like $sat(I)$, a reverse Horn theory. We thus can rewrite $sat'(I)$ similarly to a Horn theory, $sat(I)$ by reversing the propositions, where the intuitive meaning of $\pi_i$ and $\neg \pi_i$ is the converse of the meaning of $s_i$ and $s_i \cdot a$, respectively. The encoding $sat(I)$ is as follows:

(0), (1). (4) and (5) are as in $sat(I)$

(2) For any state $s \in S$ and $s'$ such that $s' \in \Phi(a, s)$ for some action $a \in exo(s)$:
\[ s_k \Rightarrow \neg \pi_k. \]

(3) For every state $s \in S \setminus E$ and for all $i, 1 \leq i \leq k$:
\[ \bigwedge_{a \in A_{ag} \cap poss(s)} \neg \pi_i \Rightarrow \pi_i; \]

(3.2) for every $a \in A_{ag} \cap poss(s)$ and $s' \in \Phi(s, a)$:
\[ s_i \Rightarrow \neg \pi_i. \]

(3.3) for every $a \in A_{ag} \cap poss(s)$, if $i < k$:
\[ s_{i+1} \Rightarrow \pi_i. \]

We thus obtain from Proposition 3 easily the following result, which is the main result of this section so far.

**Theorem 4** Let $I$ consist of a system $A = (S, A, \Phi, poss)$, a set $A_{ag} \subseteq A$, sets of states $E, S \subseteq S$, an exogenous function $exo$, and an integer $k$. Let, for any model $M$ of $sat(I)$, $C_M = \{ s | M \not\models \pi_k \}$, and let $\ell_M(s) = \min\{ j | M \not\models \pi_j, a \in A_{ag} \cap poss(a) \}$. Then,

(i) $S$ is k-maintainable w.r.t. $E$ iff the Horn SAT instance $\overline{sat(I)}$ is satisfiable;

(ii) Given any model $M$ of $\overline{sat}(I)$, every control $K$ such that $K(s)$ is defined iff $s \in C_M \setminus E$ and satisfies
\[ K(s) \in \{ a \in A_{ag} \cap poss(s) | M \not\models \pi_{\ell_M(s)} \}, \]

$k$-maintains $S$ w.r.t. $E$. \hfill \Box

**Corollary 5** Problem $k$-MAINTAIN is solvable in polynomial time.

**Example 4** Continuing Example 2, for the nondeterministic variant $I_1$ where $\Phi(c, a) = \{ d, f \}$ instead of $\{ d \}$, the formula $\overline{sat(I_1)}$ is found unsatisfiable for any $k \geq 0$. On the other hand, for the instance $I_2$ where in addition $\Phi(g, a') = \{ f, h \}$ and $a' \in poss(g)$, $\overline{sat(I_2)}$ is satisfiable and has the least model
\[ M = \{ b_0, c_0, d_0, f_0, g_0, b_1, c_1, f_1, \}
\[ b_0a_1, c_0a_1, b_0a_2, g_0a_2 \} \]

We thus obtain the super-control $K_M^{'}$ defined on the states $b, c, d, f, g, h$, where $K^{+}(s) = \{ a \}$ for $s \in \{ c, d, f \}$ and $K^{+}(s) = \{ a' \}$ for $s \in \{ b, g \}$. There is a single control $K$ which refines $K_M^{'}$, namely $K(s) = a$ for $s \in \{ c, d, f \}$ and $K(s) = a'$, for $s \in \{ b, g \}$. \hfill \Box

**Genuine procedural algorithm**

From the encoding to Horn SAT above, we can distill a direct algorithm to construct a $k$-maintainable control, if one exists. The algorithm mimics the steps which a SAT solver might take in order to solve $sat'(I)$. It uses counters $c[s]$ and $c[s, a]$ for each state $s \in S$ and possible action $a$ in state $s$, which range over $\{ -1, 0, \ldots, k \}$ and $\{ 0, 1, \ldots, k \}$, respectively. Intuitively, value $i$ of counter $c[s]$ represents that $\pi_0, \ldots, \pi_i$ are assigned true; in particular, $i = -1$ represents that no $\pi_i$ is assigned true yet. Similarly, value $i$ for $c[s, a]$ represents that $\pi_0, \ldots, \pi_i$ are assigned true (and in particular, $i = 0$ that no $\pi_i$ is assigned true yet).

