Natural Solutions for a Class of Symmetric Games*

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Abstract

We introduce a new equilibrium selection mechanism for a class of symmetric coordination games. Taking advantage of the structure of these games, we assert that the equilibrium selected by this mechanism constitutes a "natural solution" in the sense that the associated expected payoff is the highest equilibrium value that can be achieved without any player having to guess how to do so. We illustrate the concept for specific examples, discuss computational issues, and then briefly conclude with a discussion about how the concept can be generalized to other coordination games.

Introduction

The Nash equilibrium has emerged as the dominant solution concept for strategic games, where a profile of (possibly randomized) actions constitutes an equilibrium if no individual player has the incentive to deviate from his or her prescribed action. The assured existence of a Nash equilibrium in mixed strategies is one of the key results (Nash 1950) that helped to shape the field in its early days. However, in applications, it is often inadequate to simply be assured of the existence of a Nash equilibrium, or that an equilibrium can be computed in a finite number of operations. If we look to game theory for a prescription of how to act in competitive situations, the existence of multiple Nash equilibria in pure or mixed strategies is problematic. Sometimes a "salient" solution presents itself (Gauthier 1975; Gilbert 1989), and all players instinctively focus on specific equilibrium, often the option that is closer, easier, brighter, or something cognitively distinct (even if the distinction is not reflected in the payoff function). Unfortunately, notions of salience are difficult to encode within a precise mathematical framework, almost by definition. In response, researchers have sought to define alternative solution concepts for strategic games by proposing "refinements" to the Nash equilibrium that take advantage of the extensive form structure of the game with or without perfect information that can justify the choice of a specific equilibria (cf. (Fudenberg & Tirole 1995) for a comprehensive review of the subject). Other researchers have derived equilibrium selection mechanisms, including (a) payoff and risk dominance (Harsanyi & Selton 1988) and other *deductive* selection mechanisms and (b) *inductive* selection mechanisms (see (Haruvy & Stahl 2004) and references contained therein). Still others have developed alternatives to the Nash equilibrium, such as the correlated equilibrium (Aumann 1974), notions of dominance, and their relationships to Nash equilibria (again see (Fudenberg & Tirole 1995) for a review). Many of the proposed equilibrium refinements and selection mechanisms above seek to resolve a natural tension between objective individual self-interest and uncertainty about how the game will be played, and this tension seems unavoidable generally in non-cooperative games. However, for identical interest games, where self interest is synonymous with group interests, this tension is perhaps easier to resolve, and this paper seeks to exploit symmetry and common interests in deriving a new solution concept for games that is both technically precise and also endogenously salient.

We consider symmetric N-player games of identical interests, where all players have the same pure strategy action space A and the payoff function $u : A^N \mapsto \Re$ is such that (i) $u(y^1, \ldots, y^N)$ is the payoff each player receives when the group plays the profile $y^1, \ldots, y^N$ and (ii) the payoff is the same for any permutation of $y^1, \ldots, y^N$. It is well-known (Becker & Damianov 2006; Cheng *et al.* 2004) that symmetric strategic games generally have symmetric mixed strategy equilibria $\mathbf{x} = (x, \ldots, x)$, where x is a mixed strategy over A. (This true even when without identical interests.[1]) We develop a new solution concept for N-player, symmetric, identical interest games in which players are rewarded for commonality in their choice of actions. Specifically, we assert throughout the paper the following assumption.

Assumption 1 *The common set of pure strategies A is finite, and the payoff function is such that given a subset of actions $G \subseteq A$, there is a* unique *symmetric mixed strategy Nash equilibrium $\mathbf{x}(G) = (x(G), \ldots, x(G))$ for which the support of each player's equilibrium is precisely G. In ad-*

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[1] In general, a noncooperative game is symmetric if all players have the same pure strategy (action) space A and the payoff function $u : A^N \mapsto \Re$ is such that (i) that the payoff to any player who plays $x \in A$ receives $u(x, y^1, \ldots, y^{N-1})$ when all other players play $y^1, \ldots, y^{N-1}$, respectively, and (ii) the payoff is the same for any permutation of $y^1, \ldots, y^{N-1}$.

dition, using $v(G)$ to denote the expected payoff ("value") associated with the equilibrium $\mathbf{x}(G)$, the following properties hold:

P.1 *Given $G_1 \subseteq G_2 \subseteq A$, then*

$$v(G_1) \geq v(G_2), \tag{1}$$

and the inequality is strict if G_1 is a strict subset of G_2.

