Qualitative Spatial Reasoning à la Allen: An Algebra for Cyclic Ordering of 2D Orientations

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Abstract
We define an algebra of ternary relations for cyclic ordering of 2D orientations, which is a refinement of the CYCORD theory. The algebra (1) contains 24 atomic relations, hence 2^{24} general relations, of which the usual CYCORD relation is a particular relation; and (2) is NP-complete, which is not surprising since the CYCORD theory is. We then provide the following: (1) a constraint propagation algorithm for the algebra; (2) a proof that the propagation algorithm is polynomial, and complete for a subclass including all atomic relations; (3) a proof that another subclass, expressing only information on parallel orientations, is NP-complete; and (4) a solution search algorithm for a general problem expressed in the algebra. A comparison to related work indicates that the approach is promising.

Introduction
Qualitative spatial reasoning (QSR) has become an important and challenging research area of Artificial Intelligence. An important aspect of it is topological reasoning (e.g. (Cohn 1997)). However, many applications (e.g., robot navigation (Levitt & Lawton 1990), reasoning about shape (Schlieder 1994)) require the representation and processing of orientation knowledge. A variety of approaches to this have been proposed: the theory of CYCORDs for cyclic ordering of 2D orientations (Megiddo 1976; Röhrig 1994; 1997), Frank's (1992) and Hernández's (1991) sector models and Schlieder's (1993) representation of a panorama.

A cyclic ordering problem can be seen as a ternary constraint satisfaction problem of which:

1. the variables range over the points of a circle, for example the circle of centre (0, 0) and of unit radius; and
2. the constraints give for triples of variables the order in which they should appear when, say, the circle is scanned clockwise.

In real applications, information expressed by CYCORDs may not be specific enough. For instance, one may want to represent information such as "objects A, B and C are such that B is to the left of A; and C is to the left of both A and B, or to the right of both A and B", which is not representable in the CYCORD theory. This explains the need for refining the theory, which is what we propose in the paper. Before providing the refinement, which is an algebra of ternary relations, we shall define an algebra of binary relations which is much less expressive (it cannot represent the CYCORD relation). Among other things, we shall provide a composition table for the algebra of binary relations. One reason for doing this first is that it will then become easy to understand how the relations of the refinement are obtained.

So far, constraint-based approaches to QSR have mainly used constraint propagation methods achieving path consistency. These methods have been borrowed from qualitative temporal reasoning à la Allen (Allen 1983), and make use of a composition table. It is, for instance, well-known from works of van Beek that path consistency achieves global consistency for CSPs of Allen's convex relations. The proof of this result, given in (van Beek & Cohen 1990; van Beek 1992), shows that it is mainly due to the 1-dimensional nature of the temporal domain. The proof uses the specialisation of the well-known, but unfortunately not much used\(^1\) in QSR, Helly's theorem to n = 1: "If S is a set of convex regions of the n-dimensional space \(\mathbb{R}^n\) such that every \(n+1\) elements in \(S\) have a non empty intersection then the intersection of all elements of \(S\) is non empty". For the 2-dimensional space (\(n = 2\)), the theorem gets a bit more complicated, since one has to check non emptiness of the intersection of every three elements, instead of just every two. This suggests that constraint-based approaches to QSR should, if they are to be useful, devise propagation methods achieving more than just path consistency. The constraint propagation algorithm to be given for the algebra of ternary relations achieves indeed strong 4-consistency, and we shall show that it has a similar behaviour for a subclass including all atomic relations as path consistency for Allen's convex relations.

We first provide some background on the CYCORD theory; then the two algebras. Next, we consider CSPs

\(^1\)Except some works by Faltings such as in (Faltings 1995).
on cyclic ordering of 2D orientations. We then provide the following: (1) a constraint propagation algorithm for the algebra of ternary relations; (2) a proof that the propagation algorithm is polynomial, and complete for a subclass including all atomic relations; (3) a proof that another subclass, expressing only information on parallel orientations, is NP-complete; and (4) a solution search algorithm for a general problem expressed in the algebra. Before summarising, we shall discuss some related work.

