Composition for Cardinal Directions by Decomposing Horizontal and Vertical Constraints

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Abstract

In this paper, we demonstrate how to group the nine cardinal directions into sets and use them to compute a composition table. Firstly, we define each cardinal direction in terms of a certain set of constraints. This is followed by decomposing the cardinal directions into sets corresponding to the horizontal and vertical constraints. We apply two different techniques to compute the composition of these sets. The first technique is an algebraic computation while the second is the typical technique of reasoning with diagrams. The rationale of applying the latter is for confirmation purposes. The use of typical composition tables for existential inference is rarely demonstrated. Here, we shall demonstrate how to use the composition table to answer queries requiring the common forward reasoning as well as existential inference.Also, we combine mereological and cardinal direction relations to create a hybrid model which is more expressive.

Introduction

Relative positions of objects in large-scale spaces, and particularly in the geographic domain, are often described by relations referring to *cardinal directions*. These relations specify the direction from one object to another in terms of the familiar compass bearings: north, south, east and west. The intermediate directions north-west, north-east *etc*. are also often used. Two models for reasoning with cardinal directions are the *cone-shaped* and *projection-based* models (A.Frank, 1992). We shall use the latter model in this paper.

Composition tables are widely used for computing inferences involving spatial relations. Much work has been done on the composition of cardinal direction relations for point-like objects (D.Papadias & Theodoridis, 1997; Frank, 1992; Freksa, 1992) which is more suitable for describing positions point-like objects in a map. Using the direction-relation matrix, Goyal & Egenhofer (2000), composes cardinal direction relations for extended objects. Skiadopoulos & Koubarakis (2001) highlight some of the flaws in Goyal's reasoning system. Consequently, he comes up with a method for correctly computing the cardinal direction relations. However, the set of basic cardinal relations in his model consists of 218 elements which is the set of all disjunctions of the nine cardinal directions.

We shall decompose the cardinal directions into sets corresponding to horizontal and vertical constraints. Composition will be computed for these sets instead of the typical individual cardinal directions. Such a composition offers an alternative yet elegant way of representing the clumsy disjunctive relations. Also, it can be used to answer common queries using forward reasoning as well as queries using existential inference.

Some work has been done on the composition of hybrid models. M.T.Escrig & Toledo (1998) combined qualitative orientation and distance to get positional information while Sharma & Flewelling (1995) infers spatial relations from integrated topological and cardinal direction relations. We shall combine mereological and direction relations to infer the spatial relations between extended regions. Focus will only be on single-pieced regions.

In this paper, we shall firstly define each cardinal direction in terms constraints and group them into sets. This is followed by formally defining 'part and whole' cardinal direction relations. The composition table will be computed for each of the sets, using an algebraic method as well as reasoning with diagrams for confirmation purposes. Next, we shall demonstrate how to use the composition table for answering several forms of queries.

Reasoning with Cardinal Directions

According to the projection-based model for cardinal directions (A.Frank, 1992) depicted in Figure 1. The plane is partitioned into nine tiles: North-West(NW),North(N), North-East(NE), West(W), Neutral Zone(O), East(E), South-West(SW), South(S),and South-East(SE). O, is considered a neutral zone because in this tile, the relative cardinal direction between two objects cannot be determined due to their proximity (A.Frank, 1992).

Definitions

A combined algebraic method and the Cartesian co-ordinate system is used to formalise the meaning of directions for an arbitrary single-pieced extended region. The primitives used are:

i. Tile, $R(\phi)$, which is a tile of the extended region, ϕ. The set $\mathcal{R} = \{N(\phi), NE(\phi), NW(\phi), S(\phi), SE(\phi), SW(\phi), O(\phi), E(\phi), W(\phi)\}$
ii. Boundaries of the minimal bounding box of region ϕ, as illustrated in Figure 1. The set $\mathcal{B} = \{X_{min}(\phi), X_{max}(\phi), Y_{min}(\phi), Y_{max}(\phi)\}$

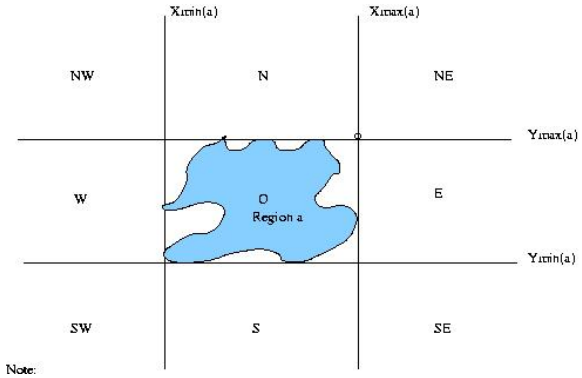

Figure 1: Boundaries

For an arbitrary extended region, ϕ, with a minimal bounding box, the two implicit constraints are:

$$X_{min}(\phi) < X_{max}(\phi)$$
$$Y_{min}(\phi) < Y_{max}(\phi)$$

Next, we shall define all the nine tiles in terms of the boundaries of the minimal bounding box of extended region ϕ.