P.2 *Given disjoint G_1, G_2, G_3 (all subsets of A), then*

$$v(G_1) = v(G_2) \iff v(G_1 \cup G_3) = v(G_2 \cup G_3). \tag{2}$$

For convenience, we refer to any subset of actions $G \subseteq A$ as an *action group*. Also, we let $X(G)$ denote the set of all mixed strategies x whose support is precisely G. One implication of Assumption 1 is that a specific (unique) mixed strategy equilibrium $x(G)$ is implied by the decision to (i) put positive probability on each of the actions $a \in G$ and (ii) put zero probability on all of the actions $b \in A \setminus G$. Thus, the problem of selecting a symmetric mixed strategy equilibrium is in a sense equivalent to the problem of selecting an action group.

The remainder of this paper is organized as follows. First we introduce the notion of *natural solutions* for games that satisfy Assumption 1. Next, we illustrate natural solutions for a simple class of games that we refer to as *static agreement* games, where positive payoff is associated with all players choosing the same action. After taking up computational issues, we conclude with a summary and offer directions for ongoing research.

Natural Solutions

The solution concept we propose for games that satisfy Assumption 1 is tightly coupled to the payoffs that can be achieved when individual players make arbitrary decisions about what actions to play. Some additional notation will be helpful. For any $a \in A$, let the function $\tilde{u}_a : A^{N-1} \mapsto \Re$ be defined by

$$\tilde{u}_a(y^2, \ldots, y^N) = u(a, y^2, \ldots, y^N). \tag{3}$$

We can interpret $\tilde{u}_a$ as the payoff function for the game that is defined by Player 1 unilaterally declaring his intent to play $a \in A$. We are now equipped to define a notion of equivalence between actions.

Definition 1 (Equivalent Actions) *Two distinct actions a and a' are* equivalent*, denoted $a \leftrightarrow a'$, if there exists a bijective function $\phi_{a,a'} : A^{N-1} \mapsto A^{N-1}$ such that*

$$\tilde{u}_a(\alpha) = \tilde{u}_{a'}(\phi_{a,a'}(\alpha)), \qquad \forall\, \alpha \in A^{N-1}. \tag{4}$$

In other words, two actions are equivalent if, after they are selected by Player 1, they offer the Players 2 through N the same opportunities to receive payoffs, i.e. if the games defined by $\tilde{u}_a$ and $\tilde{u}_{a'}$ are equivalent. We now proceed to define an important building block for our solution concept.

Definition 2 (Atomic Action Groups) *An action group G is* atomic *if it is such that for all $a \in G$*

1. *the actions a and a' are equivalent (i.e. $a \leftrightarrow a'$) for all a' in G, and*
2. *the actions a and b are not equivalent for any $b \in A \setminus G$.*

For singleton action groups $G = \{a\}$, the first requirement above holds vacuously, though the second may not. As a convention, we do not consider the empty set $\emptyset$ to be atomic. The definitions above imply that G is atomic if (i) for any action $a \in G$ the $(N-1)$-player game defined by $\tilde{u}_a$ is equivalent to the $(N-1)$-player game defined by $\tilde{u}_{a'}$ for any other $a' \in G$ *and* (ii) G contains all such actions.

Definition 3 (Proper Action Groups) *An action group G is* proper *if (i) it is a union of atomic action groups and (ii) it is such that if $F \subseteq G$ is an atomic action group then G contains all atomic action groups $H \subseteq A$ such that $v(H) = v(F)$. An action group is* improper *if it is not proper.*

Note that an atomic action group F is itself proper only if it is the unique atomic action group with the value $v(F)$. Strict subsets of atomic action groups are improper. Any action group involving a strict subset of an atomic action group is improper. By convention, the empty set is improper. The full set of actions A is itself necessarily proper. If G is proper, then $A \setminus G$ must also be proper. More generally, if G is proper and $F \subset G$ is proper, then $G \setminus F$ is proper. Finally, note that any union of proper action groups is proper.

Definition 4 (Natural Action Groups) *An action group G is* natural *if it is proper and has the property that for any proper action group $F \subseteq G$ there is no proper action group $H \subseteq A \setminus G$ such that $v((G \setminus F) \cup H) = v(G)$. If an action group is not natural, then we refer to it as* unnatural.

