Automated Quantum Reasoning: Non-Logic $\rightsquigarrow$ Semi-Logic $\rightsquigarrow$ Hyper-Logic

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Abstract

Quantum theory does *not* necessitate the breakdown of full-blown deduction. On the contrary, it comes with substantially *enhanced logical reasoning power* as compared to its classical counterpart. It features a *co-deductive mechanism* besides a deductive one, resulting in a *sound purely graphical calculus* which admits an *information-flow* interpretation. (Abramsky & Coecke 2004; Coecke 2005a; 2005b; Coecke & Pavlovic 2006). The key physical concept represented by the logic is the *interaction of quantum systems* i.e. the *tensor product* structure, contra (Birkhoff & von Neumann 1936)-logic which only addresses individual systems. The *trace structure*, important in IR applications (van Rijsbergen 2004), is an intrincic part of the logic, together with many other *quantitative concepts*, again contra BvN-logic where the trace only arises indirectly via Gleason's theorem. Hence we provide a powerful high-level formalism for designing, controlling and even automating quantum informatic tasks, which can involve multiple agents. We also mention several existing applications to non-quantum domains such as linguistics, multi-agent systems and concurrency.

Introduction

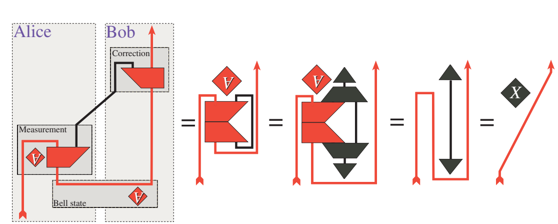
This paper is concerned with the logical mechanisms which govern the observable behaviour of physical systems subject to quantum mechanics, be it for the purpose of either:

- *simulation, analysis* and *design* of computational and communicational devices build out of quantum hardware,
- the passage from quantum informatics to *quantum AI* and more general, any form of *automated quantum reasoning*,
- or, for applying the quantum formalism to *other areas of science*, exploiting its relaxed and at the same time more versatile features as compared to its classical counterpart.

For about 70 years, following the seminal paper (Birkhoff & von Neumann 1936), there was the belief that "quantum logic" implies breakdown of deductive mechanisms. However, recent progress has shown that quantum reasoning is not about decreased but increased deductive power as compared to its classical counterpart. We provide a comprehensive story on the recent progress which has been made concerning, and discuss some of its applications to other fields. Much of the work towards "quantum hyper-logic" and its applications to quantum informatics was done jointly with, respectively, S. Abramsky (Oxford), D. Pavlovic (Kestrel) and E. O. Paquette (Montréal), the results on the preceding "quantum semi-logic" were obtained mainly with I. Stubbe (Antwerp), and the non-physics applications were developed with A. Baltag (Oxford) and M. Sadrzadeh (Southampton).

Quantum logicians have used the physical concepts of *observable* and *measurement* as vehicles to extract logical structure (Birkhoff & von Neumann 1936; Piron 1976; Foulis & Randall 1972; Randall & Foulis 1973; Coecke, Moore, & Wilce 2000). On the other hand, we focussed on the concept of *quantum interaction*, that is, any situation which involves more than one system – and, as we shall show, encompasses measurement situations. Importantly, it was exactly quantum interaction which, besides the decrease of logical mechanisms, has been a stumbling-block for quantum logicians. Indeed, even today the community's main goal is to find a useful logical counterpart to Hilbert space tensor product. As a result, the pay-off of the progress in quantum logic to the physics community has remained quite obscure, resulting in the physics community and in particular von Neumann himself to denounce the field (Rédei 1997). More specifically, while Logic has become a part of Computer Science rather than of pure mathematics, traditional quantum logic never had any pay-off to Quantum Computer Science (QCS). On the other hand, our approach recently contributed to an 1,600,000 EUR Specific Targeted Research Project to be granted by the EC to a consortium coordinated by the present author and including leading European QCS-pioneers.

