Conflict-Based Merging Operators

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Abstract

This paper deals with propositional belief merging. The key problem in this setting is to define the beliefs/goals of a group of agents from a profile of bases, gathering the beliefs/goals of each member of the group. To this aim, a well-studied family of merging operators consists of distance-based ones: the models of the merged base are the closest interpretations to the given profile. Many operators from this family are based on the Hamming distance between interpretations, which can be viewed as a degree of conflict between them. In this paper, we introduce a more general family of merging operators, based on a more primitive concept, namely the conflict between interpretations itself. We show that this family of conflict-based merging operators includes many operators from the literature, both model-based ones and syntax-based ones. We present a number of comparison relations on conflict vectors characterizing operators from this family, and study the logical properties of conflict-based merging operators.

Introduction

Reasoning under inconsistency is a major challenge in artificial intelligence. Its importance is reflected by the number of approaches considered so far for dealing with inconsistent pieces of information. Among them are a number of paraconsistent logics, argumentative logics, belief revision and belief merging operators, knowledge integration techniques, and so on, described in an abundant literature. Most of these approaches have in common to be anchored on a *notion of conflict*. In the propositional case, such conflicts can be defined on the pieces of evidence themselves (i.e., represented by formulas or sets of formulas) or on the corresponding possible worlds (i.e., the interpretations of such formulas).

Now, those approaches to reasoning under inconsistency can be discriminated by focusing on their objective and the inputs they require. Thus, belief/goal merging aims at defining the beliefs/goals of a group of agents/sources from the individual belief/goal bases of the agents, gathered into a profile, plus some integrity constraints (encoding laws of Nature, norms, etc.), which has to be satisfied. Much work has been devoted to the definition of such operators in the propositional case[1] (Revesz 1993; Liberatore & Schaerf 1998; Baral *et al.* 1992; Konieczny & Pino Pérez 2002a; Meyer, Pozos Parra, & Perrussel 2005), and to the study of their properties w.r.t. different criteria, mainly logical properties, strategy-proofness, complexity. See for instance (Konieczny & Pino Pérez 2002a; Revesz 1997; Liberatore & Schaerf 1998; Konieczny, Lang, & Marquis 2004) for postulates characterizing propositional merging operators, (Everaere, Konieczny, & Marquis 2007) for an investigation of strategy-proofness issues, and (Konieczny, Lang, & Marquis 2004; Everaere, Konieczny, & Marquis 2007) for computational complexity results.

The two main families of belief merging operators gather on the one hand the so-called "syntax-based" operators and on the other hand, the "model-based" ones. In a nutshell, syntax-based operators select subsets of formulas in the union of all bases. Typically, only these subsets which do not conflict with the given integrity constraints are selected. Some further selection ("preference") criteria can be used as well (they aim at preserving as much information as possible).

Model-based operators basically select the models of the given integrity constraints which are as close as possible to the profile of the group. Closeness is defined via a notion of profile assignment which maps each profile to a binary relation over interpretations, expressing that some interpretations are at least as close to the profile than some others. Very often, model-based operators are defined from a distance between interpretations, which intuitively indicates how conflicting they are. This distance between interpretations induce a "distance" between an interpretation and a base, which indicates the degree of conflict between them. Once such distances are computed, an aggregation function is used to define the overall "distance" of each model (of the integrity constraints) to the profile. Semantically, the models of the result of the merging are the closest ones to the profile, i.e., the models of the integrity constraints which are in some sense the less conflicting ones with the profile.

In order to define model-based merging operators, a commonly used distance between interppretations is the Ham-

[1]There are also works on merging in richer logical settings, see for instance (Meyer 2001; Benferhat *et al.* 2002; Chopra, Ghose, & Meyer 2002).

ming distance. The first use of this distance for defining belief change operators can be traced back to the revision operator defined by Dalal (Dalal 1988). The Hamming distance between two interpretations is the number of propositional variables the two interpretations disagree on. The amount of conflict between two interpretations is thus assessed as the number of atoms whose truth values must be flipped in one interpretation in order to make it identical to the second one.

The major problem with such distance-based merging operators is that evaluating the closeness of interpretations as a number can lead to lose too many information. In particular, the conflicting variables themselves (and not only how many they are) can prove significant. Especially, when variables express real-world properties, it may happen that some variables are more important than others, or that some variables are logically connected. In these cases, distances are not fully satisfactory.

In the belief revision/update literature an interesting measure used to evaluate the closeness of two interpretations is diff, the symmetrical difference between the two interpretations (see e.g., (Katsuno & Mendelzon 1991b; 1991a; Dalal 1988; Weber 1986; Satoh 1988; Borgida 1985; Winslett 1988)). Instead of evaluating the degree of conflict between two interpretations as the number of variables on which they differ (as it is the case with the Hamming distance), the diff measure assesses it as the set of such variables. Stated otherwise, instead of measuring the degree of conflict between interpretations as the size of the conflict (which does not preserve all the available information about the conflict since, e.g., two distinct conflicts may easily have the same size), such approaches use the conflict itself.

In this paper, we introduce and study the family of *conflict-based propositional merging operators*, i.e., the merging operators based on the *diff* measure. We show how conflict-based operators are parameterized by a comparison relation on conflict vectors. We present several such comparison relations, and define the corresponding merging operators. Our contribution mainly is three fold. We show that many merging operators from the literature (both syntax-based ones and model-based ones) can be easily captured in our framework. We explain how one can easily define new merging operators by combining comparison relations on conflict vectors (especially, through their lexicographic product). Finally, we investigate the connections between the properties of preference relations and the logical properties of the induced merging operators. Especially, we provide new conditions on profile assignments which prove sufficient to ensure the corresponding merging operators to satisfy some IC rationality postulates. Such conditions are then used to determine the logical properties satisfied by conflict-based merging operators, in the general case and in some specific cases (depending on the associated comparison relation on conflict vectors).

Formal Preliminaries

We consider a propositional language $\mathcal{L}$ defined from a finite set of propositional variables $\mathcal{P}$ and the usual connectives. For any subset c of $\mathcal{P}$, $|c|$ denotes the cardinality of c. An interpretation (or world) is a total function from $\mathcal{P}$ to $\{0, 1\}$, denoted by a bit vector whenever a strict total order on $\mathcal{P}$ is specified. The set of all interpretations is noted $\mathcal{W}$. An interpretation ω is a model of a formula $\phi \in \mathcal{L}$ if and only if it makes it true in the usual truth functional way. $[\phi]$ denotes the set of models of formula ϕ, i.e., $[\phi] = \{\omega \in \mathcal{W} \mid \omega \models \phi\}$. $\models$ denotes logical entailment and $\equiv$ denotes logical equivalence.

Let $\leq$ be any relation; $x \simeq y$ is a notation for $x \leq y$ and $y \leq x$, and $x < y$ is a notation for $x \leq y$ and $y \not\leq x$.

A *base* K denotes the set of beliefs/goals of an agent, it is a finite and consistent set of propositional formulas, interpreted conjunctively. Unless stated otherwise, we identify K with the conjunction of its elements.

A *profile* E denotes the group of agents that is involved in the merging process. It is a vector of belief/goal bases $E = \langle K_1, \ldots, K_n \rangle$ (observe that two agents are allowed to exhibit identical bases). We denote by $\bigwedge E$ the conjunction of bases of $E = \langle K_1, \ldots, K_n \rangle$, i.e., $\bigwedge E = K_1 \wedge \ldots \wedge K_n$. A profile E is said to be consistent if and only if $\bigwedge E$ is consistent. We say that two profiles $E = \langle K_1, \ldots, K_n \rangle$ and $E' = \langle K'_1, \ldots, K'_n \rangle$ are equivalent, noted $E_1 \equiv E_2$, if there exists a permutation π from $\{1, \ldots, n\}$ to $\{1, \ldots, n\}$ such that for every $i \in \{1, \ldots, n\}$, K_i and $K'_{\pi(i)}$ are logically equivalent. Profile $E = \langle K_1, \ldots, K_n \rangle$ is also often viewed as the corresponding multi-set $\{K_1, \ldots, K_n\}$ (when it is the case, an anonymity assumption on the merging is implicitly done: two equivalent profiles must lead to the same merged base (up to logical equivalence)).

