Hybrid Partition Machines with Disturbances

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Abstract
The notions of finite analytic partition, dynamical consistency, and partition machine were originally developed in (Wei 1995; Caines & Wei 1998) for dynamical systems on differentiable manifolds and in (Caines & Lemch 1999) for hybrid systems; in this paper they are generalized to hybrid systems with disturbances.

1. Introduction
Many complex control systems in engineering, such as air traffic management systems (Tomlin, Pappas, & Sastry 1998), motor drives (Balluchi et al. 1998) and intelligent highway systems (Lygeros, Godbole, & Sastry 1998), have a hybrid nature, in the sense that (i) the lowest level they can be characterized by continuous differential equations, (ii) at the highest level by a discrete mechanism, and (iii) the evolution of the overall system is described by the interaction of all system levels.

In this paper a hybrid (base) system is modelled as a quintuple consisting of a state space (which is the direct product of a set of discrete states and an n-dimensional manifold), sets of admissible continuous and discrete controls, a family of controlled autonomous vector fields assigned to each discrete state, and a (partially defined) map of discrete transitions. Next, generalizing the theory presented in (Caines & Wei 1998), the notion of a finite analytic partition II of a state space of a hybrid system is defined. Then the notion of dynamical consistency is generalized to that of hybrid dynamical consistency. Based on these notions, the partition machine $H_{rl}$ of a hybrid system $H$ is defined in such a way that, in the class of in-block controllable partitions, the controllability of the high level system (described by the partition machine, which is a discrete finite state machine) is equivalent (under some technical conditions) to the controllability of the low level system (described by differential equations). Within the hybrid partition machine framework, a discrete controller supervises its continuous subsystems via hierarchical feedback relations; furthermore, each continuous subsystem is itself (internally) subject to feedback control. The resulting hierarchical control structure is such that the high level controller makes supervisory decisions, while the low level controllers will actually generate the required state trajectories.

An important problem within this framework is that of finding conditions for a partition to satisfy the in-block controllability hypothesis. In Section 4, a form of local accessibility for nonlinear control systems is introduced called the continuous fountain condition. To verify the in-block controllability hypothesis using the theory of this paper it is sufficient to establish that (i) the continuous fountain condition and (ii) a recurrence condition hold. Furthermore, for the so-called energy slice partitions of Hamiltonian systems (and, in fact, partitions of more general systems), the dense recurrence condition under a distinguished control is an inherent property which does not need explicit verification (whenever each slice is precompact).

In Section 5, an application of the theory to a highly simplified air traffic management system is presented.

2. Hybrid Partition Machines for Hybrid Systems
Consider a hybrid system $H$ which, in this paper, is taken to be the quintuple

$$H = \{L \triangle Q \times M, U, \Sigma, f, \Gamma\},$$

where

- $Q = \{q_1, \ldots, q_m\}$ is a set of discrete states (which are called control locations);
- $M$ is an open connected subset of $\mathbb{R}^n$;
- $L$ is called the state space of $H$;
- $U \subset \mathbb{R}^n$ is a set of admissible controls;
- $\Sigma$ is a finite set of transition labels;
- $f : Q \times M \times U \to TM$ is an autonomous vector field assigned to each control location;
- $\Gamma : L \times \Sigma \to L$ is an untimed (partially defined) map of discrete transitions.

Similar models of hybrid systems are employed by (Branicky, Borkar, & Mitter 1994; Broucke 1998; Lygeros, Godbole, & Sastry 1998; Tomlin, Pappas, & Sastry 1998) and others.

Henceforth in this paper the notation

$$\Gamma((q_1, y_1); \sigma(t^*)) \equiv \Gamma((q_1, y_1); \sigma) = (q_2, y_2)$$
Definition 1 A finite analytic partition of the state space \( L = Q \times M \) of a hybrid system \( H \) is a collection \( \Pi = \{\pi_q; q \in Q\} \) where each \( \pi_q, q \in Q, \) is a finite analytic partition of \( M. \) By definition a finite analytic partition \( \pi_q \) is a pairwise disjoint collection of subsets \( \pi_q = \{X^q_1, X^q_2, \ldots, X^q_{k_q}\} \) such that every \( X^q_i \) is non-empty, open, path-connected, and is such that

\[
\tilde{M} = \bigcup_{i=1}^{k_q} (X^q_i \cup \partial X^q_i),
\]

where, further, the boundary \( \partial X^q_i \) of every block \( X^q_i \) is a locally finite union of connected components of \( n-p \) dimensional, \( p \geq 1, \) analytic manifolds (possibly with boundary).