- $N(\phi) = \{\langle x, y\rangle | X_{min}(\phi) \leq x \leq X_{max}(\phi) \wedge y \geq Y_{max}(\phi)\}$
- $NE(\phi) = \{\langle x, y\rangle | x \geq X_{max}(\phi) \wedge y \geq Y_{max}(\phi)\}$
- $NW(\phi) = \{\langle x, y\rangle | x \leq X_{min}(\phi) \wedge y \geq Y_{max}(\phi)\}$
- $S(\phi) = \{\langle x, y\rangle | X_{min}(\phi) \leq x \leq X_{max}(\phi) \wedge y \leq Y_{min}(\phi)\}$
- $SE(\phi) = \{\langle x, y\rangle | x \geq X_{max}(\phi) \wedge y \leq Y_{min}(\phi)\}$
- $SW(\phi) = \{\langle x, y\rangle | x \leq X_{min}(\phi) \wedge y \leq Y_{min}(\phi)\}$
- $E(\phi) = \{\langle x, y\rangle | x \geq X_{max}(\phi) \wedge Y_{min}(\phi) \leq y \leq Y_{max}(\phi)\}$
- $W(\phi) = \{\langle x, y\rangle | x \leq X_{min}(\phi) \wedge Y_{min}(\phi) \leq y \leq Y_{max}(\phi)\}$
- $O(\phi) = \{\langle x, y\rangle | X_{min}(\phi) \leq x \leq X_{max}(\phi) \wedge Y_{min}(\phi) \leq y \leq Y_{max}(\phi)\}$

Notations

The composition of two relations, $\mathcal{R}$ and $\mathcal{S}$, is written as $(\mathcal{R};\mathcal{S})$ It is defined by the following equivalence:

$$\forall xz[(\mathcal{R};\mathcal{S})xz \longleftrightarrow \exists y[\mathcal{R}xy \wedge \mathcal{S}yz]]$$

Horizontal and Vertical Constraints

Horizontal Constraints

For the horizontal sets, the range of values for y remains constant while the values for x change either in an ascending or descending order. As shown in Figure 2, the three horizontal sets of tiles for the region ϕ are: $(NW(\phi) \cup N(\phi) \cup NE(\phi))$, $(W(\phi) \cup O(\phi) \cup E(\phi))$, and $(SW(\phi) \cup S(\phi) \cup SE(\phi))$.

If there is a referent region a, and another arbitrary region, b, the possible horizontal sets of binary relations and their constraints can be written as follows:

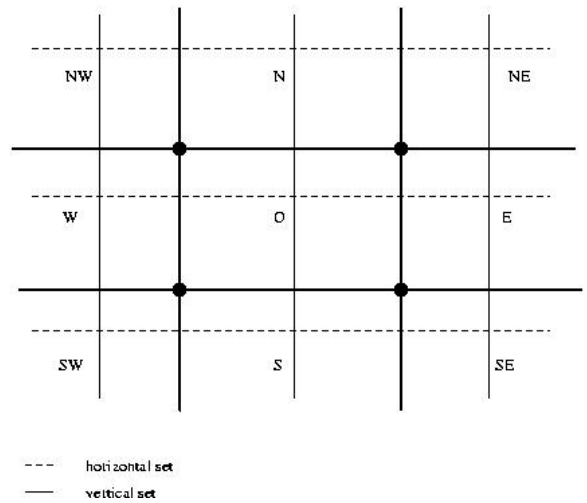

Figure 2: Sets of tiles

- If $b \subseteq (NW(a) \cup N(a) \cup NE(a))$ then $\overline{\mathcal{N}}ab = \{\mathcal{NW}(a,b), \mathcal{N}(a,b), \mathcal{NE}(a,b)\}$, and the constraints are: $Y_{max}(a) \leq Y_{min}(b) \wedge Y_{max}(a) < Y_{max}(b)$.
- If $b \subseteq (W(a) \cup O(a) \cup E(a))$ then $\overline{\mathcal{H}}ab = \{\mathcal{W}(a,b), \mathcal{O}(a,b), \mathcal{E}(a,b)\}$, and the constraints are: $Y_{max}(a) \geq Y_{max}(b) \wedge Y_{min}(a) < Y_{max}(b) \wedge Y_{min}(a) \leq Y_{min}(b) \wedge Y_{max}(a) > Y_{min}(b)$.
- If $b \subseteq (SW(a) \cup S(a) \cup SE(a))$ then $\overline{\mathcal{S}}ab = \{\mathcal{SW}(a,b), \mathcal{S}(a,b), \mathcal{SE}(a,b)\}$, and the constraints are: $Y_{min}(a) \geq Y_{max}(b) \wedge Y_{min}(a) > Y_{min}(b)$.

Vertical Constraints

As for the vertical sets, the range of values for x remains constant while the values for y change either in an ascending or descending order. The vertical sets of tiles for the region ϕ are: $(NE(\phi) \cup E(\phi) \cup SE(\phi))$, $(N(\phi) \cup O(\phi) \cup S(\phi))$, and $(NW(\phi) \cup W(\phi) \cup SW(\phi))$.

The possible vertical sets of binary relations and their constraints can be written as follows:

- If $b \subseteq (NE(a) \cup E(a) \cup SE(a))$ then $\overline{\mathcal{E}}ab = \{\mathcal{SE}(a,b), \mathcal{E}(a,b), \mathcal{NE}(a,b)\}$, and the constraints are: $X_{max}(a) \leq X_{min}(b) \wedge X_{max}(a) < X_{max}(b)$.
- If $b \subseteq (N(a) \cup O(a) \cup S(a))$ then $\overline{\mathcal{V}}ab = \{\mathcal{S}(a,b), \mathcal{O}(a,b), \mathcal{N}(a,b)\}$, and the constraints are: $X_{max}(a) \geq X_{max}(b) \wedge X_{min}(a) < X_{max}(b) \wedge X_{min}(a) \leq X_{min}(b) \wedge X_{max}(a) > X_{min}(b)$.
- If $b \subseteq (NW(a) \cup W(a) \cup SW(a))$ then $\overline{\mathcal{W}}ab = \{\mathcal{SW}(a,b), \mathcal{W}(a,b), \mathcal{NW}(a,b)\}$, the constraints are: $X_{min}(a) \geq X_{max}(b) \wedge X_{min}(a) > X_{min}(b)$.