Thus, to be "natural" an action group G must be proper and must also be such that no subset of G that is proper can be replaced by a proper action group that is disjoint to G. Note that the full set of actions itself A is itself natural vacuously since there are no proper action groups $H \subseteq A \setminus A$. Observe also that if G is natural and $H \subseteq A \setminus G$ is such that $v(H) = v(G)$, then H cannot be proper. Indeed, if $G \cap H \neq \emptyset$, then H would be a disjoint proper action group whose value is the same as G, and this would contradict the fact that G is natural. More generally, we have the following proposition.

Proposition 1 *If G is natural and $H \neq G$ is such that $v(H) = v(G)$, then H cannot be proper (and thus cannot be natural).*

Proof: Let G and H be as stated in the proposition. As we have already observed, if G and H are disjoint, then H cannot be proper. To address the remaining case in which $G \cap H \neq \emptyset$, let us suppose to the contrary that H is proper. The fact that $v(G) = v(H)$ implies through property **P.1** that neither G nor H is nested within the other. Thus, since $G \neq H$, it must be true that $G \setminus H$, $H \setminus G$, and $G \cap H$ are all nonempty. Now let a be an action in the intersection $G \cap H$. Since both G and H are proper, they must both contain all actions $b \in A$ that are equivalent to a, and thus $G \cap H$ is a union of atomic action groups. In addition, for any atomic action group $F \subseteq G \cap H$, the fact that G and H are both proper implies that they both contain all atomic action groups with value equal to $v(F)$. Thus, $G \cap H$ must itself be proper. Consequently, $G \setminus H$ and $H \setminus G$ must also be proper. Now, since G is natural and $G \setminus H$ and $H \setminus G$ are proper, we have that

$$\begin{aligned} v(H) &= v((G \cap H) \cup (H \setminus G)) \\ &= v([G \setminus (G \setminus H)] \cup (H \setminus G)) \\ &\neq v(G), \end{aligned}$$

which is a contradiction. Thus, H cannot be proper. ■

By Proposition 1, if G is natural, there can be no other natural action groups $H \neq G$ with the same value as G, and thus we are motivated to make the following definition.

Definition 5 (Natural Solutions) *A natural action group G^* for a given SA game is the* natural solution *of the game if all other natural action groups G are such that $v(G) < v(G^*)$. We use v^* to denote the value of the natural solution $v(G^*)$.*

From our earlier observations, A itself is always natural, and, from property **P.1**, A is the the only action group with value less than or equal to $v(A)$. Since there can be only finitely many distinct natural action groups, a natural solution must exist.

The following proposition describes a convenient equivalent characterization of natural action groups.

Proposition 2 *A proper action group G is natural if and only if for any proper action group $F \subseteq G$ and any proper action group $H \subseteq A \setminus G$ it is true that $v(F) \neq v(H)$.*

Proof: By property **P.2**, the existence of action groups $F \subseteq G$ and $H \subseteq A \setminus G$ such that $v(F) = v(H)$ is equivalent to $v((G \setminus F) \cup F) = v((G \setminus F) \cup H)$. Thus, the hypothesis that $v(F) \neq v(H)$ for all pairs of proper action groups $F \subseteq G$ and $H \subseteq A \setminus G$ is equivalent to the hypothesis that G is natural. ■

Discussion

Clearly, for any game satisfying Assumption 1 the natural solution is a (possibly mixed) Nash equilibrium. Thus, we may regard "natural-ness" as an equilibrium selection mechanism, in the same vein as payoff dominance (Harsanyi & Selton 1988). What we achieve in selecting a natural solution is a form of uniqueness: to paraphrase Proposition 1, *If G is the natural solution, then any other action group H with equivalent value cannot be proper, meaning that H must be comprised of some but not all elements of an atomic action group. Moreover, the natural solution G^* is the natural action group that offers the highest expected payoff.*

Assumption 1 certainly is key in deriving our main results. The assumption begins by requiring that a unique mixed strategy equilibrium be associated with the resolve (on the part of *all* players) to put positive measure on any action group $G \subseteq A$. Note that the existence of such a mixed strategy equilibrium is clear from (Becker & Damianov 2006) (Cheng *et al.* 2004), (Nash 1950), and the uniqueness requirement is what makes this an assumption. Property **P.1** requires that the equilibrium payoff associated with an action group G becomes strictly worse as new actions are added, i.e. played with positive probability. This property is the driving force behind Proposition 1 and also the assured existence of a natural solution. In a sense, **P.1** creates an essential tradeoff between (i) the value that can be achieved by all players agreeing on particular actions and (ii) the cost of uncertainty about which action to choose. Property **P.2** is a more technical requirement and is used mainly in validating the test for natural-ness in Proposition 2.