Definition 2 Let \( p = (q, x) \in L, p' = (q', x') \in L \) and let \( R \subseteq L. \) We shall say that \( p' \) is accessible from \( p \) with respect to \( R \) if there exists a finite sequence of admissible control pairs

\[
\{(u(t), t) \in [T_0, T_1); \sigma(T_1)\},
\]

\[
\cdots
\]

\[
\{(u(t), t) \in [T_{k-1}, T_k); \sigma(T_k)\},
\]

such that the state trajectory \( p(t) = (q(t), x(t)) \) defined as follows

\[
p(T_0) = p;
\]

\[
p(t) = (q(T_i), \phi_{q(T_i)}(t, x(T_i), u)),
\]

\[
t \in [T_i, T_{i+1}), 0 \leq i \leq k-1;
\]

\[
p(T_{k+1}) = \Gamma(\lim_{i \to T_{k+1}} p(t); \sigma(T_{k+1})))
\]

satisfies (i) \( p(t) \in R, \) for all \( t \in [T_0, T_k], \) and (ii) \( p(T_k) = p', \)

The set of all states accessible from \( p \) (with respect to \( R \subseteq L) \) shall be denoted by \( A_R(p). \)

The state space \( L \) of \( H \) is said to be controllable if \( A^L(p) = L, \) for all \( p \in L. \)

Definition 3 Let \( q_1, q_2 \in Q \) and let \( X^{q_1} = \pi_{q_1} \subseteq \Pi, \) \( Y^{q_2} = \pi_{q_2} \subseteq \Pi, \) where \( \Pi \) is a finite analytic partition of the state space \( L \) of \( H. \) \( (X^{q_1}, Y^{q_2}) \) is said to be a hybrid dynamical consistent pair (HDC) if and only if for any \( x \in X^{q_1} \) there exist a state \( x' \in X^{q_2} \) such that \( (q_1, x') \) is accessible from \( (q_1, x) \) with respect to \( q_1 \times X^{q_1} \subseteq L \) and at least one of the following conditions holds:

(a) there exists a discrete control \( \sigma \in \Sigma \) such that \( \Gamma((q_1, x'); \sigma) = (q_2, y), \) for some \( y \in Y^{q_2} \)

OR

(b) \( q_2 = q_1, \) i.e. \( X^{q_1}, Y^{q_2} = \pi_{q_1}, \) and there exists a control \( u \in U \) defined on a finite time interval \( [0, T) \) such that

(i) \( \phi_{q_1}(t, x', u) \in X^{q_1}, \) for all \( t \in [0, t*), 0 < t* < T; \)

(ii) \( \phi_{q_1}(t*, x', u) \) is a facial boundary state of \( \partial X^{q_1} \cap \partial Y^{q_2}, \) i.e. it lies in the relative interior of an \( (n-1) \) dimensional connected component of the boundaries \( \partial X^{q_1} \) and \( \partial Y^{q_2}; \) and

(iii) \( \phi_{q_1}(t*, x', u) \in Y^{q_2}, \) for all \( t \in (t*, T). \)

The notion of hybrid dynamical consistency for hybrid systems is evidently a generalization of the notion of dynamical consistency (DC) introduced in (Caines & Wei 1998) for control systems on differentiable manifolds.

Definition 4 Let \( H \) be a hybrid system

\[
H = \{L, Q, X, U, \Sigma, f, \Gamma\}
\]

and \( \Pi = \{\pi_q, q \in Q\} = \{\{X^q_1, \ldots, X^q_{k_q}\}; q \in Q\} \) be a finite analytic partition of \( L. \) Consider the system \( H^{\Pi} = \{L^{\Pi}, E^{\Pi}, \Phi^{\Pi}\}, \) where

\[
L^{\Pi} \Delta \{\{q, i\}; q \in Q, i \in \{1, \ldots, k_q\}\};
\]

\[
E^{\Pi} \Delta \{(\pi_q, \{i\}); q \in Q, i \in \{1, \ldots, k_q\}\};
\]

\[
\Phi^{\Pi} \Delta \{\phi_{q, i}; q \in Q, i \in \{1, \ldots, k_q\}\};
\]
\[ E = \{ E_{q_1, j}^{q_2}; q_1, q_2 \in Q, 1 \leq i \leq k_{q_1}, 1 \leq j \leq k_{q_2} \} \] is a finite set of transition labels;
\[ \Phi^H : \mathcal{L} \times E \rightarrow \mathcal{L}^H \] is a (partially defined) map such that \( \Phi^H((q_1, i); E_{q_1, j}^{q_2}) = (q_2, j) \) if and only if \( (X_1^{q_1}, X_2^{q_2}) \) is a hybrid dynamical consistent pair. Otherwise \( \Phi^H((q_1, i); E_{q_1, j}^{q_2}) \) is not defined.