This is how the "hyper-logical" description and correctness derivation of (tripartite) *quantum teleportation* looks:



An equivalent purely syntactic counterpart is also available, in which the description of quantum teleportation looks like:

$$(1_\mathtt{A}\otimes\eta_\mathtt{A})\circ\rho_\mathtt{A};\mathtt{Meas}_\mathtt{Bell}\otimes 1_\mathtt{A};1_\mathtt{X}\otimes\big((\eta_\mathtt{X}\otimes 1_\mathtt{A})\circ(1_\mathtt{X}\otimes\mathtt{U}^\dagger)\big)$$

Pictures like the above indeed admit a *sound* and *complete category-theoretic semantics* (Abramsky & Coecke 2004; Coecke & Pavlovic 2006) and corresponding *Gentzen-style proof system* (Duncan 2006). The a priori presence of $\otimes$ is essential, e.g. to capture the impossibility to copy and delete quantum data (Wootters & Zurek 1982; Pati & Braunstein 2000), usually referred to as the No-Cloning and No-Deleting theorems – these theorems were never before considered in quantum logic. Besides high-level mechanisms our graphical/symbolic setting also comprehends all the important *quantitative ingredients* of quantum theory. This combination of high-levelness and quantity resonates with recent concerns and corresponding developments in other fields (Gazdar 1996; Smolensky & Legendre 2005), and in particular with the proposal in (Clark & Pulman 2006) to use the Hilbert space tensor product for combining logical and statistical methods in Computational Linguistics. We also point to other areas in which our high-level quantum structures have been of use. Some background on relevant quantum logic can be taken from (van Rijsbergen 2004).

Logical deduction

Recall that a "full-blown" *deduction mechanism* boils down to the equivalence between the two sequents[1]

$$\frac{A, B\vdash C}{\overline{A\vdash B\Rightarrow C}}\,,$$

which, setting $[\![A]\!]=a$, in terms of *order-theoretic semantics* – e.g. see (Davey & Priestley 1990) – rewrites as

$$(a\wedge b)\leq c \iff a\leq(b\to c)\,.$$

That is, the defining property of a *Heyting algebra* is vallid.[2] The main example of such a structure is Boolean logic where one sets $b\to c:=\neg b\vee c$ (where $\neg$ is negation). At the core of deduction lies a *distributive law*. Indeed, it can be shown that *distributivity* of the operation $(-\wedge b)$ over $(-\vee-)$ for all b, that is, explicitly, for a_1, a_2 arbitrary we have

$$(a_1\vee a_2)\wedge b=(a_1\wedge b)\vee(a_2\wedge b)\,,$$

guarantees the existence of a connective $(-\to-)$ satisfying the above condition, and which is explicitly given by[3]

$$b\to c:=\bigvee\{a\,:\,(a\wedge b)\leq c\}\,.$$

Hence distributivity of conjunction over disjunction goes hand-in hand with a distributive law. For example, important logical AI-methods such as the *Robinson's resolution method* crucially exploit this distributive property.

Non-logic: breakdown of a distributive law

Unfortunately, von Neumann's analysis of quantum measurement in terms of *propositions attributable to a physical system* (von Neumann 1932) did not lead to a structure with a distributive property (Birkhoff & von Neumann 1936). Violation of distributivity was due to the existence of so-called *superposition states* $(\psi_1+\psi_2)$.[4] Indeed, when considering the subspaces of a Hilbert space ordered by inclusion, taking conjunction to be intersection and disjunction to be the subspace spanned by the union, we obtain

$$\begin{array}{l}(\psi_1+\psi_2)\ =:\ \phi\ =\ (\psi_1\vee\psi_2)\wedge\phi\\ \qquad\qquad\qquad\qquad\quad\ \nparallel\\ \qquad\qquad\qquad(\psi_1\wedge\phi)\vee(\psi_2\wedge\phi)\ =\ \mathbf{o}\vee\mathbf{o}\ =\ \mathbf{o}\,.\end{array}$$

This observation gave rise to the field now somewhat bizarrely referred to as "quantum logic".[5] It consequently has led to metaphysical considerations on logic, life, the universe and everything, which haven't caused but damage to the field of logical, and more general, structural foundations for quantum theory, and its key role it could have played in the revelation of important new structural paradigms.

Semi-logic: conjoining becomes dynamical

Deductive mechanisms need not be restricted to situations involving implication and conjunction. Many other connectives such as *modal connectives* typically also satisfy some distributive law and hence yield a deductive mechanism. In general we are looking at a situation

$$\frac{\mathcal{B}_*(A)\vdash C}{\overline{A\vdash\mathcal{B}^*(C)}}$$

which, setting $[\![\mathcal{B}_*(A)]\!]=\beta_*([\![A]\!])=\beta_*(a)$ and $[\![\mathcal{B}^*(A)]\!]=\beta^*(a)$, rewrites as a *Galois adjunction*

$$\beta_*(a)\leq c\iff a\leq\beta^*(c)$$

between the maps β_* and β^*. In fact, for quite general reasons one can show that any *computational process* or even any *physical process* satisfies such a distributive law (Abramsky & Vickers 1993; Coecke, Moore, & Stubbe 2001). This brings us within the domain of so-called *Dynamic Logic* (Pratt 1976; Harel, Kozen, & Tiuryn 2000),

[1] The double line indicates that we can read the derivation both in the upward and in the downward direction.