Combined Mereological and Cardinal Direction Relations

In this section, we shall make a distinction between part and whole direction relations between two extended regions. A direction relation $\mathcal{P}_R(a,b)$ means that only part of the destination extended region, b, is in tile $R(a)$. The direction relation $A_R(a,b)$ is used when the whole of region, b, is completely within the tile $R(a)$.

For example, if b is completely North of a, this direction relation can be represented as below:

$$\begin{aligned} A_N(a,b) = \mathcal{P}_N(a,b) \wedge \neg\mathcal{P}_{NE}(a,b) \wedge \\ \mathcal{P}_{NW}(a,b) \wedge \neg\mathcal{P}_S(a,b) \wedge \neg\mathcal{P}_{SE}(a,b) \\ \wedge\neg\mathcal{P}_{SW}(a,b) \wedge \neg\mathcal{P}_E(a,b) \wedge \\ \neg\mathcal{P}_W(a,b) \wedge \neg\mathcal{P}_O(a,b) \end{aligned}$$

We shall define the 'whole' direction relations in terms of the sets followed by a set of constraints.

- $A_N(a,b) \equiv \overline{\mathcal{N}}ab \cap \overline{\mathcal{V}}ab$
  $[Y_{max}(a) \leq Y_{min}(b) \wedge Y_{max}(a) < Y_{max}(b)] \wedge$
  $[X_{max}(a) \geq X_{max}(b) \wedge X_{min}(a) < X_{max}(b)$
  $\wedge X_{min}(a) \leq X_{min}(b) \wedge X_{max}(a) > X_{min}(b)]$
- $A_{NE}(a,b) \equiv \overline{\mathcal{N}}ab \cap \overline{\mathcal{E}}ab$
  $[Y_{max}(a) \leq Y_{min}(b) \wedge Y_{max}(a) < Y_{max}(b)] \wedge$
  $[X_{max}(a) \leq X_{min}(b) \wedge X_{max}(a) < X_{max}(b)]$
- $A_{NW}(a,b) \equiv \overline{\mathcal{N}}ab \cap \overline{\mathcal{W}}ab$
  $[Y_{max}(a) \leq Y_{min}(b) \wedge Y_{max}(a) < Y_{max}(b)] \wedge$
  $[X_{min}(a) \geq X_{max}(b) \wedge X_{min}(a) > X_{min}(b)]$
- $A_S(a,b) \equiv \overline{\mathcal{S}}ab \cap \overline{\mathcal{V}}ab$
  $[Y_{min}(a) \geq Y_{max}(b) \wedge Y_{min}(a) > Y_{min}(b)] \wedge$
  $[X_{max}(a) \geq X_{max}(b) \wedge X_{min}(a) < X_{max}(b)$
  $\wedge X_{min}(a) \leq X_{min}(b) \wedge X_{max}(a) > X_{min}(b)]$
- $A_{SE}(a,b) \equiv \overline{\mathcal{S}}ab \cap \overline{\mathcal{E}}ab$
  $[Y_{min}(a) \geq Y_{max}(b) \wedge Y_{min}(a) > Y_{min}(b)] \wedge$
  $[X_{max}(a) \leq X_{min}(b) \wedge X_{max}(a) < X_{max}(b)]$
- $A_{SW}(a,b) \equiv \overline{\mathcal{S}}ab \cap \overline{\mathcal{W}}ab$
  $[Y_{min}(a) \geq Y_{max}(b) \wedge Y_{min}(a) > Y_{min}(b)] \wedge$
  $[X_{min}(a) \geq X_{max}(b) \wedge X_{min}(a) > X_{min}(b)]$
- $A_E(a,b) \equiv \overline{\mathcal{H}}ab \cap \overline{\mathcal{E}}ab$
  $[Y_{max}(a) \geq Y_{max}(b) \wedge Y_{min}(a) < Y_{max}(b)$
  $\wedge Y_{min}(a) \leq Y_{min}(b) \wedge Y_{max}(a) > Y_{min}(b)]$
  $\wedge[X_{max}(a) \leq X_{min}(b) \wedge X_{max}(a) < X_{max}(b)]$
- $A_W(a,b) \equiv \overline{\mathcal{H}}ab \cap \overline{\mathcal{W}}ab$
  $[Y_{max}(a) \geq Y_{max}(b) \wedge Y_{min}(a) < Y_{max}(b)$
  $\wedge Y_{min}(a) \leq Y_{min}(b) \wedge Y_{max}(a) > Y_{min}(b)]$
  $\wedge[X_{min}(a) \geq X_{max}(b) \wedge X_{min}(a) > X_{min}(b)]$
- $A_O(a,b) \equiv \overline{\mathcal{H}}ab \cap \overline{\mathcal{V}}ab$
  $[Y_{max}(a) \geq Y_{max}(b) \wedge Y_{min}(a) < Y_{max}(b)$
  $\wedge Y_{min}(a) \leq Y_{min}(b) \wedge Y_{max}(a) > Y_{min}(b)]$
  $\wedge[X_{max}(a) \geq X_{max}(b) \wedge X_{min}(a) < X_{max}(b)$
  $\wedge X_{min}(a) \leq X_{min}(b) \wedge X_{max}(a) > X_{min}(b)]$

Computation of the Composition Table

The outcome of the composition of general ordered binary relations is shown in Table 1.

Table 1: Composition of binary ordered relations

	$b < c$	$b = c$	$b > c$
$a < b$	$a < c$	$a < c$	$a \top c$
$a = b$	$a < c$	$a = c$	$a > c$
$a > b$	$a \top c$	$a > c$	$a > c$

In this section, we shall compute two separate composition tables. One is for the horizontal sets (Table 2) while the other is for the vertical (Table 3) sets. We shall employ two different techniques to compute the tables. The first typical technique is reasoning with diagrams. As for the second technique, it uses algebra and the composition table in Table 1.