The finite state machine \( H^H \) is called the hybrid partition machine of \( H \).

**Definition 5** Let \( \Pi \) be a finite analytic partition of the state space \( L \) of \( H \). \( \Pi \) is called hybrid in-block controllable if for any \( X^q \in \pi_q \in \Pi, q \in Q \), and any \( x \in X^q \), the set of states accessible from \((q, x)\) with respect to \( q \times X^q \subset L \) is equal to \( q \times X^q \).

**Theorem 1** Let \( \Pi \) be a hybrid in-block controllable partition of \( L \). Then \( H^H \) is controllable (as a finite state machine)
\[ \mathcal{L} \Delta \bigcup_{q \in Q} q \times (M - \partial \pi_q) \text{ is controllable with respect to } \partial \mathcal{L} \]
\[ \mathcal{L} \Delta \bigcup_{q \in Q} q \times (M - \partial F(\pi_q)), \text{ where } NF(\pi_q), q \in Q, \text{ is the set of all non-facial states of the partition } \pi_q. \]

### 3. Hybrid Systems with Disturbances

A hybrid system with disturbances, denoted \( H + D \), is the quintuple

\[ H = \{ L \Delta Q \times M, U \Delta U_c \times U_d, \Sigma \Delta \Sigma_c \cup \Sigma_d, f, \Gamma \}, \]

where

(i) for each \( q \in Q, f_q : M \times U_c \times U_d \rightarrow TM \) is continuously differentiable with respect to its arguments;

(ii) \( Q \) and \( M \) are as defined in the previous section;

(iii) \( U_d \) is a class of disturbances which in this paper is taken to be the set of all functions \( v : \mathbb{R} \rightarrow \mathbb{R}^m \) which are

(a) \( C^1(\mathbb{R}) \) functions of time except (possibly) at each element of a sequence \( \{ t_i \} \) which does not have an accumulation point,

(b) continuous from the right at any point of discontinuity \( t_i \),

(c) bounded on any bounded subset of \( \mathbb{R} \);

(iv) \( U_c \) is a class of admissible disturbance and state feedback control functions \( u : \mathbb{R} \times \mathbb{R}^m \times L \rightarrow \mathbb{R}^m \) such that

(a) \( u \) is bounded on any bounded subset of \( \mathbb{R} \times \mathbb{R}^m \times q \times M \), for each \( q \in Q \),

(b) \( u(t, v, p) \) is continuously differentiable in \((t, v, p)\) with a bounded differential on any bounded subset of \( \mathbb{R} \times \mathbb{R}^m \times q \times M \), for each \( q \in Q \), except (possibly) at each element of a sequence \( \{ t_i \} \) which does not have an accumulation point,

(c) at any such \( t_i \), \( u(t, v, p) \) is jointly continuously differentiable with respect to \( t \) from the right;

(v) \( \Sigma_c \) and \( \Sigma_d \) denote the sets of discrete (state and disturbance) feedback controlled transitions and discrete disturbance transitions, respectively. At any instant \( t^* \) of application of a discrete feedback controlled transition \( \sigma^* \) it is the case that \( \sigma^* = \sigma^*(p^*_{t_*}, v^*_{t_*}) \).

We define the disturbance co-accessible set \( CA^{\text{dis}}(K; R) \) to \( K \subset L \) with respect to \( R \subset L \) to be

\[ CA^{\text{dis}}(K; R) = \bigcup_{n=1}^{\infty} CA^{\text{dis}}_n(K; R), \]

where \( CA^{\text{dis}}_n(K; R) = CA^{\text{dis}}_1(\cdots CA^{\text{dis}}_n(K; R), R), \) for \( n > 1 \), and, finally,

\[ CA^{\text{dis}}_1(K; R) = \{ (q, x'); \exists v \in U_d \forall T > 0 \forall u \in U \phi_q(T, x', u, v) \in K, \text{ and } \forall t \in [0, T] \phi_q(t, x', u, v) \in R \} \]

\[ \bigcup \{(q', x') \in R; \exists \mu \in \Sigma_d \Gamma((q', x'); \mu) \in K \}. \]