A merging operator is a mapping $\Delta : (E, \mu) \mapsto \Delta_\mu(E)$, which associates a profile E and a base μ, the integrity constaints, to a *merged base*.

Model-based merging operators are typically characterized by a distance between interpretations and an aggregation function. Two widely used distances between interpretations are Dalal distance (Dalal 1988), denoted d_H, that is the Hamming distance between interpretations (the number of propositional variables on which the two interpretations differ); and the drastic distance, denoted d_D, that is the simplest distance one can define: it gives 0 if the two interpretations are the same one, and 1 otherwise. And an aggregation function is:

Definition 1 *An* aggregation function f *is a total function associating a nonnegative real number to every finite tuple of nonnegative real numbers s.t. for any* $x_1, \ldots, x_n, x, y \in \mathbb{R}^+$*:*

- *if* $x \leq y$*, then* $f(x_1, \ldots, x, \ldots, x_n) \leq f(x_1, \ldots, y, \ldots, x_n)$. *(non-decreasingness)*
- $f(x_1, \ldots, x_n) = 0$ *if and only if* $x_1 = \ldots = x_n = 0$. *(minimality)*
- $f(x) = x$. *(identity)*

Finally, the following properties for propositional merging operators have been pointed out in (Konieczny & Pino Pérez 2002a). In these properties, profiles are considered as multi-sets of bases and $\sqcup$ denotes multi-set union.

(IC0) $\triangle_\mu(E) \models \mu$.

(IC1) If μ is consistent, then $\triangle_\mu(E)$ is consistent.

(IC2) If $\bigwedge E$ is consistent with μ, then $\triangle_\mu(E) \equiv \bigwedge E \wedge \mu$.

(IC3) If $E_1 \equiv E_2$ and $\mu_1 \equiv \mu_2$, then $\triangle_{\mu_1}(E_1) \equiv \triangle_{\mu_2}(E_2)$.

(IC4) If $K_1 \models \mu$ and $K_2 \models \mu$, then $\triangle_\mu(\{K_1, K_2\}) \wedge K_1$ is consistent if and only if $\triangle_\mu(\{K_1, K_2\}) \wedge K_2$ is consistent.

(IC5) $\triangle_\mu(E_1) \wedge \triangle_\mu(E_2) \models \triangle_\mu(E_1 \sqcup E_2)$.

(IC6) If $\triangle_\mu(E_1) \wedge \triangle_\mu(E_2)$ is consistent, then $\triangle_\mu(E_1 \sqcup E_2) \models \triangle_\mu(E_1) \wedge \triangle_\mu(E_2)$.

(IC7) $\triangle_{\mu_1}(E) \wedge \mu_2 \models \triangle_{\mu_1 \wedge \mu_2}(E)$.

(IC8) If $\triangle_{\mu_1}(E) \wedge \mu_2$ is consistent, then $\triangle_{\mu_1 \wedge \mu_2}(E) \models \triangle_{\mu_1}(E)$.

$\triangle$ is an IC merging operator if and only if it satisfies the postulates **(IC0)** to **(IC8)**.

Conflict-Based Merging Operators

As evoked in the introduction, conflict-based merging operators rely on the notion of conflict between interpretations. The conflict between two interpretations ω and ω' is defined as follows:

$$\mathsf{diff}(\omega, \omega') = \{p \in \mathcal{P} \mid \omega(p) \neq \omega'(p)\}.$$

As when standard distances are considered, we can straightforwardly define a *diff*-based notion of closeness between an interpretation and a base, as the minimum closeness between the interpretation and the models of the base. Of course, since *diff* gives as output a set of variables instead of a number, set-inclusion has to be considered as a minimality criterion:

$$\mathsf{diff}(\omega, K) = min(\{\mathsf{diff}(\omega, \omega') \mid \omega' \models K\}, \subseteq).$$

So the closeness between an interpretation and a base is measured as the set of the minimal sets (for set inclusion) of propositional variables which differ between the interpretation and the (models of the) base.

This notion of closeness can be extended to any profile $E = \langle K_1, \ldots, K_n \rangle$ as follows:

$$\mathsf{diff}(\omega, E) = \{\langle c_1^\omega, c_2^\omega, \ldots, c_n^\omega \rangle \mid c_i^\omega \in \mathsf{diff}(\omega, K_i)\}.$$

$\mathsf{diff}(\omega, E)$ is thus a set of vectors of sets of propositional variables; each such vector is referred to as a conflict vector. It has the same dimension as E. By construction, if $\langle c_1^\omega, c_2^\omega, \ldots, c_n^\omega \rangle$ belongs to $\mathsf{diff}(\omega, E)$, then ω is a model of the profile obtained from E by forgetting in each K_i all the variables from c_i^ω (Lang, Liberatore, & Marquis 2003).

In order to define the merging of a profile E under integrity constraints μ in a model-theoretic way, a standard approach consists in selecting the models of μ which are as close as possible to E, through a binary relation $\leq_R^E$ over interpretations, where $\omega \leq_R^E \omega'$ is interpreted as "ω is at least as close to E as ω'". In the following, we show that many such relations $\leq_R^E$ can be derived from $\mathsf{diff}(\omega, E)$. This is achieved in two steps:

- We start with a relation $\preceq_R$ which compares conflicts vectors, so that we can check whether
$$\langle c_1, \ldots, c_n \rangle \preceq_R \langle c'_1, \ldots, c'_n \rangle.$$
- We lift the relation $\preceq_R$ to sets of conflict vectors so as to derive a corresponding relation over interpretations $\leq_R^E$.

We first present a number of comparison relations $\preceq$ on vectors of subsets of the propositional variables $\mathcal{P}$, where $c \preceq c'$ intuitively means that the conflict vector c is as much as significant as the conflict vector c'.

Definition 2 *Let $c = \langle c_1, c_2, \ldots, c_n \rangle$ and $c' = \langle c'_1, c'_2, \ldots, c'_n \rangle$ be two conflict vectors of dimension n; let f be an n-ary aggregation function. We consider permutations $\pi : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$. We consider the following relations:*

- $c \preceq_{\subseteq} c' \Leftrightarrow \forall i \in \{1, \ldots, n\}, c_i \subseteq c'_i$.
- $c \preceq_{\subseteq_\pi} c' \Leftrightarrow c_i \subseteq c'_{\pi(i)}$, *where π is a permutation*
- $c \preceq_{\emptyset} c' \Leftrightarrow \forall i \in \{1, \ldots, n\}$, *if $c'_i = \emptyset$, then $c_i = \emptyset$.*
- $c \preceq_{\emptyset_\pi} c' \Leftrightarrow$ *if $c'_i = \emptyset$, then $c_{\pi(i)} = \emptyset$, where π is a permutation*
- $c \preceq_{\cup} c' \Leftrightarrow \bigcup_{i=1}^n c_i \subseteq \bigcup_{i=1}^n c'_i$.
- $c \preceq_{|\cup|} c' \Leftrightarrow \mid \bigcup_{i=1}^n c_i \mid \leq \mid \bigcup_{i=1}^n c'_i \mid$.
- $c \preceq_{\overline{f}} c' \Leftrightarrow f(\langle |c_1|, \ldots, |c_n| \rangle) \leq f(\langle |c'_1|, \ldots, |c'_n| \rangle)$.
- $c \preceq_{\overline{\overline{f}}} c' \Leftrightarrow f(\langle ||c_1||, \ldots, ||c_n|| \rangle) \leq f(\langle ||c'_1||, \ldots, ||c'_n|| \rangle)$ *where $||\emptyset|| = 0$ and $||c|| = 1$ if $c \neq \emptyset$.*

Let us illustrate some of these relations on an example:

Example 1 *Let $c = \langle \{a, b\}, \{c\}, \{b, c\}, \emptyset \rangle$ and $c' = \langle \{a\}, \{b, c\}, \{c\}, \emptyset \rangle$. We have: $c' \preceq_{\subseteq_\pi} c$, $c \preceq_{\emptyset} c'$ and $c' \preceq_{\emptyset} c$, $c \preceq_{\cup} c'$ and $c' \preceq_{\cup} c$, for $f = \Sigma$, we have $f(\langle \mid \{a, b\} \mid, \mid \{c\} \mid, \mid \{b, c\} \mid, \mid \emptyset \mid \rangle) = 5 \succeq_{\overline{\Sigma}} f(\langle \mid \{a\} \mid, \mid \{b, c\} \mid, \mid \{c\} \mid, \mid \emptyset \mid \rangle) = 4$.*

All these relations have in common to privilege conflict vectors reflecting as few conflict as possible in some sense (this will be make precise later on, cf. Proposition 12).