Starting from an initialization, the algorithm updates by demand of the clauses in $\overline{sat(I)}$ the counters (i.e., sets propositions true) using a command $upd(c, i)$ which is short for “if $c < i$ then $c := i$”, towards a fixpoint. If a counter violation is detected, corresponding to violation of a clause $\phi_0 \rightarrow \perp$ for $s \in S \setminus E$ in (1) or $\phi_k \rightarrow \perp$ for $s \in S \setminus E$ in (4), then no control is possible. Otherwise, a control is constructed from the counters.

In detail, the algorithm is as follows:
Algorithm $k$-CONTROL

Input: A system $A = (S, A, \Phi, \text{poss})$, a set $A_{ag} \subseteq A$ of agent actions, sets of states $E, S \subseteq S$, an exogenous function $exo$, and an integer $k \geq 0$.

Output: A control $K$ which $k$-maintains $S$ with respect to $E$, if any such control exists. Otherwise, output that no such control exists.

(Step 1) Initialization

(i) For every $s$ in $E$, set $c[s] := -1$.
(ii) For every $s$ in $S \setminus E$, set $c[s] := k$ if $s \in S$ and $A_{ag} \cap \text{poss}(s) = \emptyset$; otherwise, set $c[s] := 0$.
(iii) For every $s$ in $S \setminus E$ and action(a). action(e).

(Step 2) Repeat the following steps until there is no change or $c[s] = k$ for some $s$ in $S \setminus E$ or $c[s] \geq 0$ for some $s$ in $S \cap E$:

(i) For any states $s$ in $S$ and $s' \in \Phi(a, s)$ where $a \in exo(s)$ and $c[s'] = k$ do $\text{upd}(c[s], k)$.
(ii) For any state $s$ in $S \setminus E$,

(a) if $s' \in \Phi(a, s)$ for $a \in A_{ag} \cap \text{poss}(s)$ and $c[s'] = i$ such that $0 \leq i < c$ then do $\text{upd}(c[s], a, i + 1)$.
(b) if $A_{ag} \cap \text{poss}(s) \neq \emptyset$ and $i = \min(c[aiate], a \in A_{ag} \cap \text{poss}(s))$, then do $\text{upd}(c[s], a)$. 

(Step 3) If $c[s] = k$ for some $s$ in $S \setminus E$ or $c[s] \geq 0$ for some $s$ in $S \cap E$, then output that $S$ is $k$-maintainable w.r.t. $E$ and halt.

(Step 4) Output any control $K : S \setminus E \rightarrow A_{ag}$ defined on all states $s$ in $S \setminus E$ with $c[s] < k$ and such that $K(s) \subseteq \{a \in A_{ag} \cap \text{poss}(s) : c[s, a] = \min b \in A_{ag} \cap \text{poss}(s), c[s, b]\}$. □

The above algorithm is easily modifiable if we simply want to output a super-control such that each of its refinements is a $k$-maintainable control, leaving a choice about the refinement to the user. Alternatively, we can implement in Step 4 such a choice based on preference information.

The following proposition states that the algorithm works correctly and runs in polynomial time.

Proposition 6 Algorithm $k$-CONTROL solves problem $k$-MAINTAIN. Furthermore, for any input $I$ it terminates in polynomial time. □

Encoding $k$-Maintainability for an Answer Set Solver

In this section, we show how computing a $k$-maintainable control can be encoded to a logic program, based on the results of the previous section. More precisely, we describe an encoding to non-monotonic logic programs under the Answer Set Semantics (Gelfond & Lifschitz 1991), which can be executed on one of the available Answer Set Solvers such as DLV (Eiter et al. 2000) or Smoebels (Niemelä & Simons 1997). These solvers support the computation of answer sets (models) of a given program, from which solutions (in our case, $k$-maintainable controls) can be extracted.