Let \( X, Y \in \Pi \), where \( \Pi \) is a finite analytic partition of the state space \( L \) of \( H \). Define

(i) \( R = X \cup Y \cup \text{Facial}(\partial X \cap \partial Y) \), if \( X, Y \in \pi_q \in \Pi \), for some \( q \in Q \); and

(ii) \( R = X \cup Y \), otherwise (i.e. if \( X \in \pi_{q_1} \in \Pi, Y \in \pi_{q_2} \in \Pi, q_1, q_2 \in Q \) and \( q_1 \neq q_2 \).

**Definition 6** We shall say that the (undissemblable) disturbance event \( D^X \) is defined if there exists \( x \in X \) such that \( x \in CA^{\text{dis}}(Y; R) \).

We define the disturbance rejecting co-accessible set \( CA^{\text{dr}}(K; R) \) to \( K \subset R \) with respect to \( R \subset L \) to be

\[ \bigcup_{n=1}^{\infty} CA^{\text{dr}}_n(K; R), \]

where \( CA^{\text{dr}}_n(K; R) = CA^{\text{dr}}_1(\cdots CA^{\text{dr}}_n(K; R), R), \) for \( n > 1 \), and, finally,

\[ CA^{\text{dr}}_1(K; R) = \{ (q, x'); \forall v \in U_c \forall T > 0 \exists u \in U \text{ defined on } [0, T] \phi_q(T, x', u, v) \in K, \text{ and } \forall t \in [0, T] \phi_q(t, x', u, v) \in R \} \]

\[ \bigcup \{(q', x') \in R; \exists \sigma \in \Sigma_c \Gamma((q', x'); \sigma) \in K \}. \]

**Definition 7** Let \( X, Y \in \Pi \), where \( \Pi \) is a finite analytic partition of the state space \( L \) of \( H \). We shall say that the (disturbance rejecting) control event \( U_X^Y \) is defined if \( x \in CA^{\text{dr}}(Y; R) \), for all \( x \in X \).
Remark: Controlled and disturbance events are mutually exclusive in the sense that if $D_X^Y$ for some partition blocks $X, Y \in \Pi$, then $-U_Z^Y$, for any $Z \in \Pi, Z \neq Y$.

Definition 8 A hybrid partition machine with disturbances, denoted HPM+$D$, is a finite state machine $H^\Pi = \{L^\Pi, \{U_{q,i}^{q,j}\}, \{D_{q,i}^{q,j}\}\}$ where the state space $L^\Pi = \{(q,i); q \in Q, i \in \{1, \cdots, k_q\}\}$, and $D_{q,i}^{q,j}$ and $U_{q,i}^{q,j}$ are the disturbance and control events respectively.

Definition 9 A block $X \in \Pi$ is called (disturbance rejecting) in-block controllable, denoted DR-IBC, if for all $p, p' \in X$, $p' \in CA^{dr}(p; X)$.

Definition 10 A block $X \in \Pi$ is called $\varepsilon$-DR-IBC if
(i) $E_x \Delta \{p \in X; N_x(p) \subseteq X\} \neq \emptyset$, and
(ii) for all $p \in X$ and any $p' \in E_x$, $p \in CA^{dr}(N_x(p'); X)$.

Let $F \subseteq L$ denote the set of forbidden states. We shall say that there exists a safe low level feedback control law for $p, p' \in L$, if $p \in CA^{dr}(p'; L - F)$. Furthermore, we shall say that there exists an $\varepsilon$-safe low level feedback control law for $p, p' \in L$, if $p \in CA^{dr}(N_x(p'); L - F)$. It may be shown that

Theorem 2 (i) Let $X, Y \in \Pi$ and let $Y$ be DR-IBC. If there exists a sequence of blocks $X = Z_0, Z_1, \cdots, Z_r = Y$, such that $U_{i+1}^{q,i}$ is defined for any two consecutive blocks in this sequence, then, for any $x \in X, y \in Y$, there exists a safe low level feedback control.

(ii) Assume that there exists a safe low level feedback control law for some $p, p' \in L$, and assume further that a sequence of blocks $X = Z_0, Z_1, \cdots, Z_r = Y, p \in X, p' \in Y$ contains the resulting trajectories for all $v \in U_d$ and is such that each of $Z_i (0 \leq i \leq r)$ is DR-IBC. Then the sequence $\{U_i^{q,i}; 0 \leq i \leq r-1\}$ of disturbance rejecting control events is defined.