Interestingly, one can figure out many other comparison relations obtained by combining such relations $\preceq_R$ on conflict vectors. There are several ways to do it. One of them is based on the lexicographic product of relations:

Definition 3 *Let $\preceq_R$ and $\preceq_S$ be two binary relations over a set E. The lexicographic product $\preceq_{R.S} = \preceq_R . \preceq_S$ is the binary relation over E given by:*

$$x \preceq_{R.S} y \Leftrightarrow \begin{cases} x \preceq_R y \text{ and } y \not\preceq_R x \\ \text{or} \\ x \preceq_R y, y \preceq_R x \text{ and } x \preceq_S y \end{cases}$$

Once a comparison relation $\preceq_R$ is chosen, it has be lifted to sets of conflict vectors, in order to obtain $\leq_R^E$. On way to do it is as follows:

Definition 4 *Let $E = \langle K_1, \ldots, K_n \rangle$ be a profile and let $\preceq_R$ be a relation on conflict vectors of dimension n. We define the relation $\leq_R^E$ over $\mathcal{W}$ by $\omega \leq_R^E \omega' \Leftrightarrow \exists c \in \mathsf{diff}(\omega, E)$ s.t. $\forall c' \in \mathsf{diff}(\omega', E)$, we have $c \preceq_R c'$.*

For space reasons, we focus on this unique lifting principle in this paper. Other lifting principles could have been considered, using other alternations of quantifiers in Definition 4. Our investigation of these additional principles has shown that the one used here ($\exists\ \forall$) achieves a quite good compromise, in the sense that it leads to merging operators satisfying valuable properties (unlike for instance $\exists\,\exists$), without being too restrictive (like $\forall\,\forall$, which leads to merging operators with weak inferential power – i.e., almost all models of μ can be kept).

Now that $\leq^E_R$ is given, the corresponding conflict-based merging operator can be defined as usual as the selection of the models of μ which are as close as possible to E w.r.t. $\leq^E_R$:

Definition 5 *Let $E = \langle K_1, \ldots, K_n\rangle$ be a profile, μ some integrity constraints and let $\preceq_R$ be a relation on conflict vectors of dimension n. We define*

$$[\Delta^{\mathsf{diff},R}_\mu(E)] = min([\mu], \leq^E_R).$$

As one may expect, the properties satisfied by $\preceq_R$ have an impact on the properties satisfied by $\Delta^{\mathsf{diff},R}_\mu$; this concerns both the inferential power of the merging operator and its logical behaviour (as we will see in a forthcoming section).

We now explain how imposing some properties on $\preceq_R$ ensures some valuable logical properties for the corresponding conflict-based merging operator $\Delta^{\mathsf{diff},R}$. We start with properties linking $\preceq_R$ to $\leq^E_R$:

Proposition 1 *Let $E = \langle K_1, \ldots, K_n\rangle$ be a profile:*

- *If $\preceq_R$ is a transitive relation on conflict vectors of dimension n, then the corresponding lifted relation $\leq^E_R$ on interpretations is transitive.*
- *If $\preceq_R$ is a total preorder (i.e., a reflexive and transitive relation) on conflict vectors of dimension n, then the corresponding lifted relation $\leq^E_R$ on interpretations is a total preorder.*

Proof:

- If $\preceq_R$ is a transitive relation on conflict vectors of dimension n, then the corresponding lifted relation $\leq^E_R$ on interpretations is transitive.

  Transitivity: Suppose that $\omega \leq^E_R \omega_1$ and $\omega_1 \leq^E_R \omega_2$. Then $\exists c \in \mathsf{diff}(\omega, E)$, $\forall c' \in \mathsf{diff}(\omega_1, E)$, $c \preceq_R c'$. We also have $\exists c_1 \in \mathsf{diff}(\omega_1, E)$, $\forall c' \in \mathsf{diff}(\omega_2, E)$, $c_1 \preceq_R c'$. Since $c_1 \in \mathsf{diff}(\omega_1, E)$, we have $c \preceq_R c_1$ and $c_1 \preceq_R c'$. By transitivity of $\preceq_R$, we have $c \preceq_R c'$ $\forall c' \in \mathsf{diff}(\omega_2, E)$. So $\omega \leq^E_R \omega_2$.

- If $\preceq_R$ is a total preorder on conflict vectors of dimension n, then the corresponding lifted relation $\leq^E_R$ on interpretations is a total preorder.

  Reflexivity: Let ω be an interpretation. Since $\mathsf{diff}(\omega, E)$ is finite and $\preceq_R$ is a total preorder, there is at least one least vector $c \in \mathsf{diff}(\omega, E)$ w.r.t. $\preceq_R$, so we have $\exists c \in \mathsf{diff}(\omega, E)$, $\forall c' \in \mathsf{diff}(\omega, E)$, $c \preceq_R c'$ and $\omega \leq^E_R \omega$.

  Transitivity: See above.

  Completeness: Let ω and ω' be two interpretations. Let c (resp. c') be a least element of $\mathsf{diff}(\omega, E)$ (resp. $\mathsf{diff}(\omega', E)$) w.r.t. $\preceq_R$. Since $\preceq_R$ is total, we have $c \preceq_R c'$ or $c' \preceq_R c$. Assume that $c \preceq_R c'$. Since c' is a least element of $\mathsf{diff}(\omega', E)$, we have that $\forall c'' \in \mathsf{diff}(\omega', E)$, $c' \preceq_R c''$. By transitivity of $\preceq_R$, we get that $c \preceq_R c''$. Hence we have that $\exists c \in \mathsf{diff}(\omega, E), \forall c" \in \mathsf{diff}(\omega', E)$, $c \preceq_R c$", and as a consequence, $\omega \leq^E_R \omega'$. The remaining case $c' \preceq_R c$ enables to conclude in a similar way that $\omega' \leq^E_R \omega$. Finally, we get the expected conclusion: $\omega \leq^E_R \omega'$ or $\omega' \leq^E_R \omega$.

□

Observe that requiring $\preceq_R$ to be reflexive is not enough in general to ensure that the corresponding $\leq^E_R$ is reflexive. Contrastingly, as the previous proposition shows it, if $\preceq_R$ is a total preorder, then $\leq^E_R$ is reflexive.

As to inferential power, it is now easy to show that:

Proposition 2 *Let $E = \langle K_1, \ldots, K_n\rangle$ be a profile, μ an integrity constraint and let $\preceq_R$, $\preceq_S$ be two relations on conflict vectors of dimension n. If $\preceq_R \subseteq \preceq_S$ and $\preceq_R$ is a total preorder, then we have $\Delta^{\mathsf{diff},R}_\mu(E) \models \Delta^{\mathsf{diff},S}_\mu(E)$.*

Proof: Towards a contradiction assume that there exists a model ω of $\Delta^{\mathsf{diff},R}_\mu(E)$ which does not satisfy $\Delta^{\mathsf{diff},S}_\mu(E)$. Hence there exists a model ω' of $\Delta^{\mathsf{diff},S}_\mu(E)$ such that $\omega' <^E_S \omega$, so we must have $\omega \not\preceq^E_S \omega'$. From this and Definition 4, we get that $\forall c \in \mathsf{diff}(\omega, E)\ \exists c' \in \mathsf{diff}(\omega', E)$, $c \not\preceq_S c'$. Since $\preceq_R \subseteq \preceq_S$, we have that $c \not\preceq_R c'$. Since this holds for every $c \in \mathsf{diff}(\omega, E)$, we have that $\omega \not\preceq^E_R \omega'$. Since $\preceq_R$ is a total preorder, from Proposition 1, we know that $\preceq^E_R$ also is a total preorder. Since ω is a model of $\Delta^{\mathsf{diff},R}_\mu(E)$, ω must be a least model of μ w.r.t. $\preceq^E_R$: it must be the case that $\omega \preceq^E_R \omega'$ for every model ω' of μ, a contradiction. □

This proposition explains why it is important to determine which relations among the ones listed in Definition 2 are total preorders, and how they are related w.r.t. set-inclusion. We obtained the following easy proposition:

Proposition 3 *All the relations listed in Definition 2 are preorders, and $\preceq_{\emptyset_\pi}$, $\preceq_{|\cup|}$, $\preceq_{\overline{f}}$, $\preceq_{\overline{\overline{f}}}$ (for any aggregation function f), are total ones.*

Proof: The results come straightforwardly from the fact that $\subseteq$ and $\leq$ are orders, the latter one being total, plus the fact that the composition of two permutations is a permutation. □

As to the way they relate w.r.t. set-inclusion, we get:

Proposition 4 *Let f by any aggregation function. The inclusions between relations are stated in the Hasse diagram depicted on Figure 1, where each arrow $\preceq_A \leftarrow \preceq_B$ means that $\preceq_A \subseteq \preceq_B$, i.e. that $x \preceq_A y$ implies $x \preceq_B y$ (as usual with Hasse diagrams, for the sake of readibility, arrows stemming from reflexivity and transitivity of $\subseteq$ are not drawn).*

Proof:

- $\preceq_\subseteq$:

  $\preceq_\subseteq \subseteq \preceq_{\subseteq_\pi}$. Obvious.