The encoding is generic, i.e., given by a fixed program which is evaluated over the instance $I$ represented by input facts $F(I)$. It makes use of the fact that non-monotonic logic programs can have multiple models, which correspond to different solutions, i.e., different $k$-maintainable controls.

In the following, we first describe how a system is represented in a logic program, and then we develop the logic programs for both deterministic and general, nondeterministic domains. We shall follow here the syntax of the DLV system; the changes needed to adapt the programs to other Answer Set Solvers such as Smoebels are very minor.

Input representation $F(I)$

The input $I$ of problem $k$-MAINTAIN, can be represented by facts $F(I)$ as follows.

- The system $A = (S, A, \Phi, \text{poss})$ can be represented using predicates state, transition, and poss by the following facts:
  - state(s), for each $s \in S$;
  - action(a), for each $a \in A$;
  - transition(s, a, s'), for each $s, s' \in S$ and $a \in A$ such that $s' \in \Phi(a, s)$;
  - poss(s, a), for each $s \in S$ and $a \in A$ such that $a \in poss(s)$.
- the set $A_{ag} \subseteq A$ of agent actions is represented by a predicate agent by facts agent(a), for each $a \in A_{ag}$;
- the set of states $S$ is represented by using a predicate start by facts start(s), for each $s \in S$;
- the set of states $E$ is represented by using a predicate goals by facts goal(s), for each $s \in E$;
- the exogenous function $exo$ is represented by using a predicate $exo$ by facts $exo(s, a)$ for each $s \in S$ and $a \in exo(s)$.
- finally, the integer $k$ is represented using a predicate limit by the fact limit(k).

Example 5 Coming back to Example 2, the input $I$ is represented as follows:

\begin{verbatim}
state(b). state(c). state(d). state(f).
state(g). state(h).
start(b). goal(h).
poss(b, a). poss(c, a). poss(d, a).
poss(b, al). poss(f, a). poss(f, e).
action(a). action(al). action(e).
trans(b, a, c). trans(c, a, d). trans(d, a, h).
trans(b, al, f). trans(f, a, h). trans(f, e, g).
limit(3).
\end{verbatim}

Deterministic transition function $\Phi$

The following is a program, executable on the DLV engine, for deciding the existence of a $k$-control. In addition to the predicates for the input facts $F(I)$, it employs a predicate n_path(X, I), which intuitively corresponds to $X_I$, and further auxiliary predicates.
The predicate range(I) specifies the index range from 0 to K, given by the input limit(K). The rules encoding the clause groups (0) – (2) and (4), (5) are straightforward and self-explanatory. For (3), we need to encode rules with bodies of different size depending on the transition function \( \Phi \), which itself is part of the input. We use that the antecedent of any implication (3) is true if it is not falsified, where falsification means that some atom \( s_1' \in s \). The output of DLV for the input \( s_1 \) is computed in \( \alevel(X, I) \). To assess this, we use the auxiliary predicate \( \text{some_pass}(X, I) \).

To compute the super-control \( K^+ \), we may add the rule:

\[
\text{cbar}(X) :- \text{state}(X), \text{not n_path}(X, K), \text{limit}(K).
\]

% Define\( C_M \)
\[
cbar(X) :- \text{state}(X), \text{not n_path}(X, K), \text{limit}(K).
\]

Define state level \( L \)
\[
\text{level}(X, I) :- \text{cbar}(X), \text{not n_path}(X, I), I > 0, \text{n_path}(X, J), I = J+1.
\]

\[
\text{level}(X, 0) :- \text{cbar}(X), \text{not n_path}(X, 0).
\]

Define super-control \( k_{\text{plus}} \)
\[
\text{k_plus}(X, A) :- \text{agent}(A), \text{trans}(X, A, Y), \text{pos}(X, A), \text{level}(X, I), \text{level}(Y, J), J < I, \text{not goal}(X).
\]

In \( cbar(X) \), we compute the states in \( C_M \), and in \( \text{level}(X, I) \) the level \( \ell_M(s) \) of each state \( s \in C_M = \mathcal{M}_+ \) for the corresponding model \( \mathcal{M} \) of \( \text{sat}(I) \). The super-control \( K_M^+ \) is then computed in \( \text{k_plus}(X, A) \).