**CYCORDs**

Given a circle centred at \( O \), there is a natural isomorphism from the set of 2D orientations to the set of points of the circle: the image of orientation \( X \) is the point \( P_X \) such that the orientation of the directed straight line \((OP_X)\) is \( X \). A CYCORD \( X-Y-Z \) represents the information that the images \( P_X, P_Y, P_Z \) of orientations \( X, Y, Z \), respectively, are distinct and encountered in that order when the circle is scanned clockwise starting from \( P_X \).

We now provide a brief background on the CYCORD theory, taken from (Megiddo 1976; Röhrig 1994; 1997). For this purpose, we consider a set \( S = \{X_0, \ldots, X_n\} \).

**Definition 1 (cyclic equivalence)** Two linear orders \( (X_{i_0}, \ldots, X_{i_m}) \) and \( (X_{j_0}, \ldots, X_{j_k}) \) on \( S \) are called cyclically equivalent if there exists \( m \in \mathbb{N} \) such that: \( \forall k \in \{0, \ldots, n\} (j_k = (i_k + m) \bmod (n + 1)) \).

**Definition 2 (total cyclic order)** A total cyclic order on \( S \) is an equivalence class of linear orders on \( S \) modulo cyclic equivalence; \( X_{i_0} \ldots X_{i_m} \) denotes the equivalence class containing \( (X_{i_0}, \ldots, X_{i_m}) \).

**Definition 3 (partial cyclic order)** A closed partial cyclic order on \( S \) is a set \( T \) of cyclically ordered triples such that:

| (1) \( X-Y-Z \in T \Rightarrow X \neq Y \) | (irreflexivity) |
| (2) \( X-Y-Z \in T \Rightarrow Z-Y-X \notin T \) | (asymmetry) |
| (3) \( \{X-Y-Z, X-Z-W \} \subseteq T \Rightarrow X-Y-W \notin T \) | (transitivity) |
| (4) \( X-Y-Z \notin T \Rightarrow Z-Y-X \notin T \) | (closure) |
| (5) \( X-Y-Z \in T \Rightarrow Z-Y-X \in T \) | (rotation) |

**The algebra of binary relations**

The algebra is very similar to Allen’s (1983) temporal interval algebra. We describe briefly its relations and its three operations (converse, intersection and composition).

Given an orientation \( X \) of the plane, another orientation \( Y \) can form with \( X \) one of the following qualitative configurations:

1. \( Y \) is equal to \( X \) (the angle \( (X, Y) \) is equal to \( 0 \)).
2. \( Y \) is to the left of \( X \) (the angle \( (X, Y) \) belongs to \( [0, \pi) \)).
3. \( Y \) is opposite to \( X \) (the angle \( (X, Y) \) is equal to \( \pi \)).
4. \( Y \) is to the right of \( X \) (the angle \( (X, Y) \) belongs to \( [\pi, 2\pi) \)).

We denote the four configurations by \( (Y e X) \), \( (Y l X) \), \( (Y o X) \) and \( (Y r X) \), respectively. Clearly, these configurations are Jointly Exhaustive and Pairwise Disjoint (JEPD); given any two orientations of the plane, they stand in one and only one of these configurations.

**Definition 4 (relations of the algebra)** The algebra contains four atomic relations: \( e, l, o, r \). A (general) relation is any subset of the set \( BIN \) of all four atomic relations (when a relation is a singleton set (atomic), we omit the braces in its representation). A relation \( B = \{b_1, \ldots, b_n\}, n \leq 4 \), between orientations \( X \) and \( Y \), written \( (Y B X) \), is to be interpreted as \( (Y b_1 X) \vee \ldots \vee (Y b_n X) \).