Theorem 3 (i) Let $X, Y \in \Pi$ and let $Y$ be $\varepsilon$-DR-IBC. If there exists a sequence of blocks $X = Z_0, Z_1, \cdots, Z_r = Y$, such that $U_{i+1}^{q,i}$ is defined for any two consecutive blocks in this sequence, then, for all $x \in X$ and any $y \in Y$ such that $N_x(y) \subseteq Y$, there exists an $\varepsilon$-safe low level feedback control.

(ii) Assume that for some $\varepsilon^* > 0$ there exists an $\varepsilon^*$-safe low level feedback control law for some $p, p' \in L$, and assume further that a sequence of blocks $X = Z_0, Z_1, \cdots, Z_r = Y, p \in X, p' \in Y$ contains the resulting trajectories for all $v \in U_d$ and is such that each of $Z_i (0 \leq i \leq r)$ is $\varepsilon$-DR-IBC. Then the sequence $\{U_i^{q,i}; 0 \leq i \leq r-1\}$ of disturbance rejecting control events is defined.

Example 1 To illustrate the introduced in this paper theory of hybrid systems with disturbances we present the following example.

Consider the system of three tanks shown on Figure 1 (Tittus & Egardt 1998). The system has five different control modes:
- $q_1$: the fluid is pumped from tank 1 to tank 3;
- $q_2$: the fluid is pumped from tank 3 to tank 2;
- $q_3$: the fluid is pumped from tank 2 to tank 1;
- $q_4$: the fluid is added to the system via Valve;
- $q_5$: the pump and the valve are shut off.

The three tanks system can be modelled as a hybrid system in the following way. We shall distinguish five discrete locations - each corresponds to one of the five different control modes, i.e. the set of discrete states is $Q = \{q_1, q_2, q_3, q_4, q_5\}$.

The continuous dynamics at the locations are as follows:
- $q_1: \dot{x} = [-1, 0, 1]^T$;
- $q_2: \dot{x} = [0, 1, -1]^T$;
- $q_3: \dot{x} = [w, -w, 0]^T$;
- $q_4: \dot{x} = [v, 0, 0]^T$;
- $q_5: \dot{x} = [0, 0, 0]^T$,

where $v, w \in (0, 1)$. The state space for the system is taken to be

$$D = \{(x_1, x_2, x_3) \in \mathbb{R}^3; x_i > 0, i = 1, 2, 3\}.$$

The forbidden states are

$$F = \{(x_1, x_2, x_3); x_i \notin [A_i, B_i]\},$$

for some given $0 < A_i < B_i (i = 1, 2, 3)$.

For each $q \in Q$, we construct the partition $\pi_q, ||\pi_q|| = k_q > 0$, obtained by dividing $D$ by the two-dimensional planes $x_i = -x_j + C_{ij}^q (i \neq j)$ and $x_i = C_i^q$, where $1 \leq i, j \leq 3$ and $C_{ij}^q, C_i^q$ takes values in $\{0, 1, \cdots, n\}$

Hence each cube $\{(x_1, x_2, x_3) \in \mathbb{R}^3; C_i^q \leq x_i \leq C_i^q + 1\}$ is divided into eight parts.

The forbidden blocks are any $X \in \Pi$ such that $X \cap F \neq \emptyset$.
Figure 2: A two-dimensional projection on \((x_1, x_2)\) of the \(f_{q_3}\) and \(f_{q_4}\) dynamics.

(Note that \(\pi\) is not in the class of hybrid in-block controllable partitions).

In the absence of disturbances (i.e., in the case when \(v = v^0 = (0, 1), w = w^0 = (0, 1)\) are known parameters), the existence of the hybrid dynamical consistancy relation for a pair of blocks \(X, Y \in \pi_{\nu}\), such that \(Facial(\partial X \cap \partial Y) \neq \emptyset\), can be tested as follows. Let \(\hat{n}\) be the normal vector to the boundary \(\partial X \cap \partial Y\) oriented in such a way that it points to \(Y\). If the scalar product \((\hat{n}, f_q)\) (where \(f_q\) is the vector field at the location \(q\)) is positive, then the ordered pair \((X, Y)\) is hybrid dynamically consistent. Moreover, in the case when \(v, w\) are viewed as unknown disturbances, the disturbance rejecting control event \(U_{\nu}\) is defined. Further, for each subset \(X \in D\) such that \(X_i \Delta q_i \times X \in \pi_{\nu}, X_j \Delta q_j \times X \in \pi_{\nu}\), the control event \(U_{X_i}^{q_j}\) is defined (this is because switching between control modes \(q_i\) and \(q_j\) is allowed at any state).