Technique 1: Reasoning with a diagram

$\mathcal{R}(a,b) \wedge \mathcal{S}(b,c)$ where $\mathcal{R}(a,b) \in \overline{\mathcal{N}}ab$, and $\mathcal{S}(b,c) \in \overline{\mathcal{N}}bc$.

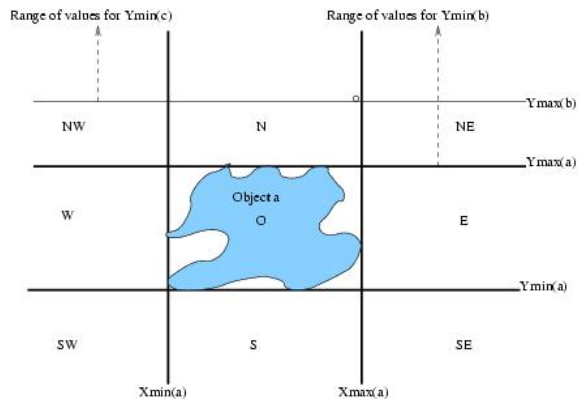

Figure 3: Composition of sets of relations $\overline{\mathcal{N}}ab$ and $\overline{\mathcal{N}}bc$

The inequalities that can be derived from Figure 3 are as follows:

$$Y_{max}(a) \leq Y_{min}(b) \tag{1}$$

$$Y_{max}(b) \leq Y_{min}(c) \tag{2}$$

By default,

$$Y_{max}(b) > Y_{min}(b) \tag{3}$$

By substituting inequality (3) into inequality (2), we get inequality (4).

$$Y_{min}(b) < Y_{min}(c) \tag{4}$$

By combining inequalities (1) and (4), we get the relation

$$Y_{max}(a) < Y_{min}(c) \tag{5}$$

Substitute this inequality $Y_{max}(a) < Y_{max}(c)$ into inequality (5), and we get another relation,

$$Y_{max}(a) < Y_{max}(c) \tag{6}$$

The solution is:

$$Y_{max}(a) < Y_{min}(c) \wedge Y_{max}(a) < Y_{max}(c)$$

Technique 2: An algebraic computation

Use the composition table in Table 1 to compute the following composition:

$$\mathcal{R}(a,b) \wedge \mathcal{S}(b,c)$$

where $\mathcal{R}(a,b) \in \overline{\mathcal{N}}ab$, and $\mathcal{S}(b,c) \in \overline{\mathcal{N}}bc$

We shall represent the above as:

$$\overline{\mathcal{N}}ab \wedge \overline{\mathcal{N}}bc$$

By using the sets of constraints listed earlier, we transform the composition into the following algebraic expression:

$$[(Y_{max}(a) \leq Y_{min}(b) \wedge (Y_{max}(a) < Y_{max}(b)] \wedge$$
$$[(Y_{max}(b) \leq Y_{min}(c) \wedge (Y_{max}(b) < Y_{max}(c)]$$

Substitute the following into the above composition:
$Y_{max}(a)$ with a,
$Y_{min}(b)$ with b_1,
$Y_{max}(b)$ with b_2,
$Y_{min}(c)$ with c_1,
$Y_{max}(c)$ with c_2.

We will now have the form:

$$[(a \leq b_1) \wedge (a < b_2)] \wedge$$
$$[(b_2 \leq c_1) \wedge (b_2 < c_2)]$$

Apply the distributive law and we get the following expression (7).

$$\begin{aligned}(a \leq b_1) \wedge [(b_2 \leq c_1) \wedge (b_2 < c_2)] \quad \wedge \\ (a < b_2) \wedge [(b_2 \leq c_1) \quad \wedge \quad (b_2 < c_2)]\end{aligned} \tag{7}$$

Part 1 of inequality (7)

$$\begin{aligned}&(a \leq b_1) \wedge [(b_2 \leq c_1) \wedge (b_2 < c_2)] \\ &= [(a \leq b_1) \wedge (b_2 \leq c_1)] \wedge [(a \leq b_1) \wedge (b_2 < c_2)]\end{aligned} \tag{8}$$

Part 1.1 of inequality (8)

$$\begin{aligned}&(a \leq b_1) \wedge (b_2 \leq c_1) \\ &= [(a < b_1) \vee (a = b_1)] \wedge [(b_2 < c_1) \vee (b_2 = c_1)] \\ &= (a < b_1) \wedge [(b_2 < c_1) \vee (b_2 = c_1)] \vee \\ &\quad (a = b_1) \wedge [(b_2 < c_1) \vee (b_2 = c_1)] \\ &= (a < b_1) \wedge (b_2 < c_1) \vee (a < b_1) \wedge (b_2 = c_1) \vee \\ &\quad (a = b_1) \wedge (b_2 < c_1) \vee (a = b_1) \wedge (b_2 = c_1)\end{aligned} \tag{9}$$

By default, $b_2 > b_1$, inequality (9) becomes:

$$\begin{aligned}&(a < b_2) \wedge (b_2 < c_1) \vee (a < b_2) \wedge (b_2 = c_1) \vee \\ &(a < b_2) \wedge (b_2 < c_1) \vee (a < b_2) \wedge (b_2 = c_1) \\ &= (a < c_1) \vee (a < c_1) \\ &= (a < c_1)\end{aligned} \tag{10}$$

Part 1.2 of inequality (8)

$$\begin{aligned}&(a \leq b_1) \wedge (b_2 < c_2) \\ &= [(a < b_1) \wedge (a = b_1)] \wedge (b_2 < c_2) \\ &= [(a < b_1) \wedge (b_2 < c_2)] \wedge [(a = b_1) \wedge (b_2 < c_2)]\end{aligned} \tag{11}$$