Games that reward commonality in action selection tend to satisfy property **P.1**. In the next section, we illustrate this for a class of static "agreement" games, where positive reward is associated only with *every* player agreeing on an action.

Static Agreement (SA) Games

Definition 6 *(Static Agreement Games) A static agreement game is an N-player symmetric game defined by a common set of actions $A = \{a_1, a_2, \ldots, a_n\}$ and a payoff vector $u = (u_{a_1}, u_{a_2}, \ldots, u_{a_n})' > 0$ such that u_{a_i} is the common payoff if all players select the same action $a_i \in A$ and the common payoff otherwise is zero.*

Note that when there is a unique maximum u^* among the payoffs $\{u_{a_1}, u_{a_2}, \ldots u_{a_n}\}$, then it is reasonable to take as an "optimal solution" the unique Nash equilibrium that achieves the value of u^* is the one in which all players put unit weight on the corresponding action in A. However, when maximum payoff is not uniquely achievable, then what constitutes a reasonable solution becomes much less clear. Unfortunately, existing equilibrium selection mechanisms, such as the payoff and risk dominance criteria of (Harsanyi & Selton 1988), which are designed to identify pure strategy equilibria, do not provide a clear answer.

Verifying Assumption 1 for SA Games

We now show that static agreement games satisfy Assumption 1. To simplify notation, let $v(x)$ denote the expected payoff associated with all players using the same mixed strategy x, i.e.

$$v(x) = \sum_{i=1}^{n} u_{a_i} x_{a_i}^N. \quad (5)$$

Similarly, let

$$v(x, \bar{x}) = \sum_{i=1}^{n} u_{a_i} x_{a_i}^{N-1} \bar{x}_{a_i}. \quad (6)$$

denote the expected payoff given that $N-1$ players agree on $x \in X$ and a single player deviates by choosing $\bar{x} \in X$.

Now given an action group $G \subseteq A$, consider the mixed strategy

$$\begin{aligned} x(G) \;\triangleq\;& k_G \cdot (1_{a_1 \in G} \cdot u_{a_1}^{-1/(N-1)}, 1_{a_2 \in G} \cdot u_{a_2}^{-1/(N-1)}, \\ & \ldots, 1_{a_n \in G} \cdot u_{a_n}^{-1/(N-1)})' \in X(G), \end{aligned} \quad (7)$$

where

$$k_G = \frac{1}{\sum_{a \in G} u_a^{-1/(N-1)}} \quad (8)$$

is a normalizing constant and $1_{a_i \in G}$ is an indicator variable that evaluates to one if $a_i \in G$ and zero otherwise.

Lemma 1 *Let G be an action group for a static agreement game. The mixed strategy profile $\mathbf{x}(G) = (x(G), \ldots, x(G))$, having value*

$$v(G) \stackrel{\triangle}{=} v(x(G)) = k_G^{N-1}, \tag{9}$$

is the unique symmetric mixed strategy Nash equilibrium in $X(G)$

Proof: We first show that $\mathbf{x}(G)$ is a Nash equilibrium. Note that

$$\begin{aligned} v(x(G), \bar{x}) &= \sum_{i=1}^{n} u_{a_i} \left(k_G \cdot 1_{a_i \in G} \cdot u_{a_i}^{\frac{-1}{N-1}} \right)^{N-1} \bar{x}_{a_i} \\ &= k_G^{N-1} \sum_{i=1}^{n} 1_{a_i \in G} \cdot \bar{x}_{a_i} \\ &\leq k_G^{N-1} \\ &= v(x(G)), \end{aligned}$$

where the third line holds with equality when $\bar{x} \in X(G)$. Since no individual player can deviate from $\mathbf{x}(G)$ to obtain a higher expected payoff, $\mathbf{x}(G)$ is a Nash equilibrium.

To establish uniqueness, suppose that $(x, \ldots .x) \in X(G)$ is a Nash equilibrium. Then, it must be true that all actions $a \in G$ have the same expected value relative to x. In particular, listing the elements of G as $a_{(1)}, \ldots, a_{(|G|)}$, it must be true that $u_{a_{(i)}} x_{a_{(i)}}^{N-1} = u_{a_{(i+1)}} x_{a_{(i+1)}}^{N-1}$ for $i = 1, \ldots, |G| - 1$, which along with the requirement that $x_{a_{(1)}} + \cdots + x_{a_{(|G|)}} = 1$ defines system of linear equations that can only be satisfied by one vector in $X(G)$, namely $\mathbf{x}(G)$. ■