**Definition 5 (converse)** The converse of an atomic relation \( B \) is the atomic relation \( B^- \) such that:

\[ \forall X, Y, (Y b X) \iff (X b^- Y) \]

The converse \( B^- \) of a general relation \( B \) is the union of the converses of its atomic relations: \( B^- = \bigcup_{b \in B} \{b\} \).

**Definition 6 (intersection)** The intersection of two relations \( B_1 \) and \( B_2 \) is the relation \( B \) consisting of the set-theoretic intersection of \( B_1 \) and \( B_2 \): \( B = B_1 \cap B_2 \).

**Definition 7 (composition)** The composition of two relations \( B_1 \) and \( B_2 \), written \( B_1 \circ B_2 \), is the strongest relation \( B \) such that:

\[ \forall X, Y, Z ((Y B_1 X) \land (Z B_2 Y) \Rightarrow (Z B X)) \]

Figure 1 gives the converse of each of the atomic relations, as well as the composition for atomic relations.

**Definition 8 (induced ternary relation)** Given three atomic binary relations \( b_1, b_2, b_3 \), we define the induced ternary relation \( b_1 \circ_2 b_3 \) as follows (see Figure 2(1)):

\[ \forall X, Y, Z (b_1 b_2 b_3 (X, Y, Z) \iff (Y b_1 X) \land (Z b_2 Y) \land (Z b_3 X)) \]

The composition table of Figure 1 (Right) has 12 entries consisting of atomic relations, the remaining four consisting of three-atom relations. Therefore any three 2D orientations stand in one of the following 24 JEPD configurations: \( ee, el, eo, err, lel, ill,illo, lrl, lrr, oeo, orl, oor, rle, rrl, rrr \). According to Definition 8, \( rol(X, Y, Z) \), for instance, means

\[ (Y r X) \land (Z o Y) \land (Z l X) \]
The ternary relation induced from three atomic binary relations:

\[ b_1 b_2 b_3(X,Y,Z) \iff ((Y b_1 X) \land (Z b_2 Y) \land (Z b_3 X)) \]

(II) The conjunction \( b_1 b_2 b_3(X,Y,Z) \land b_1' b_2' b_3'(X,Z,W) \) is inconsistent if \( b_3 \neq b_3' \).

The composition table for atomic binary relations rules out the other, \( (4 \times 4 \times 4) - 24 \), induced ternary relations \( b_1 b_2 b_3 \); these are inconsistent: no triple \((z_1, z_2, z_3)\) of orientations exists such that for such an induced relation one has

\[(z_2 b_1 z_1) \land (z_3 b_2 z_2) \land (z_3 b_3 z_1)\]

**Refining the CYCORD theory: the algebra of ternary relations**

The algebra of binary relations introduced above cannot represent a CYCORD. However, if we use the idea of what we have called an "induced ternary relation", we can easily define an algebra of ternary relations of which the CYCORD relation will be a particular relation.

**Definition 9 (ternary relation)** An atomic ternary relation is any of the 24 JEPD configurations a triple of 2D orientations can stand in. We denote by \( \text{TER} \) the set of all atomic ternary relations:

\[
\begin{align*}
\text{TER} &= \{ \text{eee, ell, eoo, err, lel, llr, lor, lre, lrl, lrr, oeo, olr, ooe, orl, rer, rle, rlr, rlo, rol, rro, rrr} \} \\
& \text{A (general) ternary relation is any subset } T \text{ of TER:} \\
& \forall X, Y, Z(T(X,Y,Z) \iff \bigvee_{t \in T} t(X,Y,Z))
\end{align*}
\]

As an example, a CYCORD \( X-Y-Z \) can be represented by the ternary relation \( \text{CR} = \{ lrl, orl, rll, rol, rrl, rro, rrr \} \) (see Figure 3):

\[
\forall X, Y, Z(X-Y-Z \iff \text{CR}(X,Y,Z))
\]