By default, $b_2 > b_1$, inequality (11) becomes:

$$\begin{aligned}&[(a < b_2) \wedge (b_2 < c_2)] \wedge [(a < b_2) \wedge (b_2 < c_2)] \\ &= (a < b_2) \wedge (b_2 < c_2) \\ &= (a < c_2)\end{aligned} \tag{12}$$

Substitute inequalities (10) and (12) into inequality (8), and we get

$$(a < c_1) \wedge (a < c_2) \tag{13}$$

Part 2 of the inequality (7)

$$\begin{aligned}&(a < b_2) \wedge [(b_2 \leq c_1) \wedge (b_2 < c_2)] \\ &= [(a < b_2) \wedge (b_2 \leq c_1)] \wedge [(a < b_2) \wedge (b_2 < c_2)] \\ &= (a < b_2) \wedge [(b_2 < c_1) \vee (b_2 = c_1)] \\ &\quad \wedge [(a < b_2) \wedge (b_2 < c_2)] \\ &= [(a < c_1) \vee (a < c_1)] \wedge (a < c_2) \\ &= (a < c_1) \wedge (a < c_2)\end{aligned} \tag{14}$$

Substitute inequalities (13) and (14) into (7), we get

$$\begin{aligned}&(a < c_1) \wedge (a < c_2) \wedge (a < c_1) \wedge (a < c_2) \\ &= (a < c_1) \wedge (a < c_2) \\ &= Y_{max}(a) < Y_{min}(c) \wedge Y_{max}(a) < Y_{max}(c)\end{aligned} \tag{15}$$

The conclusion is that for the above composition, the algebraic method yields the same results as the graphical method.

Composition table

Two composition tables computed for the sets are depicted in Table 2 and Table 3. The notation $\overline{\mathcal{N}}ac^*$ in Table 2, means that it is has the constraints of $\overline{\mathcal{N}}ac$ minus the equality $Y_{max}(a) = Y_{min}(c)$. The same goes for $\overline{\mathcal{S}}ac^*$ in Table 2, $\overline{\mathcal{E}}ac^*$, and $\overline{\mathcal{W}}ac^*$ in Table 3 . We shall use these composition tables for forward reasoning as well as existential inference.

Queries for Forward Reasoning

Query 1: $A_R(a,b) \wedge A_R(b,c)$ Composition

Example 1: Find the composition of $A_N(a,b) \wedge A_N(b,c)$.

When represented in sets,the above composition can be rewritten as:
$[\overline{\mathcal{N}}ab \cap \overline{\mathcal{V}}ab] \wedge [\overline{\mathcal{N}}bc \cap \overline{\mathcal{V}}bc] = [\overline{\mathcal{N}}ab \wedge \overline{\mathcal{N}}bc] \wedge [\overline{\mathcal{V}}ab \wedge \overline{\mathcal{V}}bc]$
Use composition tables in Table 2, and 3, we get the following outcome:

$$\overline{\mathcal{N}}ac^* \wedge \overline{\mathcal{V}}ac$$

This is equivalent to $A_N(a,c)$ with region c disjoint from the boundary $Y_{max}(a)$ of the minimum bounding box for a. This means that the extended region c is disjoint from extended region a because the region b between them is extended as well. The outcome of this composition concurs with the model presented by Skiadopoulos & Koubarakis (2001). However, our result here is more expressive because it gives us some insight into the topological relationship between a and c as well.

Example 2: Find the following composition:

$$A_{NE}(a,b) \wedge A_{SW}(b,c)$$

When represented in sets, the above composition can be rewritten as:
$[\overline{\mathcal{N}}ab \cap \overline{\mathcal{E}}ab] \wedge [\overline{\mathcal{S}}bc \cap \overline{\mathcal{W}}bc] = [\overline{\mathcal{N}}ab \wedge \overline{\mathcal{S}}bc] \wedge [\overline{\mathcal{E}}ab \wedge \overline{\mathcal{W}}bc]$
Use composition tables in Table 2, and 3, we get the following outcome:

$$[\overline{\mathcal{N}}ac \vee \overline{\mathcal{H}}ac \vee \overline{\mathcal{S}}ac] \wedge [\overline{\mathcal{E}}ac \vee \overline{\mathcal{V}}ac \vee \overline{\mathcal{W}}ac]$$

The above disjunction implies that the outcome of the composition includes all tiles and this result is also consistent with the results presented by Skiadopoulos & Koubarakis (2001).

Table 2: Composition of horizontal set relations

Note: $\overline{\mathcal{N}}ac^*$ has the constraints of $\overline{\mathcal{N}}ac$ minus the equality $Y_{max}(a) = Y_{min}(c)$
$\overline{\mathcal{S}}ac^*$ has the constraints of $\overline{\mathcal{S}}ac$ minus the equality $Y_{min}(a) = Y_{max}(c)$