Lemma 2 *Let $G_1, \ldots, G_m$ be mutually disjoint action groups for a static agreement game. Then,*

$$v(\cup_{i=1}^m G_i) = \frac{1}{\left[\sum_{i=1}^m (v(G_i))^{-1/(N-1)}\right]^{N-1}} \tag{10}$$

Proof: Since

$$(v(G_i))^{-1/(N-1)} = \left[\sum_{a \in G_i} u_a^{-1/(N-1)} \right], \quad i = 1, \ldots, m,$$

we have that

$$\begin{aligned} v\left(\cup_{i=1}^m G_i\right) &= \frac{1}{\left[\sum_{i=1}^m \sum_{a \in G_i} u_a^{-1/(N-1)}\right]^{N-1}} \\ &= \frac{1}{\left[\sum_{i=1}^m \left(v(G_i)\right)^{-1/(N-1)}\right]^{N-1}}. \end{aligned} \quad ■$$

Some easy consequences of Lemma 2 are the following.

Corollary 1 *Let G_1 and G_2 be action groups for a static agreement game such that $G_1 \subseteq G_2$. Then,*

$$v(G_1) \geq v(G_2), \tag{11}$$

and the inequality is strict if G_1 is a strict subset of G_2.

Corollary 2 *Let G_1, G_2, and G_3 be mutually disjoint action groups for a static coordination game. Then,*

$$v(G_1) = v(G_2) \quad \Longleftrightarrow \quad v(G_1 \cup G_3) = v(G_2 \cup G_3). \tag{12}$$

Corollary 3 *Let $G_1, G_2, \ldots, G_m$ be mutually disjoint action groups with equal value, i.e. $v(G_1) = v(G_2) = \ldots = v(G_m) \stackrel{\triangle}{=} \kappa$. Then,*

$$v\left(\cup_{i=1}^m G_i\right) = \frac{\kappa}{m^{N-1}}. \tag{13}$$

Corollaries 1 and 2, along with Lemma 1, imply that static agreement games satisfy the requirements of Assumption 1, and thus the solution concept of natural solutions applies.

We point out that for any $a \in A$, the value of the singleton action group $\{a\}$ is u_a. Thus, if a is such that $u_a \geq u_{\bar{a}}$ for all $\bar{a} \in A$, then u_a is the highest value an action group can have. On the other hand, from Corollary 1, thinking of A itself as an action group, $v(A)$ is the smallest value that an action group can have and no other action group can achieve that value.

Note also that an action group $G \subseteq A$ is atomic if all of the actions that it contains have individually equivalent payoffs, and no other actions $b \notin G$ have the same payoff as those represented by G. In particular, G is not atomic if it involves some, but not all, actions a that achieve a particular value.

Examples

We now illustrate our solution concept in the context of some specific examples.

Example 1 Consider the two-player SA game defined by the payoff vector

$$u = (4, 4, 2, 2).$$

Here, the only atomic action groups are $G_{1-2} \stackrel{\triangle}{=} \{a_1, a_2\}$ and $G_{3-4} \stackrel{\triangle}{=} \{a_3, a_4\}$. Since $v(G_{1-2}) \neq v(G_{3-4})$, both are proper. A itself is also proper. All three proper action groups are natural. In particular, the action group G_{1-2} is natural despite the fact that is has the same value as $G_3 \stackrel{\triangle}{=} \{a_3\}$ (and as $G_4 \stackrel{\triangle}{=} \{a_4\}$) – the action groups G_3 and G_4 are not proper.

Example 2 Consider the two-player SA game defined by the payoff vector

$$u = (4, 4, 6, 6, 6, 8, 8, 8, 8).$$

Here, using the same notation as in the preceding example, the action groups G_{1-2}, G_{3-5}, and G_{6-9} are atomic. However, since $v(G_{1-2}) = v(G_{3-5}) = v(G_{6-9})$, none of the atomic action groups are individually proper. Similarly, any pair of atomic action groups is improper. The only proper action group is A itself, which is also natural.

Example 3 Consider the two-player SA game defined by the payoff vector

$$u = (3, 6, 6, 6).$$

Here, G_1 and G_{2-4} are the atomic action groups. The proper action groups are G_1, G_{2-4}, and A itself. Note that the action group G_1 is natural since the only disjoint proper action group G_{2-4} has a different value. (It is important to note that G_1 is natural despite the fact that other disjoint action groups have the same value – all such disjoint action groups, i.e. G_{2-3}, $G_{2,4}$, and G_{3-4}, fail to be proper.) The action group G_{2-4} is also natural since its value is not the same as the disjoint proper action group G_1. Finally, the third (and final) natural action group for this game is A itself.