[2] A Heyting algebra is defined as a lattice (i.e. has both $\vee$ and $\wedge$) with an *implication connective* admitting full-blown deduction.

[3] To be truly correct, we should mention that the collection of propositions $\{a,b,c,\ldots\}$ under consideration should either be finite, or conjunctions should extend infinitarily, i.e. for arbitrary sets $\{b_i\}$ the disjunction $\bigvee_i b_i$ must make sense. This is the case for all relevant situations considered in this paper. A mathematical situation where we have such a duality between an operation which preserves $(-\vee-)$, in our case $(-\wedge b)$, and another one which turns out to always preserve $(-\wedge-)$, in our case $(b\to-)$, is called a Galois adjunction. For surveys on Galois adjunctions we refer to (Erné *et al.* 1993; Coecke & Moore 2000).

[4] While we write states in vector notation we think of them as the one-dimensional subspaces spanned by these vectors.

[5] Important contributors to the field such as C. Piron never used this term "quantum logic", and rejected any fundamental connection of their work on quantum foundations with logic. Instead, the much better terminology "quantum geometry" has been coined for example in (Varadarajan 1968), based on the fact that in finite dimensional vector spaces the subspaces constitute a modular lattice i.e. a projective geometry – see for example (Piron 1976; Stubbe & van Steirteghem 2007). This is also the terminology currently used in some pioneering non-physics applications of Hilbert space lattices (van Rijsbergen 2004; Widdows 2004).

which has become of major importance to AI and multi-agent systems research. The Galois adjoint pairs of connectives β_*, β^* admit a well-known interpretation in terms of *Hoare-style weakest precondition semantics* – see for example (Huth & Ryan 2000): for any process β, the unary connective β_* assigns to a proposition a which holds before β the strongest one which will hold after effectuating β, and the unary connective β^* assigns to a proposition c which one would like to hold after β the strongest one which needs to hold before β in order to guaranty that c will indeed hold.

The algebraic property referred to as *orthomodularity*, and which is usually conceived as what remains of distributivity for quantum logic, is exactly such a dynamic "distrubitive law" associated with the process of *quantum state reduction* in measurement. Each projector $\mathrm{P}_b : \mathcal{H} \to \mathcal{H}$ on a subspace $b \subseteq \mathcal{H}$ of a Hilbert space $\mathcal{H}$ can be extended by continuity (i.e. pointwisely) to an operation $\mathbb{P}_b : \mathbb{L}(\mathcal{H}) \to \mathbb{L}(\mathcal{H})$, where $\mathbb{L}(\mathcal{H})$ is the lattice of (closed) subspaces of $\mathcal{H}$. This operation preserves all disjunctions and hence admits a conjunction-preserving Galois adjoint

$$(b \to_s -) := \mathbb{P}_b^* : \mathbb{L}(\mathcal{H}) \to \mathbb{L}(\mathcal{H}),$$

where $(- \to_s -)$ turns out to be exactly the connective usually referred to as the *Sasaki-hook*. We can write both this Sasaki hook and the pointwisely extended projection in a form that resembles much more how one usually encounters them in the quantum logic literature:

$$(b \to_s c) = b^\perp \vee (c \wedge b) \qquad \mathrm{P}_b(a) = b \wedge (a \vee b^\perp).$$

Thus orthomodularity provides us with a distributive property and hence a deduction mechanism. It is however not one of a static nature, but of a truly dynamic one. Of course, a dynamic theory comes hand-in-hand with the notion of compositionality and it would make a lot of sense to consider for example $\mathrm{P}_{b'} \circ \mathrm{P}_b$, something which has no counterpart within $\mathbb{L}(\mathcal{H})$. That is, we would like to have a compositional logic for $\mathrm{P}_a, \mathrm{P}_b, \mathrm{P}_c, \ldots$ with binary connectives $\vee, \wedge, \circ$ rather than one for $a, b, c, \ldots$ with only $\vee, \wedge$ as binary connectives. The general order-theoretic structure in which all this can take place is that of *quantales*,[6] which in particular have recently been considered as a general mathematical setting for a purely dynamic account on quantum logic in terms of distributive laws (Coecke, Moore, & Stubbe 2001; Baltag & Smets 2006, and references therein).

Application 1a: Quantale logic in Linguistics. J. Lambek's 1958 seminal paper on the mathematics of sentence structure (Lambek 1958) was actually the first one to introduce "quantale logic". As we will see below, this is just the start of a striking parallel between the structures exposed in linguistics (and in particular those exposed by Lambek) and those which govern quantum behaviour.