	$\overline{\mathcal{N}}bc$	$\overline{\mathcal{H}}bc$	$\overline{\mathcal{S}}bc$
$\overline{\mathcal{N}}ab$	$Y_{max}(a) < Y_{min}(c) \wedge$ $Y_{max}(a) < Y_{max}(c)$ $\overline{\mathcal{N}}ac^*$	$Y_{max}(a) \leq Y_{min}(c) \wedge$ $Y_{max}(a) < Y_{max}(c)$ $\overline{\mathcal{N}}ac$	$Y_{max}(a) \top Y_{min}(c) \wedge$ $Y_{max}(a) \top Y_{max}(c)$ $\overline{\mathcal{N}}ac \vee \overline{\mathcal{H}}ac \vee \overline{\mathcal{S}}ac$
$\overline{\mathcal{H}}ab$	$Y_{max}(a) \top Y_{min}(c) \wedge$ $Y_{max}(a) \top Y_{max}(c)$ $Y_{min}(a) < Y_{min}(c)$ $\overline{\mathcal{N}}ac \vee \overline{\mathcal{H}}ac$	$Y_{max}(a) \geq Y_{max}(c) \wedge$ $Y_{min}(a) \leq Y_{min}(c) \wedge$ $Y_{max}(a) > Y_{min}(c) \wedge$ $Y_{min}(a) < Y_{max}(c)$ $\overline{\mathcal{H}}ac$	$Y_{min}(a) \top Y_{min}(c) \wedge$ $Y_{min}(a) \top Y_{min}(c)$ $Y_{max}(a) > Y_{max}(c)$ $\overline{\mathcal{H}}ac \vee \overline{\mathcal{S}}ac$
$\overline{\mathcal{S}}ab$	$Y_{max}(a) \top Y_{min}(c) \wedge$ $Y_{max}(a) \top Y_{max}(c)$ $\mathcal{N}ac \vee \overline{\mathcal{H}}ac \vee \overline{\mathcal{S}}ac$	$Y_{min}(a) \geq Y_{max}(c) \wedge$ $Y_{min}(a) > Y_{min}(c) \wedge$ $\overline{\mathcal{S}}ac$	$Y_{min}(a) > Y_{max}(c) \wedge$ $Y_{min}(a) > Y_{min}(c) \wedge$ $\overline{\mathcal{S}}ac^*$

Table 3: Composition of vertical set relations

Note: $\overline{\mathcal{E}}ac^*$ has the constraints of $\overline{\mathcal{E}}ac$ minus the equality $X_{max}(a) = X_{min}(c)$
$\overline{\mathcal{W}}ac^*$ has the constraints of $\overline{\mathcal{W}}ac$ minus the equality $X_{min}(a) = X_{max}(c)$

	$\overline{\mathcal{E}}bc$	$\overline{\mathcal{V}}bc$	$\overline{\mathcal{W}}bc$
$\overline{\mathcal{E}}ab$	$X_{max}(a) < X_{min}(c) \wedge$ $X_{max}(a) < X_{max}(c)$ $\overline{\mathcal{E}}ac^*$	$X_{max}(a) \leq X_{min}(c) \wedge$ $X_{max}(a) < X_{max}(c)$ $\overline{\mathcal{E}}ac$	$X_{max}(a) \top X_{min}(c) \wedge$ $X_{max}(a) \top X_{max}(c)$ $\overline{\mathcal{E}}ac \vee \overline{\mathcal{V}}ac \vee \overline{\mathcal{W}}ac$
$\overline{\mathcal{V}}ab$	$X_{max}(a) \top X_{max}(c) \wedge$ $X_{max}(a) \top X_{min}(c)$ $X_{min}(a) < X_{min}(c)$ $\overline{\mathcal{E}}ac \vee \overline{\mathcal{V}}ac$	$X_{max}(a) \geq X_{max}(c) \wedge$ $X_{min}(a) \leq X_{min}(c) \wedge$ $X_{max}(a) > X_{min}(c) \wedge$ $X_{min}(a) < X_{max}(c)$ $\overline{\mathcal{V}}ac$	$X_{min}(a) \top X_{max}(c) \wedge$ $X_{min}(a) \top X_{min}(c)$ $X_{max}(a) > X_{max}(c)$ $\overline{\mathcal{V}}ac \vee \overline{\mathcal{W}}ac$
$\overline{\mathcal{W}}ab$	$X_{max}(a) \top X_{min}(c) \wedge$ $X_{max}(a) \top X_{max}(c)$ $\overline{\mathcal{E}}ac \vee \overline{\mathcal{V}}ac \vee \overline{\mathcal{W}}ac$	$X_{min}(a) \geq X_{max}(c) \wedge$ $X_{min}(a) > X_{min}(c) \wedge$ $\overline{\mathcal{W}}ac$	$X_{min}(a) > X_{max}(c) \wedge$ $X_{min}(a) > X_{min}(c) \wedge$ $\overline{\mathcal{W}}ac^*$

Query 2:

$\mathcal{P}_R(a,b) \wedge A_R(b,c)$ Composition

Find the following composition:

$$[\mathcal{P}_N(a,b) \wedge \mathcal{P}_{NE}(a,b)] \wedge A_N(b,c)$$

When represented in sets, the above composition can be rewritten as follows:

$$[[\overline{\mathcal{N}}ab \cap \overline{\mathcal{E}}ab] \cup [\overline{\mathcal{N}}ab \cap \overline{\mathcal{V}}ab]] \wedge [\overline{\mathcal{N}}bc \cap \overline{\mathcal{V}}bc]$$
$$= [\overline{\mathcal{N}}ab \wedge \overline{\mathcal{N}}bc] \wedge [[\overline{\mathcal{E}}ab \wedge \overline{\mathcal{V}}bc] \vee [\overline{\mathcal{V}}ab \wedge \overline{\mathcal{V}}bc]]$$

Use composition tables in Tables 2 and 3, we get the following outcome:

$$\overline{\mathcal{N}}ac^* \wedge [\overline{\mathcal{E}}ac \vee \overline{\mathcal{V}}ac]$$

This means that the outcome of the composition is $[\mathcal{P}_N(a,c) \wedge \mathcal{P}_{NE}(a,c)] \vee A_N(a,c) \vee A_{NE}(a,c)$ but once again, with region c disjoint from the boundary $Y_{max}(a)$ of the minimum bounding box for a.

Queries for Existential Inference

In this section, we shall demonstrate how the composition tables in Table 2 and 3 can be used to answer queries using existential inference. This section will also show the outcome of existential inference with certainty and uncertainty.