  $\preceq_\subseteq \subseteq \preceq_{\overline{f}}$ for any aggregation function f. Suppose that $c \preceq_\subseteq c'$. Then, $\forall i \in \{1, \ldots, n\}, c_i \subseteq c'_i$. As a consequence, $\forall i \in \{1, \ldots, n\}, \mid c_i \mid \leq \mid c'_i \mid$. Since f is not

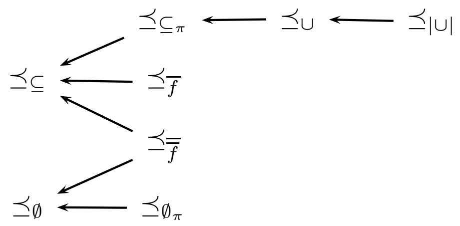

Figure 1: Inclusion of relations

decreasing, we have $f(\langle| c_1 |, | c_2 |, \dots, | c_n |\rangle) \leq f(\langle| c'_1 |, | c'_2 |, \dots, | c'_n |\rangle)$ and $c \preceq_{\overline{f}} c'$.

$\preceq_{\subseteq} \subseteq \preceq_{\overline{\overline{f}}}$ for any aggregation function f. Suppose that $c \preceq_{\subseteq} c'$. Then, $\forall i \in \{1, \dots, n\}, c_i \subseteq c'_i$. As a consequence, $\forall i \in \{1, \dots, n\}$, if $c'_i = \emptyset$ then $c_i = \emptyset$. So $\forall i \in \{1, \dots, n\}$, $\| c_i \| \leq \| c'_i \|$. Since f is not decreasing, we have $f(\langle\| c_1 \|, \| c_2 \|, \dots, \| c_n \|\rangle) \leq f(\langle\| c'_1 \|, \| c'_2 \|, \dots, \| c'_n \|\rangle)$ and $c \preceq_{\overline{\overline{f}}} c'$.

- $\preceq_{\subseteq_\pi}$:

  $\preceq_{\subseteq_\pi} \subseteq \preceq_{\cup}$. Suppose that $c \preceq_{\subseteq_\pi} c'$. Then, $\forall i \in \{1, \dots, n\}, c_i \subseteq c'_i$, and $\bigcup_{i=1}^{n} c_i \subseteq \bigcup_{i=1}^{n} c'_i$ so $c \preceq_{\cup} c'$.

- $\preceq_{\cup}$:

  $\preceq_{\cup} \subseteq \preceq_{|\cup|}$. Obvious.

- $\preceq_{\emptyset}$:

  $\preceq_{\emptyset} \subseteq \preceq_{\emptyset_\pi}$. Obvious.

  $\preceq_{\emptyset} \subseteq \preceq_{\overline{\overline{f}}}$ for any aggregation function f. Suppose that $c \preceq_{\emptyset} c'$. Then $\forall i \in \{1, \dots, n\}$, if $c'_i = \emptyset$, then $c_i = \emptyset$. So $\forall i \in \{1, \dots, n\}$, $\| c_i \| \leq \| c'_i \|$. Since f is not decreasing, we have $f(\langle\| c_1 \|, \| c_2 \|, \dots, \| c_n \|\rangle) \leq f(\langle\| c'_1 \|, \| c'_2 \|, \dots, \| c'_n \|\rangle)$ and $c \preceq_{\overline{\overline{f}}} c'$.

□

No other inclusion relation is satisfied by the preorders given in Definition 2. Especially, $\preceq_{\emptyset}$ and $\preceq_{\subseteq}$ cannot be compared w.r.t. $\subseteq$, as well as $\preceq_{\emptyset}$ and $\preceq_{\cup}$.

Many Merging Operators are Conflict-Based Ones

Let us now show that many merging operators from the literature are conflict-based ones. Our purpose is not to be exhaustive here but to show that a variety of operators, including both model-based ones and syntax-based ones, can be recovered as conflict-based operators. We first show that all the model-based merging operators based on the Hamming distance or the drastic distance are conflict-based ones:

Proposition 5 *For any aggregation function f, we have* $\Delta^{d_H, f} = \Delta^{\mathsf{diff}, \overline{f}}$.

Proof: The proof is based on the fact that for any interpretation ω and any base K_i, we have: $d_H(\omega, K_i) = min_{c_i \in \mathsf{diff}(\omega, K_i)}(| c_i |, \leq)$. Since an aggregation function is not decreasing, we have:

$$d_H(\omega, \{K_1, K_2, \dots, K_n\}) =$$
$$min_{c_i \in \mathsf{diff}(\omega, K_i), 1 \leq i \leq n}(f(\langle| c_1 |, | c_2 |, \dots, | c_n |\rangle), \leq).$$

We have:

$$\omega \leq^{E}_{\overline{f}} \omega' \Leftrightarrow \exists c(\omega, E) \in \mathsf{diff}(\omega, E), \text{ s.t.}$$
$$\forall c(\omega', E) \in \mathsf{diff}(\omega', E), c(\omega, E) \preceq_{\overline{f}} c(\omega', E).$$

This is equivalent to:

$$\omega \leq^{E}_{\overline{f}} \omega' \Leftrightarrow \exists c(\omega, E) \in \mathsf{diff}(\omega, E), \text{ s.t.}$$
$$\forall c(\omega', E) \in \mathsf{diff}(\omega', E),$$
$$f(\langle| c_1 |, | c_2 |, \dots, | c_n |\rangle) \leq f(\langle| c'_1 |, | c'_2 |, \dots, | c'_n |\rangle).$$

Hence, the models of μ that are minimal w.r.t. $\leq_{\mu}^{\mathsf{diff}, \overline{f}}$ are exactly the models of μ minimal w.r.t. $\leq^{E}_{d_H}$.

□

Proposition 6 *For any aggregation function f, we have* $\Delta^{d_D, f} = \Delta^{\mathsf{diff}, \overline{\overline{f}}}$.

Proof: For any interpretation ω, we have $d_D(\omega, K_i) = min_{c_i \in \mathsf{diff}(\omega, K_i)}(\| c_i \|), \leq)$. Then we have:

$$d_D(\omega, \{K_1, K_2, \dots, K_n\}) =$$
$$min_{c_i \in \mathsf{diff}(\omega, K_i), 1 \leq i \leq n}(f(\langle\| c_1 \|), \| c_2 \|, \dots, \| c_n \|\rangle), \leq).$$

Hence, the models of μ that are minimal w.r.t. $\leq^{E}_{\overline{\overline{f}}}$ are exactly the models of μ minimal w.r.t. $\leq^{E}_{d_D}$. □

Syntax-based operators can be also easily recovered. Let us show now how to define in the conflict-based framework the operators Δ^{C_1} and Δ^{C_4} considered in (Baral, Kraus, & Minker 1991; Baral *et al.* 1992; Konieczny 2000):

Proposition 7 $\Delta^{C_1} = \Delta^{\mathsf{diff}, \emptyset}$.