Example 4 Consider the two-player SA game defined by the payoff vector

$$u = (3, 6, 12, 12, 18, 18, 18)$$

for which the atomic action groups are G_1, G_2, G_{3-4}, and G_{5-7}. The action group G_1 is proper since no other atomic action groups have the same value. Any action group involving some but not all of G_2, G_{3-4}, and G_{5-7} is improper. On the other hand G_{2-7} and A itself are proper. Since $v(G_1) \neq v(G_{2-7})$, all three proper action groups are natural.

Example 5 Consider the two-player SA game defined by the payoff vector

$$u = (2, 2, 4, 4, 5, 5, 5, 5, 5, 6, 6, 6)$$

for which the atomic action groups are G_{1-2}, G_{3-4}, G_{5-9}, and G_{10-12}. The proper action groups are $G_{1-2,5-9}$, $G_{3-4,10-12}$ and A itself. Since $v(G_{3-4,10-12}) > v(G_{1-2,5-9})$, all three proper action groups are natural.

Example 6 Consider the two-player SA game defined by the payoff vector

$$u = (8, 8, 8, 8, 2, 4, 4, 5, 5, 5, 5, 5, 6, 6, 6)$$

for which the atomic action groups are G_{1-4}, G_5, G_{6-7}, G_{8-12}, and G_{13-15}. The proper action groups are $G_{1-7,13-15}$, G_{8-12} and A itself. Since $v(G_{8-12}) > v(G_{1-7,13-15})$, all three proper action groups are natural.

Example 7 Consider the two-player SA game defined by the payoff vector

$$u = (2, 2, 8, 8, 12, 12, 12, 5, 5, 5, 5, 5, 6, 6, 6)$$

for which the atomic action groups are G_{1-2}, G_{3-4}, G_{5-7}, G_{8-12} and G_{13-15}. The proper action groups are G_{13-15}, $G_{1-2,8-12}$, $G_{1-2,8-15}$, G_{1-12}, A, G_{3-7}, and $G_{3-7,13-15}$, of which only $G_{1-2,8-12}$, A, and $G_{3-7,13-15}$ are natural.

Example 8 Consider the two-player SA game defined by the payoff vector

$$u = (18, 18, 18, 12, 12, 6, 2, 4, 8, 16, 32, 32),$$

for which the only natural action group is A itself (even though there are many proper action groups).

Summary results for Examples 1-8 are shown in Table 1 below. All of the examples point to an important property of "natural solutions," namely the discontinuity of the solution. For instance, in Example 1, the natural solution is G_{1-2} with value $v(G_{1-2}) = 2$. By infinitesimally reducing the payoff associated with action a_2, the natural solution becomes G_1, with value 4. Thus, the concept itself is inherently extremely sensitive to small variations in payoffs.

Computational Issues

As discussed in the previous section, a natural solution is guaranteed to exist under Assumption 1. Clearly, the natural solution for an instance can be computed by brute force enumeration of the natural action groups, although the worst-case complexity of this task is likely to be prohibitive. (We have not yet attempted to characterize the complexity of this problem.) In the following subsections we report some experiences with a heuristic that we refer to as the "parallel reduction algorithm" for SA games.

The Parallel Reduction Algorithm (PRA) for SA Games

The parallel reduction algorithm involves aggregating unions of atomic action groups of like value in a stagewise process.

Algorithm 1 (PRA) *Given an SA game with action set A and payoff vector u.*

1. *(Initialization) Partition A into atomic action groups*

$$\{G_{\alpha_1^1}, G_{\alpha_2^1}, \ldots, G_{\alpha_{m_1}^1}\},$$

where $v(G_{\alpha_i^1}) \geq v(G_{\alpha_{i+1}^1})$, for $i = 1, \ldots, m_1 - 1$.

2. *In the k-th iteration:*
   (a) *(Termination Condition) If $v(G_{\alpha_1^k}) > v(G_{\alpha_2^k})$, then stop and output $G_{\alpha_1^k}$ as the solution.*
   (b) *(Reduction Step) Otherwise, aggregate action groups to obtain a coarser partition*

$$\{G_{\alpha_1^{k+1}}, G_{\alpha_2^{k+1}}, \ldots, G_{\alpha_{m_{k+1}}^{k+1}}\},$$

where (i) each $G_{\alpha_2^{k+1}}$ is an exhaustive union of k-th stage action groups with like value and (ii) $v(G_{\alpha_i^{k+1}}) \geq v(G_{\alpha_{i+1}^{k+1}})$, for $i = 1, \ldots, m_{k+1} - 1$.