**Definition 10 (converse)** The converse of an atomic ternary relation \( t = a \beta \gamma \) can be expressed in terms of the atomic binary relations \( a, \beta, \gamma \) and their converses in the following way:

\[
(a \beta \gamma)' = \gamma \beta' \alpha
\]

This is so because if \( (a \beta \gamma)' = a' \beta' \gamma' \) then \( a \beta \gamma(X,Y,Z) \) iff \( a' \beta' \gamma'(X,Z,Y) \). But \( a \beta \gamma(X,Y,Z) \) stands for the conjunction \( ((YaX) \land (Z \beta Y) \land (Z \gamma X)) \), and \( a' \beta' \gamma'(X,Z,Y) \) for the conjunction \( ((Z \alpha X) \land (Y \beta' Z) \land (Y \gamma' X)) \). A simple comparison of the atomic binary relations in the two conjunctions leads to \( a' = \gamma, \beta' = \beta, \gamma' = \alpha \).

**Definition 11 (rotation)** The rotation of an atomic ternary relation \( t = a \beta \gamma \) can be expressed in terms of the atomic ternary relation \( t' = a' \beta' \gamma' \) such that:

\[
\forall X, Y, Z(t(X,Y,Z) \iff t'(X,Z,Y))
\]

The rotation \( T' \) of a general ternary relation \( T \) is the union of the rotations of its atomic relations:

\[
T' = \bigcup_{t \in T} \{ t' \}
\]

Similarly to the converse, the rotation of an atomic ternary relation \( t = a \beta \gamma \) can be expressed in terms of...
the atomic binary relations \(\alpha, \beta, \gamma\) and their converses in the following way:

\[(\alpha_\beta \gamma)^{-\circ} = \beta \gamma \alpha^{-1}\]

Figure 4 gives the converse and rotation for each of the 24 atomic ternary relations.

**Definition 12 (intersection)** The intersection of two ternary relations \(T_1\) and \(T_2\) is the ternary relation \(T\) consisting of those atomic relations belonging to both \(T_1\) and \(T_2\) (set-theoretic intersection):

\[\forall X, Y, Z (T(X, Y, Z) \iff T_1(X, Y, Z) \land T_2(X, Y, Z))\]

**Definition 13 (composition)** The composition of two ternary relations \(T_1\) and \(T_2\), written \(T_1 \circ T_2\), is the most specific ternary relation \(T\) such that:

\[\forall X, Y, Z, W (T_1(X, Y, Z) \land T_2(X, Z, W) \Rightarrow T(X, Y, W))\]

If we know the composition for atomic ternary relations, we can compute the composition of any two ternary relations \(T_1\) and \(T_2\):

\[T_1 \circ T_2 = \bigcup_{t_1 \in T_1, t_2 \in T_2} t_1 \circ T_2\]

In other words, what we need is to give a composition table for atomic ternary relations, similar to Allen's (1983) composition table for temporal interval relations.

Given four 2D orientations \(X, Y, Z, W\) and two atomic ternary relations \(t_1 = b_1 \circ b_2 \circ b_3\) and \(t_2 = b'_1 \circ b'_2 \circ b'_3\), the conjunction \(t_1 \land t_2 = b_1 \circ b_2 \circ b_3\) is inconsistent if \(b_3 \neq b'_3\) (see Figure 2(II) for illustration). Stated otherwise, when \(b_3 = b'_3\) we have \(t_1 \circ t_2 = \emptyset\). Therefore, in defining composition for atomic ternary relations, we have to consider four cases:

1. **Case 1**: \(b_3 = b'_3 = e\) \((t_1 \in \{\text{e}, \text{e}, \text{l}, \text{e}, \text{o}, \text{e}, \text{r}\}\) and \(t_2 \in \{\text{e}, \text{e}, \text{l}, \text{o}, \text{e}, \text{r}\}\)).