Finally, by the following rules we can nondeterministically generate any control which is a refinement of \( K_M^+ \):

% Selecting a control from \( k_{\text{plus}} \)
\[
\text{control}(X, Y) :- \text{k_plus}(X, Y), \text{exclude_k_plus}(X, Y).
\]

\[
\text{exclude_k_plus}(X, Y) :- \text{k_plus}(X, Y), \text{control}(X, Z), Y <> Z.
\]

The first rule enforces that any possible choice for \( K(s) \) must be taken unless it is excluded, which by the second rule is the case if some other choice has been made. In combination the two rules effect that one and only one element from \( K_M^+(s) \) is chosen for \( K(s) \).

**Example 6** The output of DLV for the input \( I \) and the above program, filtered to \( \text{control} \{ \text{control}(b, a), \text{control}(c, a), \text{control}(d, a) \} \). This corresponds to the “maximal” control \( K \) mentioned earlier.

**Nondeterministic transition function \( \Phi \)**

As for deciding the existence of a \( K \)-maintaining control, the only change in the code for the deterministic case affects Step (3). The modified code is as follows, where \( \text{n_apath}(X, A, I) \) intuitively corresponds to \( X \notin A_I \).

% Rules for (3); different from above
\[
\text{n_path}(X, I) :- \text{state}(X), \text{not goal}(X), \text{range}(I), I > 0, \text{not some_pass}(X, I).
\]

\[
\text{some_pass}(X, I) :- \text{range}(I), I > 0, \text{trans}(X, A, Y), \text{agent}(A), \text{pos}(X, A), \text{not n_path}(Y, J), I = J+1.
\]

% Rule for (4)
\[
\text{n_path}(X, K) :- \text{trans}(X, A, Y), \text{exo}(X, A), \text{n_path}(Y, K), \text{limit}(K).
\]

**Example 7** The output of DLV for the input input \( I_2 \) and the above program, filtered to \( \text{control} \{ \text{control}(b, a), \text{control}(c, a), \text{control}(d, a), \text{control}(e, a), \text{control}(g, a1) \} \). Here, the value of \( \ell_M(s) \) is computed in \( \text{level}(X, I) \), using the auxiliary predicate \( \text{level}(X, I) \) which intuitively means that \( \ell_M(X) \leq I \).

For computing the controls refining \( K_M^+ \), we can add the two rules for selecting a control from \( k_{\text{plus}} \) from the program for the deterministic case.

% Define \( C_M \)
\[
cbar(X) :- \text{state}(X), \text{not n_path}(X, K), \text{limit}(K).
\]

% Define state action level \( \text{alevel} \)
\[
\text{alevel}(X, I) :- \text{cbar}(X), \text{not goal}(X), I = J+1, \text{n_path}(X, J), \text{not some_apass}(X, I).
\]

\[
\text{some_apass}(X, I) :- \text{range}(I), I > 0, \text{agent}(A), \text{pos}(X, A), \text{not n_apath}(X, A, I), \text{not goal}(X).
\]

% Rule for (2)
\[
\text{n_path}(X, K) :- \text{trans}(X, A, Y), \text{exo}(X, A), \text{n_path}(Y, K), \text{limit}(K).
\]

% Rule for (5)
\[
\text{n_path}(X, 0) :- \text{state}(X), \text{not goal}(X).
\]
State descriptions by variables

In many cases, states of a system are described by a vector of values for parameters which are variable over time. It is easy to incorporate such compact state descriptions into the LP encoding from above, and to evaluate them on Answer Set Solvers provided that the variables range over finite domains. In fact, if any state \( s \) is given by a (unique) vector \( s = (s_1, \ldots, s_m) \) \( m > 0 \), of values \( s_i \), \( 1 \leq i \leq m \), for variables \( X_i \) ranging over nonempty (finite) domains, then we can represent \( s \) as fact \( \text{state}(v_1, \ldots, v_m) \) and use a vector \( X_1, \ldots, X_m \) of state variables in the DLV code, in place of a single variable, \( X \). No further change of the programs from above is needed.