Application 2: Quantale logic in observational semantics. (Abramsky & Vickers 1993; Resende 2000) This is a line of research which traces back to the work of D. S. Scott and C. Strachey and which in terms of operational interpretation is very much kin to von Neumann's take on the structure of quantum mechanics, which led him to quantum logic.

Application 3: Extended quantale logic for information update in multi-agent systems. In recent work we extended quantale logic with additional connectives, to be able to deal with information update in situations involving not necessarily honest interacting agents (Baltag, Coecke, & Sadrzadeh 2005; 2006; Sadrzadeh 2006a). Remarkably, none of the methods, which for example enabled an easy proof of the muddy children puzzle, relied on some underlying Boolean structure and hence extend to situations where no distributivity of the logic of propositions can be assumed e.g. *quantum agents*. We understand that a more detailed account on this work has been submitted to this symposium (Sadrzadeh 2006b).

Hyper-logic: co-deduction besides deduction

While semi-logic admits a distributive law and hence a deductive mechanism, it is not the full-blown logical mechanism which involves the key logical connectives of conjunction and implication. Hence it does not involve the "comma's" which one encounters in sequents and does not internalize entailment. In this semi-logic there is also no obvious counterpart for quantum interaction. Below we show that "quantumness" is not reflected in terms of loss, but in terms of gain of logical properties. In other words, we will capture quantum *weirdness* in logical terms, by having "weird additional properties", not in terms of loosing some non-weird ones. Moreover, this story will involve quantum interaction as the main actor. Let *co-deduction* be

$$\frac{A \vdash B, C}{\overline{A \Rightarrow B \vdash C}}$$

which in terms of order-theoretic semantics rewrites as

$$a \leq (b \wedge c) \iff (a \to b) \leq c.$$

In Boolean logic, besides deduction, we do encounter something which looks a bit like co-deduction, namely

$$a \leq (b \vee c) \iff (a \rightsquigarrow b) \leq c,$$

where $(a \rightsquigarrow c) := a \wedge \neg c = \neg(a \to c)$ and with disjunction playing the role of conjunction, as can easily be seen in

$$\frac{A \vdash B \vee C}{\overline{A \wedge \neg B \vdash C}}.$$

But the *real deal* will be to have both deduction and co-deduction with respect to the same connectives, that is, from the perspective of Boolean logic, when conjunction and disjunction coincide, as well as the implication and its negation. In terms of *Gentzen-systems* – e.g. see (Troelstra & Schwichtenberg 1996) – this means symmetric left and right combined introduction and elimination rules for implication:

$$\frac{A \vdash B, C}{\overline{A \Rightarrow B \vdash C}}\ (\Rightarrow \mathsf{I}, \mathsf{R_L}) \qquad \frac{A, B \vdash C}{\overline{A \vdash B \Rightarrow C}}\ (\Rightarrow \mathsf{I}, \mathsf{R_R})$$

However absurd all this may sound at first, examples of such situations are plenty. The absurdity is tightly connected with

[6] This name was initially coined by C. Mulvey in 1986 referring to the (non-commutative) quantum counterpart of topology in terms of so-called *locales*, within the general theory of C^*-algebra.

our usual static view of of logic, but vanishes when taking a *dynamic* (actually categorical) together with a *resource sensitive* perspective (see also below). One such situation is

$$\frac{\mathcal{H}_1 \vdash \mathcal{H}_2 \otimes \mathcal{H}_3}{\mathcal{H}_1 \multimap \mathcal{H}_2 \vdash \mathcal{H}_3} \qquad \frac{\mathcal{H}_1 \otimes \mathcal{H}_2 \vdash \mathcal{H}_3}{\mathcal{H}_1 \vdash \mathcal{H}_2 \multimap \mathcal{H}_3}$$

where $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ are Hilbert spaces describing three quantum systems, where the Hilbert space tensor product $\otimes$ is the semantic counterpart within quantum theory to jointly (cf. *and*) describe several systems, and where

$$\mathcal{H}_i \multimap \mathcal{H}_j := \mathcal{H}_i^* \otimes \mathcal{H}_j$$

with $(-)^*$ assigning the dual space. As compared to the previous subsection we have passed from considering subspaces of a Hilbert space to connectives which combine Hilbert spaces into new ones. At the same time we tackle the above mentioned stumbling-block of quantum logic, namely the inability to deal with compound quantum systems.