2. **Case 2**: \(b_3 = b'_3 = l\) \((t_1 \in \{\text{e}, \text{e}, \text{l}, \text{l}, \text{l}, \text{e}, \text{r}, \text{r}, \text{r}\}\) and \(t_2 \in \{\text{e}, \text{e}, \text{l}, \text{l}, \text{o}, \text{l}, \text{l}, \text{i}, \text{r}, \text{r}\}\)).

3. **Case 3**: \(b_3 = b'_3 = o\) \((t_1 \in \{\text{e}, \text{e}, \text{o}, \text{o}, \text{e}, \text{o}\})\) and \(t_2 \in \{\text{e}, \text{e}, \text{o}, \text{o}, \text{e}, \text{o}\}\)).

4. **Case 4**: \(b_3 = b'_3 = r\) \((t_1 \in \{\text{e}, \text{e}, \text{o}, \text{r}, \text{r}, \text{r}, \text{o}, \text{r}, \text{r}\}\) and \(t_2 \in \{\text{e}, \text{e}, \text{o}, \text{r}, \text{r}, \text{r}, \text{o}, \text{r}, \text{r}\}\)).

Composition for the CYCOCRD theory as introduced in (Megiddo 1976; Röhrig 1994; 1997) (see Definition 3, rule (3)) consists of one single rule. Again, this is due to the fact that the theory is not specific enough. The algebra of ternary relations has a much finer level of granularity, and hence is much more specific: composition splits into many more cases, which are grouped together in four composition tables (one composition table for each of the above four cases). See Figure 5 for details: the entries \(E_1, E_2, E_3, E_4\) stand for the relations \(\{\text{e}, \text{e}, \text{l}, \text{r}\}, \{\text{e}, \text{l}, \text{r}, \text{r}\}, \{\text{r}, \text{r}, \text{r}, \text{r}\}, \{\text{e}, \text{e}, \text{l}, \text{r}, \text{r}\}\), respectively.

Alternatively, one can define a single composition table for the algebra of ternary relations. Such a table would have 24 \(\times\) 24 entries, most of which (i.e., 24 \(\times\) 24 - 16 + 64 + 64) would be the empty relation.

**Definition 14 (projection)** Let \(T\) be a ternary relation. The 1st, 2nd and 3rd projections of \(T\), which we shall refer to as \(\text{proj}_1(T), \text{proj}_2(T), \text{proj}_3(T)\), respectively, are the binary relations defined as follows:

\[\text{proj}_1(T) = \{b_1 \in \text{BIN} \mid \exists b_2, b_3 \in \text{BIN} \mid b_1 \circ b_2 \circ b_3 \in T\}\]
\[\text{proj}_2(T) = \{b_2 \in \text{BIN} \mid \exists b_1, b_3 \in \text{BIN} \mid b_1 \circ b_2 \circ b_3 \in T\}\]
\[\text{proj}_3(T) = \{b_3 \in \text{BIN} \mid \exists b_1, b_2 \in \text{BIN} \mid b_1 \circ b_2 \circ b_3 \in T\}\]

**Definition 15 (cross product)** The cross product of three binary relations \(B_1, B_2, B_3\), written \(\Pi(B_1, B_2, B_3)\), is the ternary relation consisting of those atomic relations \(b_1 \circ b_2 \circ b_3\) such that \(b_1 \in B_1, b_2 \in B_2, b_3 \in B_3\):

\[\Pi(B_1, B_2, B_3) = \{b_1 \circ b_2 \circ b_3 \mid (b_1 \in B_1, b_2 \in B_2, b_3 \in B_3) \land \text{TER}\}\]
CSPs of 2D orientations

A CSP of 2D orientations (hereafter 2D-OCSP) consists of

1. a finite number of variables ranging over the set 2DO
2. relations on cyclic ordering of these variables, standing for the constraints of the CSP.

A binary (resp. ternary) 2D-OCSP is a 2D-OCSP of which the constraints are binary (resp. ternary). We shall refer to binary 2D-OCSPs as BOCSPs, and to ternary 2D-OCSPs as TOCSPs.