Proof: We know that $\Delta^{C_1}_{\mu} \equiv \bigvee\{M \in MAXCONS(\bigcup_{K_i \in E} K_i, \mu)\}$. $MAXCONS(\bigcup_{K_i \in E} K_i, \mu)$ is the set of all maximal (for inclusion) consistent subsets of formulas of $\bigcup_{K_i \in E} K_i \cup \mu$. Any $M \in MAXCONS(\bigcup_{K_i \in E} K_i, \mu)$ corresponds to a conflict vector containing a maximum (w.r.t. pointwise inclusion) of coordinates equal to $\emptyset$. So the models of $\Delta^{C_1}_{\mu}$ are exactly the models of $\Delta^{\mathsf{diff}, \emptyset}_{\mu}$. □

Proposition 8 $\Delta^{C_4} = \Delta^{\mathsf{diff}, \emptyset_\pi}$.

Proof: We know that $\Delta^{C_4}_{\mu} \equiv \bigvee\{M \in MAXCONS_{card}(\bigcup_{K_i \in E} K_i, \mu)\}$. $MAXCONS_{card}(\bigcup_{K_i \in E} K_i, \mu)$ is the set of all maximal (for cardinality) consistent subsets of formulas of $\bigcup_{K_i \in E} K_i \cup \mu$. Any $M \in MAXCONS_{card}(\bigcup_{K_i \in E} K_i, \mu)$ corresponds to a conflict vector containing a maximum (w.r.t. cardinality) of coordinates equal to $\emptyset$. So the models of $\Delta^{C_4}_{\mu}$ are exactly the models of $\Delta^{\mathsf{diff}, \emptyset_\pi}_{\mu}$. □

ω	$\mathsf{diff}(\omega, K_1)$	$\mathsf{diff}(\omega, K_2)$	$Gmax(\langle \mid c_1 \mid, \mid c_2 \mid \rangle)$	$\mid c_1 \cup c_2 \mid$
000	$\{\{a\}, \{b, c\}\}$	$\{\emptyset\}$	$(1, 0)$	1
111	$\{\emptyset\}$	$\{\{a\}\}$	$(1, 0)$	1

Table 1: $\Delta_\mu^{\mathsf{diff}, \overline{Gmax}.|\cup|}(E_1)$

ω	$\mathsf{diff}(\omega, K_1)$	$\mathsf{diff}(\omega, K_3)$	$Gmax(\langle \mid c_1 \mid, \mid c_3 \mid \rangle)$	$\mid c_1 \cup c_3 \mid$
000	$\{\{a\}, \{b, c\}\}$	$\{\emptyset\}$	$(1, 0)$	1
111	$\{\emptyset\}$	$\{\{b\}\}$	$(1, 0)$	1

Table 2: $\Delta_\mu^{\mathsf{diff}, \overline{Gmax}.|\cup|}(E_2)$

ω	$\mathsf{diff}(\omega, K_1)$	$\mathsf{diff}(\omega, K_2)$	$\mathsf{diff}(\omega, K_1)$	$\mathsf{diff}(\omega, K_3)$	$Gmax(\langle \mid c_i \mid \rangle)$	$\mid \bigcup c_i \mid$
000	$\{\{a\}, \{b, c\}\}$	$\{\emptyset\}$	$\{\{a\}, \{b, c\}\}$	$\{\emptyset\}$	$(1, 1, 0, 0)$	1
111	$\{\emptyset\}$	$\{\{a\}\}$	$\{\emptyset\}$	$\{\{b\}\}$	$(1, 1, 0, 0)$	2

Table 3: $\Delta_\mu^{\mathsf{diff}, \overline{Gmax}.|\cup|}(E_1 \sqcup E_2)$

Generating New Merging Operators

As evoked in a previous section, one can easily generate new conflict-based operators by combining comparison relations $\preceq_R$ on conflict vectors using lexicographic product.

Obviously enough, the lexicographic product of $\preceq_R$ by $\preceq_S$ leads to a relation $\preceq_{R.S} = \preceq_R . \preceq_S$ refining $\preceq_R$: $\preceq_{R.S} \subseteq \preceq_R$. As shown by Proposition 2, if $\preceq_{R.S}$ is a total preorder (which is ensured whenever both $\preceq_R$ and $\prec_S$ are total preorders), then more information is typically preserved by the conflict-based merging based on $\preceq_{R.S}$ in the sense that $\Delta_\mu^{\mathsf{diff}, R.S}(E) \models \Delta_\mu^{\mathsf{diff}, R}(E)$ for any E and μ.

One of the main motivations for introducing conflict-based merging operators is that they include new merging operators, refining existing ones. Thus using the lexicographic product, one can define new operators based on usual model-based operators (Konieczny & Pino Pérez 2002b), like $\Delta^{d_H, Gmax}$, $\Delta^{d_H, \Sigma}$, etc. so that the new operators have a stronger inferential power.

The gain of inferential power achieved by such refinements may easily lead to get rid of some logical properties, i.e., $\Delta^{\mathsf{diff}, R.S}$ does not always satisfy all the postulates satisfied by $\Delta_\mu^{\mathsf{diff}, R}$. For the sake of illustration let us consider the conflict-based merging operator given by $\Delta_\mu^{\mathsf{diff}, \overline{Gmax}.|\cup|}$ ($\preceq_{\overline{Gmax}} . \preceq_{|\cup|}$ is a total preorder).[2]

Proposition 9 $\Delta^{\mathsf{diff}, \overline{Gmax}.|\cup|}$ *satisfies* **(IC0)**, **(IC1)**, **(IC2)**, **(IC3)**, **(IC4)**, **(IC7)**, **(IC8)**. *It does not satisfy* **(IC5)** *or* **(IC6)**.

Proof:

- **(IC0)**, **(IC1)**, **(IC7)** and **(IC8)** come from Proposition 11.
- **(IC2)**: If $\bigwedge E \wedge \mu$ is consistent, then any model ω of $\bigwedge E \wedge \mu$ satisfies:

  $$min_{c_i \in \mathsf{diff}(\omega, K_i), 1 \leq i \leq n}(Gmax(\langle \mid c_1 \mid, \ldots, \mid c_n \mid \rangle), \leq)$$
  $$= (0, \ldots, 0)$$

  and $min_{c_i \in \mathsf{diff}(\omega, K_i)} \mid \bigcup_{i=1}^n c_i \mid = 0$.

  So the models of $\Delta_\mu^{\mathsf{diff}, \overline{Gmax}.|\cup|}(E)$ are exactly the models of $\bigwedge E \wedge \mu$.
- **(IC3)**: Obvious.
- **(IC4)**: $\Delta^{\mathsf{diff}, \overline{Gmax}}$ satisfies **(IC4)**. We have to show that its refinement by $\mid \cup \mid$ preserves this property. Suppose that $K_1 \models \mu$, $K_2 \models \mu$ and that $\Delta_\mu^{\mathsf{diff}, \overline{Gmax}.|\cup|}(\{K_1, K_2\}) \wedge K_1$ is consistent.