Assume, in addition, that there is a leak at the second tank at the level \(x_2 = E\) (\(E < A_2\)). Then, for each block \(X \in \pi_{\nu}\) (1 \(\leq q_i \leq 5\)) such that \(X \cap \{(x_1, x_2, x_3) ; x_2 > E\} \neq \emptyset\), an undisablable transition is defined to another discrete location \(q_i\) at which the dynamics of the fluid are described by the differential equation

\[
\dot{x} = \begin{bmatrix} a_i & (b_i - 1) & c_i \end{bmatrix}^T,
\]

where \([a_i b_i c_i]^T\) is the vector field at the location \(q_i\).

The objective for the three tank system could be formulated in terms of reachability of a finite state or region from an initial state or region while avoiding, if possible, the set of forbidden states and, at the high level, the set of blocks with undisablable transitions leading to forbidden blocks. Using the partition machines with disturbances framework, this problem has been reduced to the question of reachability for a finite state machine. As can be seen from Figure 2, the sequence of control modes (together with the interleaved high level control events \(U_\alpha, U_\beta, U_\gamma, U_\delta\)) which drives the three tanks system from the initial state \(S_0\) to the final region \(S_f\) can be chosen to be, for instance, \(q_1, q_4, q_3, q_3, q_0\).

4. On Construction of IBC Partitions for Systems with First Integrals

This section of the paper concerns the global controllability of nonlinear systems of the form (2). There is an extensive literature on various forms of the accessibility and controllability problems for nonlinear systems. In particular, in (Jurdjevic 1997; Kunita 1979; Lobry 1974), it has been shown that control affine systems satisfying a dense recurrence condition and a full rank Lie algebra condition are (globally) controllable. In (Manikonda & Krishnaprasad 1997), sufficient conditions for controllability of affine nonlinear control systems (where the drift vector field is a Lie-Poisson reduced Hamiltonian vector field) are presented. In (Hammer 1998), an open function condition is employed to prove global reachability.

A state \(x \in M\) is said to be a continuous fountain if (i) there exists an open ball neighbourhood \(B_p(x) \subset M\) such that for all open ball neighbourhoods \(B_\delta(x)\), where \(\delta < p\), the accessibility and co-accessibility sets from \(x\) relative to \(B_\delta(x)\) are open when the state \(x\) is deleted; and (ii) whenever (i) holds throughout an open neighbourhood \(N(x)\) of \(x\), the limit supremum of the radii \(\rho\) of the open ball neighbourhoods of \(y \in N(x)\) for which (i) holds is continuous at \(x\).

A state \(x \in M\) is called control recurrent if it lies in a non-trivial positive limit set under some control \(u \in U\), i.e. \(x\) is not an equilibrium point under \(u\) and \(x = \lim t_n x, u\), for some sequence \(\{t_n ; n = 1, 2, \cdots\}\) such that \(\lim n t_n = \infty\).

Theorem 4 (Caines & Lemch 1998) Assume that the system \(S\) on the open connected state space \(M\) is such that every state \(x \in M\) is a continuous fountain and through each \(x \in M\) there exists a non-trivial orbit. Then \(M\) is controllable.

Theorem 5 (Caines & Lemch 1998) Assume that the system \(S\) on the open connected state space \(M\) is such that every state \(x \in M\) is a continuous fountain and for every \(x \in M\) there exists a state dependent control \(u_x(-) \in U^N\) such that \(x\) is control recurrent under \(u_x\). Then \(M\) is controllable.

4.1 The Fountain Condition for Control Affine Systems

Consider a system \(S\) of the input-linear or control affine class of nonlinear time-invariant control systems

\[
S : \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i,
\]
where \( f, g_1, g_2, \ldots, g_m \) are smooth mappings from \( \mathbb{R}^n \) into \( \mathbb{R}^n \).

It can be shown that the fountain condition for control affine systems does not imply the Lie Algebra Rank Condition.

Conversely, the Lie Algebra Rank Condition does not imply the fountain condition (Caines & Lemch 1998; 1999).