Application 4: Linguistics. In an intriguing coincidence, not only we abandoned the dynamic quantale semi-logic for a far more powerful hyper-logic, but also J. Lambek, independently and in parallel, developed his version of hyper-logic to describe sentence structure (Lambek 2004). In this case, negation is trivial (i.e. identity) but conjunction is non-commutative. Again a more detailed account on this work has been submitted to this workshop (Sadrzadeh 2006b).

Linearity: accounting for available resources

In lattice logic, where meet is conjunction, we always have $b \leq b \wedge b$ and $b \leq 1$, that is, we can *copy* and *delete* premisses. In term of sequent calculus this becomes

$$\frac{A, B, B \vdash C}{A, B \vdash C} \qquad \text{and} \qquad \frac{A \vdash C}{A, B \vdash C} .$$

When dropping these rules, that is, treating propositions as *resources*, we enter the domain of *Linear Logic* (Girard 1987; Seely 1989; Abramsky 1993). Linear logic has been extremely influencial over the past 15 years in programming language semantics and many other fields within Theoretical Computer Science. Since quantum data is subject to *No-Cloning* and *No-Deleting Theorems*, it should come as no surprise that our hyper-logic is a specialisation of linear logic. It is in some sense even surprising that Linear Logic was invented within the Computer Science community, and not as a Quantum Structure by the Physics community.

Application 1b: Linear Logic in Linguistics. In fact, while the name and refined development of linear logic is due to J.-Y. Girard, the first paper which provided a (non-commutative) linear logic was again Lambek's seminal paper on sentence structure (Lambek 1958). In this perspective linearity is not at all a surprise. Indeed, we have[7]

$$not \cdot not \cdot X \neq not \cdot X \qquad \text{and} \qquad not \cdot X \neq X .$$

[7]Note that here we treat *not* just as "some word" with a particular meaning and not as logical connective.

Categoricity: processes and types of systems

Consider a physical system of *type* A (e.g. qubit, 2 qubits, quantum agent, 2 quantum agents, classical data, ...) and perform an operation f on it (e.g. perform a measurement on it) which results in a system possibly of a different type B (e.g. the system together with the measurement data). So typically we have $A \xrightarrow{f} B$ where A is the initial type of the system, B is the resulting type, and f is the operation. One can perform an operation $B \xrightarrow{g} C$ after f since the resulting type B of f is also the initial type of g, and we write $g \circ f$ for the consecutive application of these two operations. Clearly we have $(h \circ g) \circ f = h \circ (g \circ f)$ since putting the brackets merely adds the superficial data of conceiving two operations as one. If we further set $A \xrightarrow{1_A} A$ for the operation "doing nothing on a system of type A" we have $1_B \circ f = f \circ 1_A = f$. We can also conceive two systems or operations as on by writing $A \otimes B$ and $f \otimes g$, imposing some coherences between $(- \circ -)$ and $(- \otimes -)$. We obtain a so-called (symmetric) monoidal category – for a low-level introduction consult (Coecke 2006). *Categorical logic* considers the actual structure of proofs, and how they compose, and not only "provability" (Lambek & Scott 1986; Troelstra & Schwichtenberg 1996). Similarly, *categorical semantics* considers the actual structure of programs and how they compose, with I/O-types as "types of systems". Analogously, our hyper-logic focusses on physical processes, comprising the notion of state as the corresponding preparation procedure, with kinds of systems as types.

Application 5: Concurrency. Structures similar to our hyper-logic were used as a structural foundation for typed concurrent programming (Abramsky, Gay, & Nagarajan 1996).

Application 6: High-level methods for quantum informatics – of course! Our work as so far mainly been motivated by applications to quantum informatics (Abramsky & Coecke 2004; Abramsky 2004; Coecke 2005b; 2005a; Coecke & Pavlovic 2006; Duncan 2006; Coecke, Paquette, & Pavlovic 2006).[8] The interaction of quantum systems is still, after some 100 years of quantum theory, an extremely badly understood area, and is desperate for high-level tools. We feel that we achieved a major breakthrough by recasting the entire quantum mechanical formalism, including quantitative notions and classical data flow, as a hyper-logical system. The remainder of this paper discusses this system.

The hyper-logic of quantum interaction

Our logic combines all the above mentioned together: dynamics/processes/categoricity, linearity, co-deduction and in particular multiplicity of systems. Having co-deduction gives rise to a purely graphical calculus.[9] In this calculus we

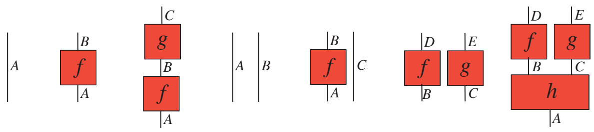
[8]Instances of the ideas developed in these papers appeared independently also in (Baez 2004) and (Kauffman 2005).