  Let ω be a model of $\Delta_\mu^{\mathsf{diff}, \overline{Gmax}.|\cup|}(\{K_1, K_2\}) \wedge K_1$ and let c be the minimal vector of $\mathsf{diff}(\omega, \{K_1, K_2\})$ w.r.t. $\preceq_{\overline{Gmax}.|\cup|}$. Let ω_2 be the model of K_2 such that $c = \mathsf{diff}(\omega, \{K_1, \{\omega_2\}\}) = \langle \emptyset, c_2 \rangle$. Then $\langle \emptyset, c_2 \rangle \in \mathsf{diff}(\omega_2, \{K_1, K_2\})$. Since $\langle \emptyset, c_2 \rangle$ is minimal for $\preceq_{\overline{Gmax}.|\cup|}$, $\langle c_2, \emptyset \rangle$ is minimal as well. So $\omega_2 \models \Delta_\mu^{\mathsf{diff}, \overline{Gmax}.|\cup|}(\{K_1, K_2\})$ and $\Delta_\mu^{\mathsf{diff}, \overline{Gmax}.|\cup|}(\{K_1, K_2\}) \wedge K_2$ is consistent.
- **(IC5)**: We consider the bases $K_1 = (a \wedge \neg b \wedge \neg c) \vee (a \wedge b \wedge c)$, $K_2 = (\neg a \wedge b \wedge c) \vee (\neg a \wedge \neg b \wedge \neg c)$, $K_3 = (a \wedge \neg b \wedge c) \vee (\neg a \wedge \neg b \wedge \neg c)$, and two profiles $E_1 = \{K_1, K_2\}$ and $E_2 = \{K_1, K_3\}$. The integrity constraint is $\mu = (a \wedge b \wedge c) \vee (\neg a \wedge \neg b \wedge \neg c)$. Details of computations are in Tables 1, 2 and 3. We have $[\Delta_\mu^{\mathsf{diff}, \overline{Gmax}.|\cup|}(E_1)] = \{000, 111\}$, $[\Delta_\mu^{\mathsf{diff}, \overline{Gmax}.|\cup|}(E_2)] = \{000, 111\}$. We also have $[\Delta_\mu^{\mathsf{diff}, \overline{Gmax}.|\cup|}(E_1 \sqcup E_2)] = \{000\}$, and $\Delta_\mu^{\mathsf{diff}, \overline{Gmax}.|\cup|}(E_1) \wedge \Delta_\mu^{\mathsf{diff}, \overline{Gmax}.|\cup|}(E_2) \not\models \Delta_\mu^{\mathsf{diff}, \overline{Gmax}.|\cup|}(E_1 \sqcup E_2)$.
- **(IC6)**: We consider the bases $K_1 = (\neg a \wedge \neg b \wedge \neg c) \vee (a \wedge b \wedge c)$, $K_2 = (a \wedge b \wedge \neg c) \vee (a \wedge b \wedge c)$, $K_3 = (\neg a \wedge \neg b \wedge \neg c) \vee (\neg a \wedge \neg b \wedge c)$, and

[2] Remind that when f is an aggregation function, $\overline{f}$ denotes the aggregation of the cardinalities of the input sets. See Definition 2.

ω	$\mathsf{diff}(\omega, K_1)$	$\mathsf{diff}(\omega, K_2)$	$Gmax(\langle \mid c_1 \mid, \mid c_2 \mid \rangle)$	$\mid c_1 \cup c_2 \mid$
100	$\{\{a\},\{b,c\}\}$	$\{\{b\}\}$	$(1,1)$	2
011	$\{\{a\},\{b,c\}\}$	$\{\{a\}\}$	$(1,1)$	1

Table 4: $\Delta_\mu^{\mathsf{diff},\overline{Gmax}.|\cup|}(E_1)$

ω	$\mathsf{diff}(\omega, K_3)$	$\mathsf{diff}(\omega, K_4)$	$Gmax(\langle \mid c_3 \mid, \mid c_4 \mid \rangle)$	$\mid c_3 \cup c_4 \mid$
100	$\{\{a\}\}$	$\{\{\{b\},\{a,c\}\}\}$	$(1,1)$	2
011	$\{\{\{b\}\}\}$	$\{\{b\},\{a,c\}\}$	$(1,1)$	1

Table 5: $\mathsf{diff}_\mu^{\mathsf{diff},\overline{Gmax}.|\cup|}(E_2)$

ω	$\mathsf{diff}(\omega, K_1)$	$\mathsf{diff}(\omega, K_2)$	$\mathsf{diff}(\omega, K_3)$	$\mathsf{diff}(\omega, K_4)$	$Gmax(\langle \mid c_i \mid \rangle)$	$\mid \bigcup c_i \mid$
100	$\{\{a\},\{b,c\}\}$	$\{\{b\}\}$	$\{\{a\}\}$	$\{\{\{b\},\{a,c\}\}\}$	$(1,1,1,1)$	2
011	$\{\{a\},\{b,c\}\}$	$\{\{a\}\}$	$\{\{\{b\}\}\}$	$\{\{b\},\{a,c\}\}$	$(1,1,1,1)$	2

Table 6: $\Delta_\mu^{\mathsf{diff},\overline{Gmax}.|\cup|}(E_1 \sqcup E_2)$

$K_4 = (a \wedge b \wedge \neg c) \vee (\neg a \wedge \neg b \wedge c)$. We consider also two profiles $E_1 = \{K_1, K_2\}$ and $E_2 = \{K_3, K_4\}$. The integrity constraint is $\mu = (a \wedge \neg b \wedge \neg c) \vee (\neg a \wedge b \wedge c)$. Details of computations are in Tables 4, 5 and 6. We have $[\Delta_\mu^{\mathsf{diff},\overline{Gmax}.|\cup|}(E_1)] = \{011\}$, $[\Delta_\mu^{\mathsf{diff},\overline{Gmax}.|\cup|}(E_2)] = \{011\}$. We also have $[\Delta_\mu^{\mathsf{diff},\overline{Gmax}.|\cup|}(E_1 \sqcup E_2)] = \{100, 011\}$, and $\Delta_\mu^{\mathsf{diff},\overline{Gmax}.|\cup|}(E_1 \sqcup E_2) \not\models \Delta_\mu^{\mathsf{diff},\overline{Gmax}.|\cup|}(E_1) \wedge \Delta_\mu^{\mathsf{diff},\overline{Gmax}.|\cup|}(E_2)$.

□

Compared to $\Delta^{d_H,Gmax}$ **(IC5)** and **(IC6)** are lost by $\Delta^{\mathsf{diff},\overline{Gmax}.|\cup|}$. This is the price to be paid for a more accurate conflict evaluation in some scenarios.

Nonetheless, some examples show that, although such refined operators do not satisfy all the expected logical properties for merging, they can prove more adequate than usual model-based operators in some cases. For instance, let us consider the following profile $E = \langle K_1, K_2, K_3, K_4 \rangle$ where K_i are reported in Table 7 and the constraint μ is such that $[\mu] = \{\omega_1, \omega_2\}$. Clearly the Hamming distance d_H does not discriminate between the two possible worlds, which can be problematic. For ω_1, all the agents agree on what they disagree (i.e., the conflict is on a), while this is not the case for ω_2. Operators based on the Hamming distance cannot make this distinction. Although the Hamming distances of the two interpretations to the bases are all identical and equal to 1, the diff distance exhibits the fact that there is less conflict on ω_1 than on ω_2 (while flipping the variable a in ω_1 is enough to obtain a model of all the bases, it is not the case with ω_2). We believe that this kind of examples opens the way for discussions on the scenarios where this behaviour is necessary, and on the logical characterization of this behaviour.

From Properties on $\preceq_R$ to Logical Properties on $\Delta^{\mathsf{diff},R}$

The price to be paid by the generality of the family of conflict-based merging operators is that only few logical postulates can be guaranteed if no conditions are imposed on the underlying preference relation. Especially none of **(IC4)**, **(IC5)**, and **(IC6)** can be guaranteed in the general case since Δ^{C_4} does not satisfy them, while it is a conflict-based merging operator (cf. Proposition 8). Contrastingly, as a direct consequence of Propositions 5 and 6, the family of conflict-based merging operators also includes a number of "fully rational" merging operators (i.e., IC merging ones).

Let us now present the logical properties satisfied by conflict-based merging operators. (Konieczny & Pino Pérez 2002a) give a representation theorem allowing to define IC merging operators from assignments which associate a pre-order on interpretations to each profile. For the representation theorem to hold, the assignment has to satisfy a set of properties that are not satisfied by all conflict-based merging operators. So it is interesting to determine the properties which are guaranteed by conflict-based operators. To this purpose, it is useful to recall first the representation theorem from (Konieczny & Pino Pérez 2002a). It is based on the notion of syncretic assignment:

Definition 6 *A* profile assignment *is a function φ mapping each profile E to a relation $\leq_E$ over interpretations. let us consider the following properties on such assignments, for any $\omega, \omega' \in \mathcal{W}$:*

(0). $\leq_E$ *is a total preorder.*

(1). If $\omega \models \bigwedge E$ and $\omega' \models \bigwedge E$, then $\omega \simeq_E \omega'$.

(2). If $\omega \models \bigwedge E$ and $\omega' \not\models \bigwedge E$, then $\omega <_E \omega'$.

(3). If $E_1 \equiv E_2$, then $\leq_{E_1} = \leq_{E_2}$.

(4). $\forall \omega \models K \ \exists \omega' \models K' \ \omega' \leq_{\{K,K'\}} \omega$.