Computational Complexity

In this section, we give some results regarding the complexity of constructing \( k \)-maintainable controls under various assumptions.

We consider here the decision problem associated with \( k \)-MAINTAIN (deciding \( k \)-maintainability of \( S \) w.r.t. \( E \) in \( A \), which we refer to as \( k \)-MAINTAINABILITY), and deciding the maintainability of \( S \) w.r.t. \( E \) in \( A \), which we refer to as \( \text{MAINTAINABILITY} \).

We first consider the problems in the setting where the constituents of an instance are explicitly given, i.e., the sets in enumerative form and the functions by their graphs in tables.

Theorem 7 Problem \( k \)-MAINTAINABILITY is \( \text{PTIME} \)-complete (under logspace reductions). Furthermore, the \( \text{PTIME} \)-hardness holds for \( 1 \)-MAINTAINABILITY (i.e., \( k \) fixed to 1), even if all actions are deterministic and there is only one (deterministic) exogenous action.

Proof. (Sketch) The membership of \( k \)-MAINTAINABILITY in \( \text{PTIME} \) has been established above. The \( \text{PTIME} \)-hardness is shown by a reduction from deciding entailment of an atom \( q \) from a propositional Horn logic program \( \pi \).

Briefly, the idea is to represent backward rule application through agent actions; i.e., for a rule \( r \) of \( b_0 \leftarrow b_1, \ldots, b_m \), there is an agent action \( a.r \) which applied to a state \( s_{b_0} \) representing \( b_0 \), brings the agent nondeterministically to any state \( s_i \) representing \( b_i \), \( i \in \{1, \ldots, m\} \). In order to deal with cycles through rules, the states also carry level information. Given a state \( s_0 \) encoding \( q, S = \{s_0\} \) is maintainable w.r.t. a set of states \( E \) encoding the facts in \( \pi \) if \( q \) is provable from \( S \). Given that \( k \) rules exist in \( \pi \), this is equivalent to \( k \)-maintainability of \( S \) w.r.t. \( E \), and a \( k \)-maintaining control corresponds to a proof of \( q \) from \( \pi \).

By using a special form of the rules in \( \pi \) and an exogenous action, it is possible with some coding tricks to emulate nondeterministic agent actions and sequences of agent actions by alternating sequences of agent and exogenous actions, such that provability of \( q \) from \( \pi \) corresponds to \( 1 \)-maintainability of \( S \) w.r.t. a set \( E \) in a system \( A \) constructible in logarithmic workspace from \( q \) and \( \pi \).

Theorem 8 MAINTAINABILITY is \( \text{PTIME} \)-complete. The \( \text{PTIME} \)-hardness holds even in absence of exogenous actions, or if all actions are deterministic and there is only one exogenous action.

Proof. (Sketch) The membership in \( \text{PTIME} \) is immediate from the previous theorem and the fact that \( S \) is maintainable w.r.t. \( E \) iff \( S \) is \( k \)-maintainable w.r.t. \( E \) for \( k = |S| \).

The \( \text{PTIME} \)-hardness for the stated restrictions is shown by reductions similar to the one in the proof of Theorem 7.

The problems are thus of the same difficulty as Horn SAT, which is \( \text{PTIME} \)-complete (Papadimitriou 1994). In some cases, the complexity is lower:

Theorem 9 Problem MAINTAINABILITY for systems with only deterministic actions and no exogenous actions is \( \text{NLOG-Space} \)-complete.

Proof. (Sketch) In this case, it can be shown that the problem amounts to deciding whether each \( s \in S \) reaches some \( s' \in E \) in the graph whose nodes are all states in \( S \) and which has an edge \( s \to s' \) whenever \( s' \in \Phi(s, a, s') \) for some \( a \in A_{ag} \cap \text{poss}(s) \).