4.2 Hamiltonian Control Systems

In this section an application of the results of Section 4 to affine Hamiltonian control systems is considered. Such systems have a smooth Hamiltonian of the form

\[
H(q, p, u) = H_0(q, p) - \sum_{j=1}^{m} H_j(q, p) u_j,
\]

where \( H_0(q, p) \) is the internal Hamiltonian (energy) and \( H_j, j = 1, 2, \ldots, m, \) are the interaction or coupling Hamiltonians.

Definition 11 A \( u_0 \) energy slice \( ES(H^-, H^+) \) of a Hamiltonian control system is the set of states for which the value of \( H \), under the state dependent control \( u_0 \in \mathcal{U}(\mathbb{R}^n; \mathbb{R}^n) \), lies between some fixed values \( H^- \) and \( H^+ \).

\[
ES(H^-, H^+) = \{(p, q); (p, q) \in E \text{ and } H^- < H(p, q, u)|_{u=u_0} < H^+\}.
\]

Theorem 6 (Caines & Lemch 1998) A Hamiltonian control system for which all states are continuous fountains and all equilibrium points under some constant control \( u_0 \in \mathcal{U} \) are isolated is such that any precompact connected component of a \( u_0 \) energy slice is controllable.

Note that arguments analogous to those used in the proof of Theorem 6 can be applied to any dynamical system for which there is a measure preserving flow and precompact slices based on any (not necessarily energy) first integral of the system motion. The theorem is stated for affine Hamiltonian systems and energy based slices because of their great importance.

5. Application to a Planar Aircraft System

Current air traffic control management is highly centralized; it provides quite detailed instructions to all aircraft regardless of the short term objectives of an individual aircraft (Perry 1997). Air traffic control is an example of a hybrid control system which consists of differential equations coupled with a discrete event system. There is an extensive literature on this subject, see, for example, (Tomlin, Pappas, & Sastry 1998), and the references therein.

Using the approach developed in earlier this paper, we propose a decomposition of the state space of the system, which models the relative motion of two planar aircraft, \( A_1 \) and \( A_2 \). The motion of an aircraft is described by the left invariant vector field (Tomlin, Pappas, & Sastry 1998)

\[
\dot{y} = gX,
\]

where

\[
g = \begin{bmatrix}
\cos \phi & -\sin \phi & x \\
\sin \phi & \cos \phi & y \\
0 & 0 & 1
\end{bmatrix},
\]

\[
X = \begin{bmatrix}
0 & -\omega & v \\
\omega & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

\( x, y, \phi \) represent the planar position and orientation of the aircraft; \( v, \omega \) represent the translational and rotational velocities.

Let \( g_1(x_1, y_1, \phi_1), g_2(x_2, y_2, \phi_2), \) and \( g(x, y, \phi) \) denote the configuration of aircraft \( A_1 \), the configuration of aircraft \( A_2 \), and the relative configuration of \( A_2 \) with respect to \( A_1 \), respectively. Then

\[
\dot{y}_1 = g_1 X_1, \quad \dot{y}_2 = g_2 X_2, \quad \text{and} \quad g_2 = g_1 g,
\]

where \( X_1 \) and \( X_2 \) are given by the velocity \((v_1, w_1)\) of \( A_1 \) and the velocity \((v_2, w_2)\) of \( A_2 \). The relative motion of two aircraft is described by

\[
\begin{aligned}
\dot{x} &= v_2 \cos \phi + w_1 y - v_1 \\
\dot{y} &= v_2 \sin \phi - w_1 x \\
\dot{\phi} &= w_2 - w_1.
\end{aligned}
\]

The state space for the system (5) is taken to be

\[
M = \mathbb{R}^1 \times \mathbb{R}^1 \times [0, 2\pi).
\]

For any given \( u = (v_1, w_1, v_2, w_2) \) such that \( v_1 \neq 0, w_2 \neq 0 \), the function

\[
F(x, y, \phi; u) = (y - \frac{w_1}{w_2} x + \frac{w_1}{w_2} \cos \phi)^2 + (x - \frac{w_1}{w_2} \sin \phi)^2
\]

stays constant along the relative (with respect to \( A_1 \)) trajectory of the aircraft \( A_2 \). In other words, \( F(x, y, \phi; u) \) is a first integral of (5).