(5). If $\omega \leq_{E_1} \omega'$ and $\omega \leq_{E_2} \omega'$, then $\omega \leq_{E_1 \sqcup E_2} \omega'$.

(6). If $\omega <_{E_1} \omega'$ and $\omega \leq_{E_2} \omega'$, then $\omega <_{E_1 \sqcup E_2} \omega'$.

	$\mathsf{diff}(\omega, K_1)$	$\mathsf{diff}(\omega, K_2)$	$\mathsf{diff}(\omega, K_3)$	$\mathsf{diff}(\omega, K_4)$
ω_1	$\{\{a\}\}$	$\{\{a\}\}$	$\{\{a\}\}$	$\{\{a\}\}$
ω_2	$\{\{a\}\}$	$\{\{b\}\}$	$\{\{c\}\}$	$\{\{d\}\}$

Table 7: How to discriminate between ω_1 and ω_2?

A syncretic assignment *is a profile assignment which satisfies properties (0-6).*

The representation theorem states that:

Proposition 10 (Konieczny & Pino Pérez 2002a) $\triangle$ *is an IC merging operator if and only if there exists a syncretic assignment which maps each profile E to a total preorder $\leq_E$ such that*

$$[\triangle_\mu(E)] = \min([\mu], \leq_E).$$

Note that the conditions required on the assignment by this theorem are numerous, in particular it asks the relations given by the syncretic assignments to be total preorders.

The problem is that some comparison relations at work in in Definition 2, and used in Definitions 4 lead to relations which are not total preorders. So a key issue is to determine the properties ensured when this assumption on syncretic assignments is relaxed. The following proposition addresses it:

Proposition 11 *Let φ be a profile assignment which associates to each profile E a relation $\leq_E$ on interpretations. Let $\triangle$ be the merging operator given by $[\triangle_\mu(E)] = \min([\mu], \leq_E)$. Then $\triangle$ satisfies:*

- **(IC0)**, **(IC1)**, **(IC7)**, *and* **(IC8)**.
- **(IC2)** *if φ satisfies conditions (1) and (2).*
- **(IC3)** *if φ satisfies condition (3).*
- **(IC4)** *if φ satisfies condition (4).*
- **(IC5)** *if φ satisfies condition (5'): if $\omega <_{E_1 \sqcup E_2} \omega'$, then $\omega <_{E_1} \omega'$ or $\omega <_{E_2} \omega'$.*[3]
- **(IC6)** *if φ satisfies conditions (0) and (6).*

Proof:

(IC0): By definition $[\triangle_\mu(E)] \subseteq [\mu]$.
(IC1): If μ is consistent, then $[\mu] \neq \emptyset$ and, as there is a finite number of interpretations, there is no infinite descending chains of strict inequalities, so $\min([\mu], \leq_E) \neq \emptyset$. Then $\triangle_\mu(E)$ is consistent.
(IC2): Assume that $\bigwedge E \wedge \mu$ is consistent. We want to show that $\min([\mu], \leq_E) = [\bigwedge E \wedge \mu]$. First note that if $\omega \models E$ then from conditions (1) and (2), $\omega \in \min([\mu], \leq_E)$. So $\min([\mu], \leq_E) \supseteq [\bigwedge E \wedge \mu]$. For the other inclusion consider $\omega \in \min([\mu], \leq_E)$. Suppose towards a contradiction that $\omega \not\models E \wedge \mu$. So $\omega \not\models E$, by condition (2) we know that $\forall \omega' \models E \wedge \mu$ ($\bigwedge E \wedge \mu$ is consistent by assumption) $\omega' <_E \omega$. So $\omega \notin \min([\mu], \leq_E)$. Contradiction.
(IC3): Direct from condition (3) and the definition of $\triangle$.

[3]Note that for assignments satisfying condition (0), like syncretic assignments, conditions (5) and (5') are equivalent.

(IC4): Assume that $K \models \mu$, $K' \models \mu$, and $\triangle_\mu(\{K, K'\}) \wedge K \not\models \bot$, we want to show that $\triangle_\mu(\{K, K'\}) \wedge K' \not\models \bot$. Consider $\omega \models \triangle_\mu(\{K, K'\}) \wedge K$. Then $\nexists \omega' \models \mu$ s.t. $\omega' <_{\{K,K'\}} \omega$. But from condition (4) $\exists \omega' \models K'$ s.t. $\omega' \leq_{\{K,K'\}} \omega$. Since $\omega' \models \mu$, this means that $\omega' \simeq_{\{K,K'\}} \omega$. So $\omega' \in \min([\mu], \leq_{\{K,K'\}})$. Hence $\omega' \models \triangle_\mu(\{K, K'\})$ and therefore $\triangle_\mu(\{K, K'\}) \wedge K' \not\models \bot$.
(IC5): If $\omega \models \triangle_\mu(E_1) \wedge \triangle_\mu(E_2)$ then $\omega \in \min([\mu], \leq_{E_1})$ and so $\nexists \omega' \models \mu$ s.t. $\omega' <_{E_1} \omega$. We have in the same way $\nexists \omega' \models \mu$ s.t. $\omega' <_{E_2} \omega$. So we have that $\nexists \omega' \models \mu$ s.t. $\omega' <_{E_1 \sqcup E_2} \omega$ (otherwise by condition (5') a contradiction would follow). So $\omega \in \min([\mu], \leq_{E_1 \sqcup E_2})$. So by definition $\omega \models \triangle_\mu(E_1 \sqcup E_2)$.
(IC6): Assume that $\triangle_\mu(E_1) \wedge \triangle_\mu(E_2)$ is consistent. We want to show that $\triangle_\mu(E_1 \sqcup E_2) \models \triangle_\mu(E_1) \wedge \triangle_\mu(E_2)$ holds. Take $\omega \models \triangle_\mu(E_1 \sqcup E_2)$, so $\nexists \omega' \models \mu$ s.t. $\omega' <_{E_1 \sqcup E_2} \omega$. Suppose towards a contradiction that $\omega \not\models \triangle_\mu(E_1) \wedge \triangle_\mu(E_2)$. So $\omega \not\models \triangle_\mu(E_1)$ or $\omega \not\models \triangle_\mu(E_2)$. Suppose that $\omega \not\models \triangle_\mu(E_1)$ (the other case is symmetrical). So $\exists \omega' \models \mu$ s.t. $\omega' <_{E_1} \omega$ (*). As $\triangle_\mu(E_1) \wedge \triangle_\mu(E_2)$ is consistent $\exists \omega'' \models \triangle_\mu(E_1) \wedge \triangle_\mu(E_2)$. So $\nexists \omega' \models \mu$ s.t. $\omega' <_{E_1} \omega''$ and $\nexists \omega' \models \mu$ s.t. $\omega' <_{E_2} \omega''$. By condition (0) the two last inequalities are equivalent to respectively $\forall \omega' \models \mu$ s.t. $\omega'' \leq_{E_1} \omega'$ and $\forall \omega' \models \mu$ s.t. $\omega'' \leq_{E_2} \omega'$. Then by (*) and transitivity (from condition (0)) we have that $\omega'' <_{E_1} \omega$. And by condition (6) we obtain $\omega'' <_{E_1 \sqcup E_2} \omega$. Contradiction.
(IC7): Let us take $\omega \models \triangle_{\mu_1}(E) \wedge \mu_2$. We have $\nexists \omega' \models \mu_1$ s.t. $\omega' <_E \omega$. So $\nexists \omega' \models \mu_1 \wedge \mu_2$ s.t. $\omega' <_E \omega$, so $\omega \models \triangle_{\mu_1 \wedge \mu_2}(E)$.
(IC8): Assume that $\triangle_{\mu_1}(E) \wedge \mu_2$ is consistent, so $\exists \omega' \models \triangle_{\mu_1}(E) \wedge \mu_2$. Consider $\omega \models \triangle_{\mu_1 \wedge \mu_2}(E)$ and suppose that $\omega \not\models \triangle_{\mu_1}(E)$. So $\omega' <_E \omega$. But $\omega' \models \mu_1 \wedge \mu_2$ then $\omega \notin \min([\mu_1 \wedge \mu_2], \leq_E)$. Thus $\omega \not\models \triangle_{\mu_1 \wedge \mu_2}(E)$. Contradiction.