[9]The formal justification for this sound graphical calculus requires some quite advanced category theory (Kelly & Laplaza 1980; Freyd & Yetter 1989; Joyal & Street 1991; Selinger 2006). The first one to (informally) use this kind of pictures for calculating with tensor products was Penrose in (Penrose 1971).

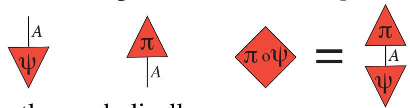
depict physical processes by boxes, of which the inputs and outputs are labelled by *types* of the corresponding system. Sequential composition is depicted by connecting matching outputs and inputs by wires, and parallel composition (cf. tensor) by locating entities side by side. E.g. operations

$$1_A \quad f \quad g\circ f \quad 1_A\otimes 1_B \quad f\otimes 1_C \quad f\otimes g \quad (f\otimes g)\circ h$$

for $f: A\to B$, $g: B\to C$ and $h: E\to A\otimes B$ depict as:



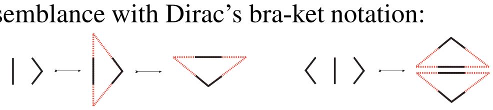
Hence the 'upward' vertical direction represents progress of time. A special role is played by boxes with either no input or no output, respectively representing *states* and "*costates*" (cf. Dirac's *ket*s and *bra*s) which we depict by triangles. Finally, we also need to consider diamonds which arise by post-composing a state with a matching costate (cf. Dirac's *bra-ket*) and which represent numbers (cf. *probabilities*):[10]



or equivalently, symbolically,

$$\psi:\mathbb{C}\to A \qquad \pi: A\to\mathbb{C} \qquad \pi\circ\psi:\mathbb{C}\to\mathbb{C}$$

where $\mathbb{C}$ is the *tensor unit* i.e. $A\otimes\mathbb{C}\simeq A\simeq\mathbb{C}\otimes A$.[11] Note the resemblance with Dirac's bra-ket notation:



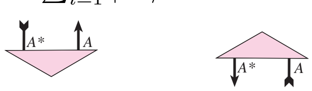
Quantum interaction as information-flow

Now we introduce the structure which truly captures quantum intearction, e.g. which suffices to derive multiparty protocols such as *quantum teleportation* and *entanglement swapping* (Bennet *et al.* 1993; Zukowski *et al.* 1993).

1. We assign *directions* to the wires, and reversal of this direction is denoted by an involution $(-)^*$ – cf. negation.
2. For each box there exists another one obtained by reversing the first one, denoted by an involution $(-)^\dagger$ – cf. the *adjoint* in Hilbert space e.g. ket $= |\psi\rangle \overset{\dagger}{\longleftrightarrow} \langle\psi| =$ bra.
3. We assume that for each type A there exists of a special bipartite *Bell-state* and its adjoint *Bell-costate*

$$\eta_A:\mathbb{C}\to A^*\otimes A \qquad \eta_A^\dagger: A^*\otimes A\to\mathbb{C}\,,$$

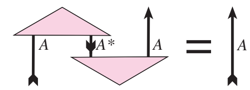
with $\eta_A := 1\mapsto\sum_{i=1}^{i=n}|\,i\,i\,\rangle$, and which we depict as:



[10]We think here of vectors and numbers as linear maps of respective types $\mathbb{C}\to A$ and $\mathbb{C}\to\mathbb{C}$, using the fact that vectors ψ faithfully represent as linear maps $\mathbb{C}\to A :: 1\mapsto\psi$, and that complex numbers c faithfully represent as linear maps $\mathbb{C}\to\mathbb{C} :: 1\mapsto c$.

[11]We depict this tensor as $\mathbb{C}$ to indicate that in the Hilbert space formalism its role is played by the complex number field. But in the picture calculus it is merely a structure-less primitive type.

We impose a single axiom, namely:

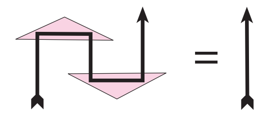


Validity of this statement in the Hilbert space formalism can be easily checked using Dirac notation:

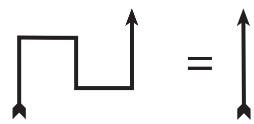
$$\Big(\sum_j\langle jj|\otimes 1_A\Big)\circ\Big(1_A\otimes\sum_i|ii\rangle\Big) \;=\; \sum_{ij}\delta_{ij}\langle j|\otimes|i\rangle$$
$$=\; \sum_i|i\rangle\langle i| = 1_A\,.$$

But if we extend the graphical notation of Bell-(co)states to:

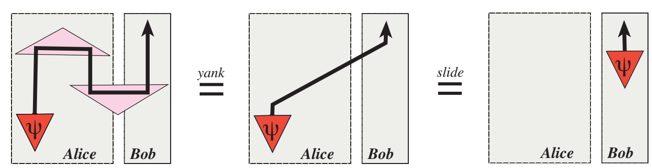
we obtain a far more lucid interpretation for the axiom:



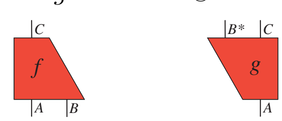
which now tells us that we are allowed to *yank a line*:



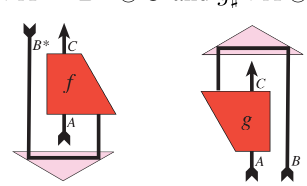
We called this line *quantum information flow* (Coecke 2005b). Ignoring indeterminism in quantum measurements it exposes the capability of (conditional) teleportation:



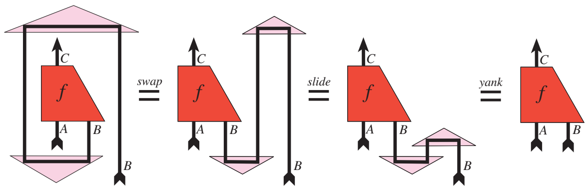
The yanking mechanism combines the deduction and the co-deduction mechanisms. In categorical logic, to show that one can derive a consequent from given premisses, one has to provide an explicit witness of this derivation i.e. a proof. We will explicitly construct a bijective passage between proofs of $A, B\vdash C$ and proofs of $A\vdash B\Rightarrow C$, and one between proofs of $A\vdash B, C$ and proofs of $A\Rightarrow B\vdash C$. First we show that each proof of $A, B\vdash C$ can be transformed into one of $A\vdash B\Rightarrow C$, and vice versa. Given $f: A\otimes B\to C$ and $g: A\to B^*\otimes C$ which we depict as



we produce $f^\sharp: A\to B^*\otimes C$ and $g_\sharp: A\otimes B\to C$ as

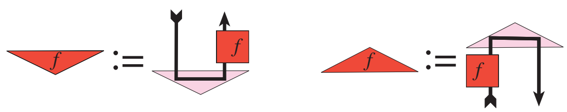


The transformations $(-)^\sharp$ and $(-)_\sharp$ indeed establish a bijective correspondence since they are mutual inverses. By

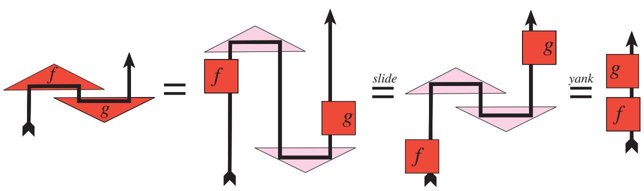


we have $(f^\sharp)_\sharp = f$ and similarly we obtain $(g_\sharp)^\sharp = g$. Co-deduction is exposed by the same argument as above merely by reversing all pictures involved – in Hilbert space terms this means taking the adjoint of everything.

Also much of the *quantum mysticism* associated with multipartite protocols as for example the *seemingly violated causality* is captured by the yanking mechanism. Setting



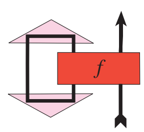
we obtain



On the left-hand-side we have *first* a state labelled g and *then* a costate labelled f, but on the right-hand-side it is stated that the net effect of this is *first* applying f and *then* only g.

Trace: combining quantity and high-levelness

While at first sight the above calculus might seem purely quantitative, it comprises *quantum probabilistic structure*, both in terms of *indeterminism* for outcomes in measurements on pure states and in terms of *mixedness* of states (Abramsky & Coecke 2004; Selinger 2006; Coecke & Paquette 2006; Coecke, Paquette, & Pavlovic 2006). E.g. the partial trace of $f : C \otimes A \to C \otimes B$ is

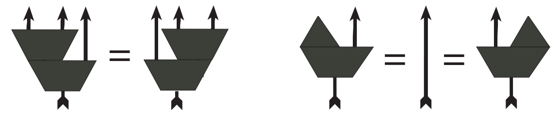


i.e., symbolically, $\mathrm{Tr}^C_{A,B}(f) : A \to B$. For (mixed) state $\rho : C \to C$ and projector $\mathrm{P} : C \to C$ we obtain the probability $\mathrm{Tr}^C_{A,B}(\rho \circ \mathrm{P}) : \mathbb{C} \to \mathbb{C}$ of obtaining the outcome associated to projector P in a measurement on a system in state ρ. As already mentioned in the abstract, this trace structure has shown up in IR in (pseudo-)relevance feedback (van Rijsbergen 2004). The need for combining high-level symbolic and quantitative methods has also been put forward in Natural Language Processing and Cognitive Theories (Gazdar 1996; Smolensky & Legendre 2005). In (Clark & Pulman 2006), submitted to this symposium, the authors propose to use the Hilbert space tensor product for this purpose.