We now consider a 2D-OCSP $P$ (either binary or ternary) on $n$ variables $X_1, \ldots, X_n$.

**Remark 1 (normalised 2D-OCSP)** If $P$ is a BOCSP, we assume that for all $i, j, k$, at most one constraint involving $X_i$ and $X_j$ is specified. The network representation of $P$ is the labelled directed graph defined as follows:

1. The vertices are the variables of $P$.
2. There exists an edge $(X_i, X_j)$, labelled with $B$, if and only if a constraint of the form $(X_j B X_i)$ is specified. If $P$ is a TOCSP, we assume that for all $i, j, k$, at most one constraint involving $X_i, X_j, X_k$ is specified.

**Definition 16 (matrix representation)** If $P$ is a BOCSP, it is associated with an $n \times n$-matrix, which we shall refer to as $P$, and whose elements will be referred to as $P_{ij}, i,j \in \{1, \ldots, n\}$. The matrix $P$ is constructed as follows:

1. Initialise all entries of $P$ to the universal relation $BIN$: $P_{ij} := BIN, \forall i, j \in \{1, \ldots, n\}$.
2. $P_{ii} := 1, \forall i = 1 \ldots n$.
3. For all $i, j \in \{1, \ldots, n\}$ such that $P$ contains a constraint of the form $(X_j B X_i)$: $P_{ij} := P_{ji} \cap B; P_{ji} := P_{ij}^B$.

If $P$ is a TOCSP, it is associated with an $n \times n \times n$-matrix, which we shall refer to as $P$, and whose elements will be referred to as $P_{ijk}, i,j,k \in \{1, \ldots, n\}$. The matrix $P$ is constructed as follows:

1. Initialise all entries of $P$ to the universal relation $TER$: $P_{ijk} := TER, \forall i,j,k \in \{1, \ldots, n\}$.
2. $P_{iii} := e, \forall i = 1 \ldots n$.
3. For all $i,j,k \in \{1, \ldots, n\}$ such that $P$ contains a constraint of the form $T(X_i, X_j, X_k)$:

   (a) $P_{ijk} := P_{ijk} \cap T; P_{kij} := P_{kij}^T$
   (b) $P_{jki} := P_{jki} \cap T; P_{kji} := P_{kji}^T$
   (c) $P_{kij} := P_{kij}^T; P_{jki} := P_{jki}^T$

4. For all $i,j \in \{1, \ldots, n\}, i < j$:

   (a) $B := \bigcap_{k=1}^n \text{proj}_1(P_{ijk})$
   (b) $P_{ij} := \Pi(B, B, B); P_{ji} := P_{ij}^B; P_{ij} := P_{ij}^B$
   (c) $P_{ij} := \Pi(B, B, B); P_{ij} := P_{ij}^B; P_{ij} := P_{ij}^B$

**Definition 17 (closure under projection)** The TOCSP $P$ is closed under projection if:

$$\forall i, j, k, l \left( \text{proj}_1(P_{ijk}) = \text{proj}_1(P_{ijl}) \right)$$

A TOCSP $P$ can always be transformed into an equivalent TOCSP which is closed under projection. This can be achieved using a loop such as the following:

repeat

(a) consider four variables $X_i, X_j, X_k, X_l$ such that $\text{proj}_1(P_{ijk}) \neq \text{proj}_1(P_{ijl})$
(b) $B := \text{proj}_1(P_{ijk}) \cap \text{proj}_1(P_{ijl})$
(c) If $B = \emptyset$ then exit (the TOCSP is inconsistent)
(d) $P_{ijk} := \Pi(B, \text{proj}_2(P_{ijk}), \text{proj}_3(P_{ijk})) \cap P_{ijk}$
(e) $P_{ijl} := \Pi(B, \text{proj}_2(P_{ijl}), \text{proj}_3(P_{ijl})) \cap P_{ijl}$

until($\forall i, j, k, l \left( \text{proj}_1(P_{ijk}) = \text{proj}_1(P_{ijl}) \right)$)

From now on, we make the assumption that a TOCSP is closed under projection.