$\square$

As a consequence, we easily get that:

Proposition 12 *Conflict-based merging operators $\Delta^{\mathsf{diff}, R}$ satisfy* **(IC0)**, **(IC1)**, **(IC7)**, **(IC8)**.
Furthermore let us consider the two following properties on the relation $\preceq_R$:

- *for any $n > 0$, $\langle \emptyset, \ldots, \emptyset \rangle$ is the unique minimal element w.r.t. $\preceq_R$ of the set of all conflict vectors of dimension n,* **(minimality of empties)**
- *for any conflict vectors c and c' of dimension n we have $c \simeq_R c'$ when there exists a permutation $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that for every $i \in 1 \ldots n$, $c'_i = c_{\pi(i)}$,* **(permutation irrelevance)**

Then:

- *If $\preceq_R$ satisfies minimality of empties then $\Delta^{\mathsf{diff},R}$ satisfies* **(IC2)**.
- *If $\preceq_R$ satisfies permutation irrelevance then $\Delta^{\mathsf{diff},R}$ satisfies* **(IC3)**.

Since it is easy to show that all the relations listed in Definition 2 satisfy minimality of empties, we obtained that the corresponding conflict-based merging operators satisfy **(IC2)**. Similarly, it is easy to prove that all the relations listed in Definition 2, except $\preceq_{\subseteq}$, $\preceq_{\emptyset}$, and $\preceq_{\overline{f}}$ or $\preceq_{\overline{\overline{f}}}$ when f is not symmetric in any argument, satisfy premutation irrelevance; hence the corresponding conflict-based merging operators satisfy **(IC3)**.

More specifically, we have studied the logical properties of many conflict-based operators relying on preorders given in Definition 2. In the following, we refrain from considering $\Delta^{\mathsf{diff},\emptyset_\pi}$, $\Delta^{\mathsf{diff},\overline{f}}$, or $\Delta^{\mathsf{diff},\overline{\overline{f}}}$ since such operators are equivalent to operators already considered in the literature (and such an investigation has already been achieved for them), and just report the results for two operators:

Proposition 13 • $\Delta^{\mathsf{diff},\subseteq_\pi}$ *satisfies* **(IC0)**, **(IC1)**, **(IC2)**, **(IC3)**, **(IC4)**, **(IC7)** *and* **(IC8)**. *It does not satisfy* **(IC5)** *or* **(IC6)**.

- $\Delta^{\mathsf{diff},\cup}$ *satisfies* **(IC0)**, **(IC1)**, **(IC2)**, **(IC3)**, **(IC4)**, **(IC7)** *and* **(IC8)**. *It does not satisfy* **(IC5)** *or* **(IC6)**.

The proof of Proposition 13 is mainly based on results induced by Propositions 11 and 12, and on counterexamples to **(IC5)** and **(IC6)**. Since the rest of the proof is easy, we just give the counterexamples to **(IC5)** and **(IC6)** below.

Proof:

- $\Delta^{\mathsf{diff},\subseteq_\pi}$

(IC5): We consider the bases $K_1 = a \wedge \neg b \wedge \neg c$ and $K_2 = \neg a \wedge \neg b \wedge c$. The profiles are $E_1 = \{K_1\}$ and $E_2 = \{K_2\}$. The integrity constraint is $\mu = (a \wedge b \wedge c) \vee (\neg a \wedge \neg b \wedge \neg c)$. Details of computations are in Table 8.

ω	$\mathsf{diff}(\omega, K_1)$	$\mathsf{diff}(\omega, K_2)$	$\mathsf{diff}(\omega, E_1 \sqcup E_2)$
000	$\{\{a\}\}$	$\{\{c\}\}$	$\langle\{a\},\{c\}\rangle$
111	$\{\{b,c\}\}$	$\{\{a,b\}\}$	$\langle\{b,c\},\{a,b\}\rangle$

Table 8: $\Delta^{\mathsf{diff},\subseteq_\pi}$ does not satisfy **(IC5)**

We have $[\Delta_\mu^{\mathsf{diff},\subseteq_\pi}(E_1)] = \{000, 111\}$, $[\Delta_\mu^{\mathsf{diff},\subseteq_\pi}(E_2)] = \{000, 111\}$, but $[\Delta_\mu^{\mathsf{diff},\subseteq_\pi}(E_1 \sqcup E_2)] = \{000\}$, and $\Delta_\mu^{\mathsf{diff},\subseteq_\pi}(E_1) \wedge \Delta_\mu^{\mathsf{diff},\subseteq_\pi}(E_2) \not\models \Delta_\mu^{\mathsf{diff},\subseteq_\pi}(E_1 \sqcup E_2)$.

(IC6): We consider the bases $K_1 = a \wedge \neg b \wedge \neg c \wedge \neg d$ and $K_2 = a \wedge b \wedge \neg c \wedge \neg d$. The profiles are $E_1 = \{K_1\}$ and $E_2 = \{K_2\}$. The integrity constraint is $\mu = (a \wedge b \wedge c \wedge \neg d) \vee (\neg a \wedge \neg c \wedge \neg d)$. Details of computations are in Table 9.

We have $[\Delta_\mu^{\mathsf{diff},\subseteq_\pi}(E_1)] = \{0000, 1110\}$, $[\Delta_\mu^{\mathsf{diff},\subseteq_\pi}(E_2)] = \{0100, 1110\}$, but $[\Delta_\mu^{\mathsf{diff},\subseteq_\pi}(E_1 \sqcup E_2)] = \{0000, 0100, 1110\}$, and $\Delta_\mu^{\mathsf{diff},\subseteq_\pi}(E_1 \sqcup E_2) \not\models \Delta_\mu^{\mathsf{diff},\subseteq_\pi}(E_1) \wedge \Delta_\mu^{\mathsf{diff},\subseteq_\pi}(E_2)$.

ω	$\mathsf{diff}(\omega, K_1)$	$\mathsf{diff}(\omega, K_2)$	$\mathsf{diff}(\omega, E_1 \sqcup E_2)$
0000	$\{\{a\}\}$	$\{\{a,b\}\}$	$\langle\{a\},\{a,b\}\rangle$
0100	$\{\{a,b\}\}$	$\{\{a\}\}$	$\langle\{a,b\},\{a\}\rangle$
1110	$\{\{b,c\}\}$	$\{\{c\}\}$	$\langle\{b,c\},\{c\}\rangle$

Table 9: $\Delta^{\mathsf{diff},\subseteq_\pi}$ does not satisfy **(IC6)**

□

Conclusion

In this paper we introduced conflict-based merging operators. These operators are similar to usual model-based operators, in the sense that they select in the set of models of the integrity constraints, the models that are the closest ones to the profile under consideration. However, while usual model-based operators definition relies on a definition of distance between interpretations (very often the Hamming distance), conflict-based merging operators take into account the conflict itself, through the diff measure, instead of its size.

This diff measure has been exploited previously for defining revision/update operators (Katsuno & Mendelzon 1991b; 1991a; Weber 1986; Satoh 1988; Borgida 1985; Winslett 1988), but as far as we know, it has not been considered before for defining merging operators.

Our framework for conflict-based merging operators appears as a very general setting for defining merging operators; especially, many merging operators from the literature can be recovered in this framework. Furthermore, the possibility to combine comparison relations allows to define refinements (with respect to inference) of many well-known operators.

We have shown that such refinements may satisfy less logical properties for merging than their original counterparts. But we have also shown that they allow to discriminate conflicts in a subtle way, not achievable by distance-based operators. At a first glance, it looks that such distinctions are incompatible with postulates **(IC5)** and **(IC6)**. It is a perspective for further research to determine whether there exist conflict-based merging operators enabling a fine-grained discrimination of conflicts and satisfying **(IC5)** and **(IC6)**.

One interesting issue for further work would be to determine a representation theorem, in order to fully characterize the set of conflict-based merging operators from a logical point of view. Propositions 11 and 12 already state some of the properties they satisfy, but do not characterize them in an accurate way. Though interesting, it seems to be a difficult task, since today there is no such representation theorem for any family of merging operators. Especially, while model-based merging operators (definable from a distance and an aggregation function) are often taken as examples to illustrate the representation theorem of (Konieczny & Pino Pérez 2002b) for characterizing IC merging operators in terms of syncretic assignements, determining whether the set of IC merging operators is exactly the set of model-based merging operators is still an open issue.

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