Definition 12 Let \( F(x, y, \phi; u) \), where \( u = (v_1, w_1, v_2, w_2) \), be a first integral of the system (5). \( u_0 \) F-slice \( S(A^-, A^+) \) of the control system (5) is the set of states for which the value of \( F \) under given controls \( u_0 = (v_0^1, \omega_0^1, v_0^2, \omega_0^2) \) lies between some fixed values \( A^- \) and \( A^+ \), i.e.

\[
S(A^-, A^+) = \{(x, y, \phi); (x, y, \phi) \in M \text{ and } A^- < F(x, y, \phi; u)|_{u=u_0} < A^+\}.
\]

Lemma 1 (Caines & Lemch 1998)

(i) For any \( 0 \leq A^- < A^+ < \infty \), \( S(A^-, A^+) \) is non-empty, precompact, and path-connected.
(ii) Consider the system (5) on some finite time interval $[T_0, T_1]$. Assume that the velocities $v_1, w_1$ of the aircraft $A_1$ are constant on $[T_0, T_1]$ and $\omega_1 \neq 0$. Then each state $(x, y, \phi) \in M$ is a continuous fountain with respect to controls $(v_2, \omega_2)$.

(iii) The flow of the system (5) is measure preserving.

Applying the methodology described in Section 4, it can be shown that each $u_0 F$-slice (with $\omega_1 \neq 0, \omega_2 \neq 0$) constitutes a controllable subset of the system (5), where $(v_1, \omega_1)$ are treated as constant parameters and $(v_2, \omega_2)$ are treated as control inputs.

The air traffic system described above can be modelled as a hybrid system, in which each control location $q_i \in Q$ is characterized by the velocity $(v_i^1, \omega_i^1)$ of aircraft $A_i$. Then, as has been shown in the previous section, it is possible to construct a partition $\pi_{u_0}$ in such a way that each block $X^v \in \pi_{u_0}$ is a $u_0 F$-slice of the system (5), for some $u_0 = (v_i^1, \omega_i^1, v_2, \omega_2)$. Hence $\Pi = \{\pi_{u_0}; q_i \in Q\}$ is in-block controllable.

The partition machine with disturbances can be constructed as follows. The set of forbidden states $F$ is taken to be the relative protected zone, which is the 5-mile radius cylinder around aircraft $A_1$. In (Tomlin, Pappas, & Sastry 1998), a methodology is developed in order to compute the predecessor $Pre(F)$; $Pre(F)$ is the set of initial states for which, regardless of the control input, there exists a disturbance input which would drive the system into the forbidden set. We declare a block $X^v \in \Pi$, for some $q_i \in Q$, to be forbidden or safe depending on whether the set of states $\{X^v\} \subset M$ satisfies, respectively,

(i) $Pre(F) \cap \{X^v\} \neq \emptyset$, or

(ii) $Pre(F) \cap \{X^v\} = \emptyset$.

For any disturbance input $(v_1, \omega_1)$ which is constant on $[T_0, T_1]$, each safe block $X \in \pi_q$ (where the discrete state $q \in Q$ corresponds to $(v_1, \omega_1)$) is in-block controllable. Moreover, for any pair of safe blocks $X, Y \in \pi_q$ ($q \in Q$), a disturbance rejecting control event $U_X^Y$ is defined.

A control objective for aircraft $A_2$ may be formulated as follows: reach a certain point in the state space $M$, while (i) avoiding the protected zone $F$ (or, for the high level controller, avoiding the set of forbidden blocks) (ii) minimizing a cost function (which could characterize, for example, fuel consumption, time, or any other resource). In other words, the objective is to balance the safety of each individual aircraft with the optimal utilization of resources.

**Conclusion**

Given a partition $\pi$, where each block $X \in \pi$ is a connected component of an energy slice, one can generate finer partitions using (i) $u = constant \neq 0$ energy slices, (ii) different integrals of motion, or (iii) controllable transversal manifolds, in particular transversal foliations (Broucke 1998). The resulting blocks may constitute neighbourhoods of system states of possible interest and hence form candidate targets for high level trajectory control. Even though, in general, the hybrid in-block controllability property is not preserved under the transversal decomposition of an energy slice block, the associated hybrid partition machine stays in the class of hybrid between-block controllable machines.

In conclusion, the notions of state aggregation described in this paper promise to facilitate the analysis and design of complex control systems; in addition to the traffic management example introduced above, we also mention in that context mechanical systems with many degrees of freedom and large space structures (Goh & Caughey 1985a; 1985b).

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**References**