Query 1: $R(a,b) \wedge A_R(b,c) = A_R(a,c)$
If given the constraints for $A_R(b,c)$ and $(A_R(a,c)$, we have to find what $R(a,b)$ is.

Example 1: Find $R(a,b)$ when given $A_{NW}(b,c)$ and $A_{NW}(a,c)$.

The sets for the relations $A_{NW}(b,c)$ are $\overline{\mathcal{N}}bc$ and $\overline{\mathcal{W}}bc$ and $A_{NW}(a,c)$ can be $\{\overline{\mathcal{N}}ac, \overline{\mathcal{N}}ac^*\}$ and $\{\overline{\mathcal{W}}bc, \overline{\mathcal{W}}bc^*\}$. We shall tabulate the given information in Table 4.

Table 4: Query for $A_{NW}(b,c)$ and $A_{NW}(a,c)$

$R1(a,b)$	$R2(b,c)$	$R3(a,c)$
?	$\overline{\mathcal{N}}bc$	$\overline{\mathcal{N}}ac^*$
		$\overline{\mathcal{N}}ac$
?	$\overline{\mathcal{W}}bc$	$\overline{\mathcal{W}}ac^*$
		$\overline{\mathcal{W}}ac$
From Tables 2 and 3		
With certainty		
$\overline{\mathcal{N}}ab$	$\overline{\mathcal{N}}bc$	$\overline{\mathcal{N}}ac^*$
$\overline{\mathcal{W}}ab$	$\overline{\mathcal{W}}bc$	$\overline{\mathcal{W}}ac^*$
With uncertainty		
$\overline{\mathcal{H}}ab$	$\overline{\mathcal{N}}bc$	$\overline{\mathcal{N}}ac \vee \overline{\mathcal{H}}ac$
$\overline{\mathcal{S}}ab$	$\overline{\mathcal{N}}bc$	$\overline{\mathcal{N}}ac \vee \overline{\mathcal{H}}ac \vee \overline{\mathcal{S}}ac$
$\overline{\mathcal{V}}ab$	$\overline{\mathcal{W}}bc$	$\overline{\mathcal{V}}ac \vee \overline{\mathcal{W}}ac$
$\overline{\mathcal{E}}ab$	$\overline{\mathcal{W}}bc$	$\overline{\mathcal{E}}ac \vee \overline{\mathcal{V}}ac \vee \overline{\mathcal{W}}ac$

Based on the results in Table 4, with the given constraints $A_{NW}(b,c)$ and $A_{NW}(a,c)$, $R(a,b)$ is either $A_{NW}(a,c)$ or $[\mathcal{P}_{NW}(a,c) \wedge [\mathcal{P}_{NE}(a,c) \vee \mathcal{P}_N(a,c) \vee \mathcal{P}_E(a,c) \vee \mathcal{P}_O(a,c) \vee \mathcal{P}_W(a,c) \vee \mathcal{P}_{SE}(a,c) \vee \mathcal{P}_S(a,c) \vee \mathcal{P}_{SW}(a,c)]]$. The latter relation is subject to the 'single-piece' condition. It is true when the existing parts are connected.

Query 2:

$R(a,b) \wedge A_R(b,c) = \mathcal{P}_R(a,c)$

If given the constraints for $A_R(b,c)$ and $\mathcal{P}_R(a,c)$, we have to find what $R(a,b)$ is.

Example 2: Find $R(a,b)$ when given $A_N(b,c)$ and $\mathcal{P}_N(a,c) \wedge \mathcal{P}_{NE}(a,c)$.

The sets for the relation $A_N(b,c)$ are $\overline{\mathcal{N}}bc$ and $\overline{\mathcal{V}}bc$. $\mathcal{P}_N(a,c)$ are $\{\overline{\mathcal{N}}ac, \overline{\mathcal{N}}ac^*\}$ and $\overline{\mathcal{V}}ac$. $\mathcal{P}_{NE}(a,c)$ can be $\{\overline{\mathcal{N}}ac, \overline{\mathcal{N}}ac^*\}$ and $\overline{\mathcal{E}}ac$. We shall tabulate the given information in Table 5.

Table 5: Query $A_N(b,c)$ and $\mathcal{P}_N(a,c) \wedge \mathcal{P}_{NE}(a,c)$

$R1(a,b)$	$R2(b,c)$	$R3(a,c)$
?	$\overline{\mathcal{N}}bc$	$\overline{\mathcal{N}}ac^*$
		$\overline{\mathcal{N}}ac$
?	$\overline{\mathcal{V}}bc$	$\overline{\mathcal{E}}ac$
?		$\overline{\mathcal{V}}ac$
From Tables 2 and 3		
With certainty		
$\overline{\mathcal{N}}ab$	$\overline{\mathcal{N}}bc$	$\overline{\mathcal{N}}ac^*$
$\overline{\mathcal{E}}ab$	$\overline{\mathcal{V}}bc$	$\overline{\mathcal{E}}ac$
$\overline{\mathcal{V}}ab$	$\overline{\mathcal{V}}bc$	$\overline{\mathcal{V}}ac$
With uncertainty		
$\overline{\mathcal{H}}ab$	$\overline{\mathcal{N}}bc$	$\overline{\mathcal{N}}ac \vee \overline{\mathcal{H}}ac$
$\overline{\mathcal{S}}ab$	$\overline{\mathcal{N}}bc$	$\overline{\mathcal{N}}ac \vee \overline{\mathcal{H}}ac \vee \overline{\mathcal{S}}ac$

Based on the results in Table 5, with the given constraints $A_N(b,c)$ and $\mathcal{P}_N(a,c) \wedge \mathcal{P}_{NE}(a,c)$, the possible outcome for $R(a,b)$ is either $[\mathcal{P}_{NE}(a,b) \wedge \mathcal{P}_N(a,b)]$ or $[[\mathcal{P}_{NE}(a,b) \wedge \mathcal{P}_N(a,b)] \wedge [\mathcal{P}_{NE}(a,b) \vee \mathcal{P}_O(a,b) \vee \mathcal{P}_{SE}(a,b) \vee \mathcal{P}_S(a,b)]]$.