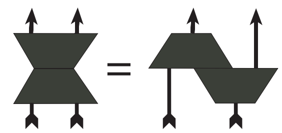
Classicality & measurement from resource control

We implement classicality within the graphical calculus by exploiting the fact that, while quantum data can't be copied, classical data can. Classical data will be conceived as a triple (X, δ, ϵ) where $\delta : X \to X \otimes X$ is a *copying* operation and $\epsilon : X \to \mathbb{C}$ is a *deleting* operation, respectively depicted as:

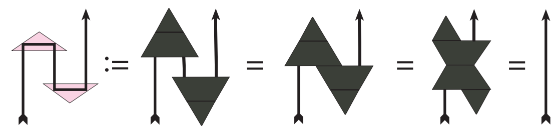
Hence the following properties are obvious:



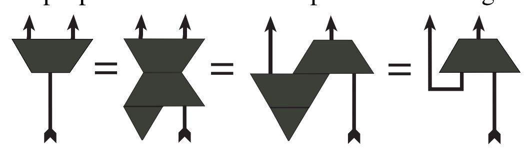
The Frobenius identity also captures classicality (Carboni & Walters 1987; Coecke & Pavlovic 2006):



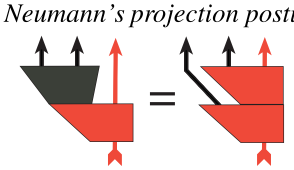
We can now coherently set:



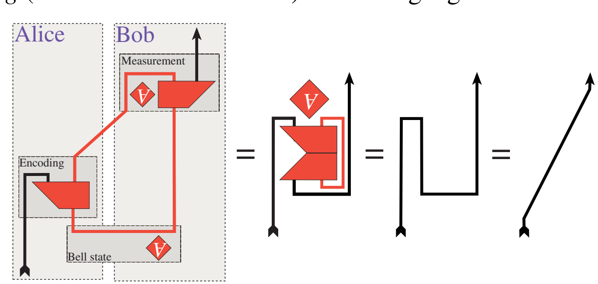
and new properties arise from the previous ones e.g.:



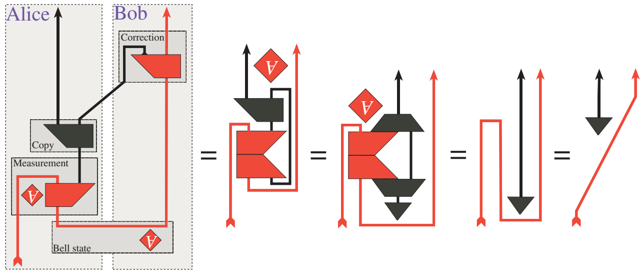
These properties together are a sufficient characterisation of classicality and *quantum measurements*, which are defined as those $\mathcal{M} : A \to X \otimes A$ which satisfy a "resource sensitive version" of *von Neumann's projection postulate*:



where we depicted quantum data in red and classical data in black. A theorem which confirms that this characterisation truly coincides with the corresponding concepts in the Hilbert space formalism is in (Coecke & Pavlovic 2006). We can now design protocols in the information-flow based hyper-logical graphical language. Deriving *superdense coding* (Bennet & Wiesner 1992) in our language looks like:



Here is a slight variantion on the teleportation protocol depicted on the 1st page, where the measurement data is *copied* before *consuming* one copy of it in the measurement:



Currently ongoing work exposed that also classical information-theoretic such as for example von Neumann entropy and majorization ordering live within this quantum formalism (Coecke, Paquette, & Pavlovic 2006).

Conclusion and outlook

We described a structural foundation for quantum informatic devices and languages which is both diagrammatic and symbolic, and combines high-level mechanisms with quantitative content. In contrast with (Birkhoff & von Neumann 1936)-style quantum logic this setting has not less, but increased deductive power as compared to its classical counterpart, and in particular captures the interaction within *multipartite* situations. The validity of this setting for quantum informatics has already been established. We are keen to see how this setting applies beyond pure quantum informatic applications, for example to the research programs outlined in (van Rijsbergen 2004) and (Clark & Pulman 2006). We expect to have made progress in this direction by the time the AAAI *Quantum Interaction Symposium* will take place.

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