**Definition 18 (Freuder 1982)** An instantiation of $P$ is any $n$-tuple $(z_1, z_2, \ldots, z_n)$ of $[0, 2\pi]^n$, representing an assignment of an orientation value to each variable.

A consistent instantiation, or solution, is an instantiation satisfying all the constraints. A sub-CSP of size $k$, $k \leq n$, is any restriction of $P$ to $k$ of its variables and the constraints on the $k$ variables. $P$ is $k$-consistent if every solution to every sub-CSP of size $k - 1$ extends to every $k$-th variable; it is strongly $k$-consistent if it is $j$-consistent, for all $j \leq k$.

1-, 2- and 3-consistency correspond to node-, arc- and path-consistency, respectively (Mackworth 1977; Montanari 1974). Strong $n$-consistency of $P$ corresponds to global consistency (Dechter 1992). Global consistency facilitates the exhibition of a solution by backtrack-free search (Freuder 1982).

**Remark 2** If we make the assumption that a 2D-OCSP does not include the empty constraint, which indicates a trivial inconsistency, then:

1. A BOCSP is strongly 2-consistent:

   (a) A 1-variable BOCSP has no constraint, so its unique variable can be consistently instantiated to any value in $[0, 2\pi]$ (1-consistency).

   (b) On the other hand, a 2-variable BOCSP has two variables, say $X_1$ and $X_2$, and one constraint, say $(X_1 B X_2)$. If $X_1$ is instantiated to any value, say $z_1$, then that instantiation is a solution of the 1-variable sub-CSP consisting of variable $X_1$, and we can still find an instantiation to $X_2$, say $z_2$, in such a way that the relation $(z_2 B z_1)$ holds. Similarly, any instantiation $z_2$ to $X_2$ is solution to the 1-variable sub-CSP consisting of variable $X_2$, and this can always be extended to an instantiation $z_1$ of $X_1$ such that $(X_1, X_2) = (z_1, z_2)$ satisfies the constraint $(X_1 B X_2)$.

2. A TOCSP is strongly 3-consistent:

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3The set $2DO$ is isomorphic to the set $[0, 2\pi)$.
Since a BOCSP is strongly 2-consistent, it follows that if it is path-consistent (3-consistent) then it is strongly 3-consistent. The atomic relations of the algebra of ternary relations are obtained from the composition table of the algebra of binary relations, which records all possible 24 3-variable BOCSPs of atomic relations which are strongly 3-consistent. Strong 4-consistency of a TOCSP follows.

We now assume that to the plane is associated a reference system \((O, x, y)\); and refer to the circle centred at \(O\) and of unit radius as \(C_{O,1}\). Given an orientation \(z\), we denote by \(\text{rad}(z)\) the radius \([O, P_z]\) of \(C_{O,1}\), excluding the centre \(O\), such that the orientation of the directed straight line \((OP_z)\) is \(z\). An orientation \(z\) can be assimilated to \(\text{rad}(z)\).

Definition 19 (sector of a binary relation)
The sector determined by an orientation \(z\) and a binary relation \(B\), written \(\text{sect}(z, B)\), is the sector of circle \(C_{O,1}\), excluding the centre \(O\), representing the set of orientations \(z'\) related to \(z\) by relation \(B\):

\[
\text{sect}(z, B) = \{ \text{rad}(z') | z' B z \}
\]

Remark 3 The sector determined by an orientation and a binary relation does not include the centre \(O\) of circle \(C_{O,1}\). Therefore, given \(n\) orientations \(z_1, \ldots, z_n\) and \(n\) binary relations \(B_1, \ldots, B_n\), the intersection \(\bigcap \text{sect}(z_i, B_i)\) is either the empty set or a set of radii: i.e., this cannot be equal to the centre \(O\), which would be possible if the sector determined by an orientation and a binary relation included \(O\).