Query 3:

$R(a,b) \wedge \mathcal{P}_R(b,c) = A_R(a,c)$

If given the constraints for $\mathcal{P}_R(b,c)$ and $A_R(a,c)$, we have to find what $R(a,b)$ is.

Example 3: Find $R(a,b)$ when given $\mathcal{P}_W(b,c) \wedge \mathcal{P}_{SW}(b,c) \wedge \mathcal{P}_S(b,c)$ and $A_{SW}(a,c)$.

The sets for the relation $\mathcal{P}_W(b,c)$ are $\overline{\mathcal{H}}bc$ and $\overline{\mathcal{W}}bc$, $\mathcal{P}_{SW}(b,c)$ are $\overline{\mathcal{S}}bc$ and $\overline{\mathcal{W}}bc$, and lastly, $\mathcal{P}_S(b,c)$ are $\overline{\mathcal{S}}bc$ and $\overline{\mathcal{V}}bc$, As for $A_{SW}(a,c)$, it is $\overline{\mathcal{S}}ac$ and $\overline{\mathcal{W}}ac$.We shall tabulate the given information in Table 6.

Table 6: Query for $\mathcal{P}_W(b,c) \wedge \mathcal{P}_{SW}(b,c) \wedge \mathcal{P}_S(b,c)$ and $A_{SW}(a,c)$

$R1(a,b)$	$R2(b,c)$	$R3(a,c)$
?	$\overline{\mathcal{H}}bc$	$\overline{\mathcal{S}}ac^*$
?	$\overline{\mathcal{S}}bc$	$\overline{\mathcal{S}}ac$
?	$\overline{\mathcal{V}}bc$	$\overline{\mathcal{W}}ac^*$
?	$\overline{\mathcal{W}}bc$	$\overline{\mathcal{W}}ac$
From Tables 2 and 3		
With certainty		
$\overline{\mathcal{S}}ab$	$\overline{\mathcal{S}}bc$	$\overline{\mathcal{S}}ac^*$
$\overline{\mathcal{S}}ab$	$\overline{\mathcal{H}}bc$	$\overline{\mathcal{S}}ac$
$\overline{\mathcal{W}}ab$	$\overline{\mathcal{W}}bc$	$\overline{\mathcal{W}}ac^*$
$\overline{\mathcal{W}}ab$	$\overline{\mathcal{V}}bc$	$\overline{\mathcal{W}}ac$

Based on the results in Table 6, with the given constraints $\mathcal{P}_W(b,c) \wedge \mathcal{P}_{SW}(b,c) \wedge \mathcal{P}_S(b,c)$ and $A_{SW}(a,c)$, the only possible relation $R(a,b)$ is $A_{SW}(a,b)$.

Example 4: Find $R(a,b)$ when given $\mathcal{P}_N(b,c) \wedge \mathcal{P}_{NW}(b,c) \wedge \mathcal{P}_W(b,c)$ and $A_N(a,c)$.

The sets for the relation $\mathcal{P}_N(b,c)$ are $\overline{\mathcal{N}}bc$ and $\overline{\mathcal{V}}bc$, $\mathcal{P}_{NW}(b,c)$ are $\overline{\mathcal{N}}bc$ and $\overline{\mathcal{W}}bc$, and lastly, $\mathcal{P}_W(b,c)$ are $\overline{\mathcal{H}}bc$ and $\overline{\mathcal{W}}bc$, As for $A_N(a,c)$, it is $\overline{\mathcal{N}}ac$ and $\overline{\mathcal{V}}ac$.We shall tabulate the given information in Table 7.

Table 7: Query for $\mathcal{P}_W(b,c) \wedge \mathcal{P}_{SW}(b,c) \wedge \mathcal{P}_S(b,c)$ and $A_{SW}(a,c)$

$R1(a,b)$	$R2(b,c)$	$R3(a,c)$
?	$\overline{\mathcal{N}}bc$	$\overline{\mathcal{N}}ac^*$
?	$\overline{\mathcal{H}}bc$	$\overline{\mathcal{N}}ac$
?	$\overline{\mathcal{V}}bc$	$\overline{\mathcal{V}}ac$
?	$\overline{\mathcal{W}}bc$	
From Tables 2 and 3		
With certainty		
$\overline{\mathcal{N}}ab$	$\overline{\mathcal{N}}bc$	$\overline{\mathcal{N}}ac^*$
$\overline{\mathcal{N}}ab$	$\overline{\mathcal{H}}bc$	$\overline{\mathcal{N}}ac$
$\overline{\mathcal{V}}ab$	$\overline{\mathcal{V}}bc$	$\overline{\mathcal{V}}ac$
With uncertainty		
$\overline{\mathcal{V}}ab$	$\overline{\mathcal{W}}bc$	$\overline{\mathcal{V}}ac \vee \overline{\mathcal{W}}ac$

Based on the results in Table 7, with the given constraints $\mathcal{P}_N(b,c) \wedge \mathcal{P}_{NW}(b,c) \wedge \mathcal{P}_W(b,c)$ and $A_N(a,c)$, the only possible relation $R(a,b)$ is $A_N(a,b)$.

Conclusion

In this paper, we have shown how to decompose the nine cardinal directions into sets corresponding to horizontal and vertical constraints. Using these constraints, we formally define 'part and whole' direction relations between extended regions. 3x3 composition tables for sets have been computed using an algebraic method confirmed by a graph. Such composition tables can be used to answer queries using forward reasoning or existential inference.

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