We use the term "parallel" to indicate that in Step 2.(b) *all* possible unions of like-valued action groups are aggregated simultaneously, as opposed to say only aggregating the like-valued action groups with highest value.

Our empirical experience with PRA is that it tends to identify natural solutions for SA games with small numbers of atomic groups. For example, PRA correctly identifies natural solutions for Examples 1-7. More generally, we have the following analytical result.

Proposition 3 *If the PRA algorithm terminates within $k = 2$ stages, then it produces a natural solution.*

Proof: The first round of parallel reduction results in the partitioning of A into atomic action groups

$$\{G_{\alpha_1^1}, G_{\alpha_2^1}, \ldots, G_{\alpha_{m_1}^1}\},$$

where we may assume that $v(G_{\alpha_i^1}) \geq v(G_{\alpha_{i+1}^1})$, for $i = 1, \ldots, m_1 - 1$. The PRA algorithm terminates at this point with $G_{\alpha_1^1}$ as the "answer" if $v(G_{\alpha_1^1}) > v(G_{\alpha_2^1})$. Would this answer be correct? Well, termination implies that $G_{\alpha_1^1}$ is necessarily proper; it would in fact have to be natural, since

Table 1: Natural Solutions for the Examples

Example	Proper Action Groups	Natural Action Groups	Natural Solution
1	G_{1-2}, G_{3-4}, A	G_{1-2}, G_{3-4}, A	G_{1-2}
2	A	A	A
3	G_1, G_{2-4}, A	G_1, G_{2-4}, A	G_1
4	G_1, G_{2-7}, A	G_1, G_{2-7}, A	G_1
5	$G_{1-2,5-9}$, $G_{3-4,10-12}$, A	$G_{1-2,5-9}$, $G_{3-4,10-12}$, A	$G_{3-4,10-12}$
6	$G_{1-7,13-15}$, G_{8-12}, A	$G_{1-7,13-15}$, G_{8-12}, A	G_{8-12}
7	G_{13-15}, $G_{1-2,8-12}$, $G_{1-2,8-15}$, G_{1-12}, A, G_{3-7}, $G_{3-7,13-15}$	$G_{1-2,8-12}$, A, $G_{3-7,13-15}$	$G_{3-7,13-15}$
8	too many to list	A	A

(i) there are no strict subsets of $G_{\alpha_1^1}$ that are proper and (ii) any other disjoint proper action group would have to have value less than $v(G_{\alpha_1^1})$. Thus, the "answer" would indeed be the natural solution of the SA game.

If the PRA fails to terminate in the first iteration, then it must have been the case that $v(G_{\alpha_1^1}) = v(G_{\alpha_2^1})$, and the second stage of reduction results in a new, coarser partition of A:

$$\{G_{\alpha_1^2}, G_{\alpha_2^2}, \ldots, G_{\alpha_{m_2}^2}\},$$

where we may assume that $v(G_{\alpha_i^2}) \geq v(G_{\alpha_{i+1}^2})$, for $i = 1, \ldots, m_2 - 1$. Note that by definition of the PRA each $G_{\alpha_i^2}$ is an exhaustive union of like-valued atomic action groups and is thus necessarily proper. The PRA algorithm terminates with $G_{\alpha_1^2}$ as the "answer" if $v(G_{\alpha_1^2}) > v(G_{\alpha_2^2})$. To see if this answer would be correct, suppose that the termination condition is satisfied. Note that no strict subset of $G_{\alpha_1^2}$ can be proper. Thus, to show that $G_{\alpha_1^2}$ is natural, we must show that no disjoint proper subset has the same value. We now establish that this is the case. First, note that any action group that involves a part but not all of $G_{\alpha_i^2}$ for any $i > 1$ cannot be proper. Thus, proper action groups that are disjoint to $G_{\alpha_1^2}$ must involve unions of $G_{\alpha_i^2}$ for $i > 1$. Since $v(G_{\alpha_1^2}) > v(G_{\alpha_2^2})$ all such unions must have strictly lower value, and $G_{\alpha_1^2}$ and must be natural. To see that $G_{\alpha_1^2}$ is the natural solution, we must show that all other natural action groups have lower value. The only way this could possibly happen is if one of the atomic action groups in the preceding stage $G_{\alpha_i^1}$ for any $i > 1$ happened to be natural with value higher than $G_{\alpha_1^2}$. However, any such action group, being natural, would have to be listed before $G_{\alpha_1^2}$ in the second-stage partition, and this would be a contradiction. Thus, the termination condition would imply that $G_{\alpha_1^2}$ is the natural solution. ∎