Thus, decidability of reachability of a node \( s' \) from a node \( s \in S \) in a directed graph is well-known \( \text{NLOG-Space} \)-complete (Papadimitriou 1994), the result follows.

Another case in which the complexity is lower is if the maintenance phase is short and exogenous actions are excluded.

Theorem 10 Problem \( k \)-MAINTAINABILITY for systems without exogenous actions is in \( \text{LOG-Space} \), if \( k \) is constant.

Proof. (Sketch) In this case, the problem consists in deciding whether for every state \( s \in S \), some state \( s' \in E \) is reachable within \( k \) steps by executing appropriate actions. More precisely, define inductively

\[
\begin{align*}
r_0(s) &= s \in E, \\
r_{i+1}(s) &= s \in E \lor \exists a \in A_{ag} \cap \text{poss}(s) \quad \forall s' \in S, a \in \Phi(s', a, s') \Rightarrow r_i(s'),
\end{align*}
\]

for \( i \geq 0 \). Then, in absence of exogenous actions \( k \)-MAINTAINABILITY is equivalent to \( r_k(s) \) for all \( s \in S \). This can be checked, for constant \( k \), in logarithmic workspace.

Thus, in the two cases above, the problems can be reduced to deciding reachability of a node \( t \) from a node \( s \) in graph, where in the latter each node has at most one outgoing edge.

State descriptions by variables

We note that under compact state representation as described above, in which a system state \( s \) is represented by a vector \( s = (v_1, \ldots, v_m) \) of values for fluents \( f_1, \ldots, f_m \) ranging over given finite domains, the complexity of the problem increases by an exponential in general.

In this setting, we assume that the following membership predicates, evaualable in polynomial time, are available: \( \text{in}_\Phi(s, a, s') \), \( \text{in}_\text{poss}(s, a) \), and \( \text{in}_\text{exo}(s, a) \) respectively for \( s' \in \Phi(s, a, s') \), \( a \in \text{poss}(s) \), and \( a \in \text{exo}(s) \), respectively. Furthermore, \( \text{in}_S(s) \) and \( \text{in}_E(s) \) for deciding whether \( s \in S \) and \( s \in E \), respectively. We thus obtain the following result.
Theorem 11 \( k \)-Maintainability and maintainability are \text{EXPTIME}-complete, when the input is given in the compact representation from above. The \text{EXPTIME}-hardness holds even for 1-Maintainability (i.e., \( k \) fixed to 1) and also for Maintainability, even if all actions are deterministic and there is only one exogenous action.

Proof. (Sketch) The membership in \text{EXPTIME} follows easily from unpacking the compact state representation to an explicit (enumerative) one, which leads to an exponential increase in the worst case, and which can be constructed in exponential time. On the enumerative representation, the problem can then be solved in polynomial time as shown above. In total, this means that the problem is solvable in exponential time.

The \text{EXPTIME}-hardness can be shown by a reduction from deciding inference of an atom from a Horn logic program with variables (a datalog program). The construction lifts a similar one for propositional programs, showing \text{PTIME}-hardness for the respective problems under enumerative representation, to the Datalog case.

In absence of exogenous actions, under compact representation \( k \)-Maintainability for constant \( k \) and Maintainability are in \text{PSPACE} provided that for the latter problem all actions are deterministic.

A more detailed discussion of complexity issues, with full proofs of all results will be given in the extended paper.

Conclusion

In this paper, we presented a polynomial time algorithm to compute the control for \( k \)-maintainability, if one exists. We then analyzed the complexity of constructing such controls under various assumptions. One interesting aspect of our polynomial time algorithm is the approach that led to its finding: use of \text{SAT} encoding, and complexity results regarding the special Horn sub-class of propositional logic.

In other related work, (Jensen, Veloso & Bryant 2003) considers the somewhat opposite problem of developing policies that achieve a given goal assuming at most \( k \) interferences from the environment. A formal connection, if any, between this problem and \( k \)-maintainability remains open. Also, in recent years there have been several work, such as (Cimatti et al. 2003), on planning with non-deterministic actions, but in none of those papers, agents actions and exogenous actions are viewed separately.

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References