One implication of Proposition 3 is that the parallel reduction algorithm will always correctly identify natural solutions in games with three or fewer distinct atomic action groups. It is not clear at this point whether we can or how to extend the argument above to situations where PRA terminates with $k \geq 3$ iterations. Unfortunately, PRA fails to produce natural solution in general. To see this consider the two-player SA game of Example 8 in Section , defined by the payoff vector

$$u = (18, 18, 18, 12, 12, 6, 2, 4, 8, 16, 32, 32).$$

This game reduces by PRA as follows[2]:

$$\begin{aligned} u^0 &= (18, 18, 18, 12, 12, 6, 2, 4, 8, 16, 32, 32) \\ u^1 &= (6, 6, 6, 2, 4, 8, 16, 16) \\ u^2 &= (2, 2, 4, 8, 8) \\ u^3 &= (1, 4, 4) \\ u^4 &= (1, 2), \end{aligned}$$

Terminating with G_{8-12}, whose value is $v(G_{8-12}) = 2$. While G_{8-12} is proper, it is unnatural since it has the same value as the disjoint proper action group G_7. It turns out for this game that the only natural solution is A itself, whose value is $v^* = v(A) = 2/3$. (Interestingly, PRA produces a natural action group for the SA game defined by $u = (6, 6, 6, 2, 4, 8, 16, 16)$, i.e. u^1 above.)

Discussion

The fact that PRA may fail to produce a natural solution raises some interesting points. By definition, PRA always terminates finitely with a mixed strategy Nash equilibrium. Moreover, PRA is "robust" in the sense that whenever independent players implement the algorithm for a game it will always produce the same set of actions (and corresponding mixed strategy Nash equilibrium) regardless of how the players have labeled the actions. Consequently, we could drop the notion of "natural solutions" altogether and adopt PRA as a purely algorithmic approach to equilibrium selection. However, for a number of reasons, a purely algorithmic mechanism for equilibrium selection can be unsatisfying. First, PRA is not the only algorithm that can unambiguously identify mixed strategy Nash equilibria. For example, any algorithm that always produces the action group A itself as the answer can be interpreted as a one that robustly selects Nash equilibria, whereas clearly PRA may not always elect to put positive measure on all actions. Indeed, there

[2] In implementing the PRA algorithm "by hand" we often find it convenient to not reorder action groups according to value.

are many robust algorithms for selecting action groups, all of which may produce distinct solutions, and consequently the equilibrium selection problem is "pushed up" one level to a problem of *algorithm selection*.[3] Another problem with a purely algorithmic response to equilibrium selection is the fact that, unless the algorithm happens to produce natural solutions, the resulting solution can be such that no player is motivated to implement the solution. Consider again Example 8 where the PRA algorithm outputs the action group G_{8-12} with value $v(G_{8-12}) = 2$. In this case, a rational player may well consider playing the proper action group G_7, which as a singleton offers the same value as the five-action group G_{8-12}.

Conclusions

We have defined a solution concept, namely *natural solutions*, for SA games as an equilibrium selection rule. We have established that the PRA in general fails to produce natural solutions, although it provably works for special cases (e.g. games for which PRA terminates within two stages.) Ultimately, the failure of PRA in general does not interfere with the proposed solution concept, since natural solutions can still be finitely computed by enumeration.

In future work, we will show how the notion of a *natural solution* applies more generally than just for SA games. Indeed, Assumption 1 is framed in general-enough terms to capture non-diagonal games, as well as certain dynamic extensions to "agreement" games. In addition, we believe that the notion of "natural" solutions to games can be extended to other types of coordination games, including those that reward action diversity (as opposed to action commonality) for which (i) property $\mathbf{P.1}$ of Assumption 1 holds in reverse and the monotonicity of Equation (1) may not be strict.

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[3]Perhaps it is possible to define a solution concept in terms of an optimization over equilibria that can be computed robust, i.e. out of all robust algorithms that can be applied to a particular game, hopefully there is one that uniquely produces an equilibrium with highest value, and we would call that equilibrium the solution to the game. Of course, there are problems with this approach if the maximum is not unique.

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