Definition 20 The projection, \(\text{proj}(P)\), of a TOCSP \(P\) is the BOCSP \(P'\) having the same set of variables and such that:

\[
\forall i, j, k (P'_{ij} = \text{proj}_1(P_{ijk}))
\]

A ternary relation, \(T\), is projectable if \(T = \Pi(\text{proj}_1(T), \text{proj}_2(T), \text{proj}_3(T))\). A TOCSP is projectable if for all \(i, j, k\), \(P_{ijk}\) is a projectable relation.

Definition 21 The dimension of a binary relation is the dimension of its sector. A binary relation, \(B\), is convex if the sector determined by \(B\) and any orientation is a convex part of the plane; it is holed if

1. it is equal to \(\text{BIN}\); or
2. the difference \(\text{BIN} - B\) is a binary relation of dimension 1 (is equal to \(e, o\) or \(\{e, o\}\).

The subclass of all binary relations which are either convex or holed will be referred to as \(\text{BCH}\). There are:

1. eight convex binary relations: \(e, l, o, r, \{e, l\}, \{e, r\}, \{l, o\}, \{o, r\}\); and
2. four holed binary relations: \(\{l, r\}, \{e, l, r\}, \{l, o, r\}, \{e, l, o, r\}\).

A ternary relation is \(\{\text{convex, holed}\}\) if

1. it is projectable; and
2. each of its projections belongs to \(\text{BCH}\).

The subclass of all \(\{\text{convex, holed}\}\) ternary relations will be referred to as \(\text{TCH}\).

Example 1 (the 'Indian tent') The 'Indian tent' consists of a clockwise triangle \((ABC)\), together with a fourth point \(D\) which is to the left of each of the directed lines \((AB)\) and \((BC)\) (see Figure 6(I)). The knowledge about the 'Indian tent' can be represented as a BOCSP on four variables, \(X_1, X_2, X_3\) and \(X_4\), representing the orientations of the directed lines \((AB), (AC), (BC)\) and \((BD)\), respectively. From \((ABC)\) being a clockwise triangle, we get a first set of constraints: \(\{(X_2 r X_1), (X_3 r X_1), (X_3 r X_2)\}\). From \(D\) being to the left of each of the directed lines \((AB)\) and \((BC)\), we get a second set of constraints: \(\{(X_4 l X_1), (X_4 l X_3)\}\).

If we add the constraint \((X_1 r X_2)\) to the BOCSP, this clearly leads to an inconsistency. Röhrig (1997) has shown that using the CYCORD theory one can detect such an inconsistency, whereas this cannot be detected using classical constraint-based approaches such as those in (Frank 1992; Hernández 1991).

The BOCSP is represented graphically in Figure 6(II). The CSP is path-consistent; i.e.: \(\forall i, j, k (P_{ij} \subseteq \{P_{ik} \otimes P_{kj}\})\). However, as mentioned above, the CSP is inconsistent. Therefore:

Theorem 1 Path-consistency does not detect inconsistency even for BOCSPs of atomic relations.

The algebra of ternary relations is NP-complete:

Theorem 2 Solving a TOCSP is NP-complete.

Proof: Solving a TOCSP of atomic relations will be shown to be polynomial. Hence, all we need to show is that there exists a deterministic polynomial transformation of an NP-complete problem to a TOCSP.

The CYCORD theory is NP-complete (Galil & Megiddo 1977). The transformation of a problem ex-

\[4\] This can be easily checked using the composition table for atomic binary relations.
Theorem 4 performs is removed from Queue for propagation, the algorithm achieves strong 4-consistency for the input TOCSP P.

Theorem 3

The process terminates when there) since it may in turn constrain the relations on neighbouring triples: this is done by add-to-queue(). Every time a triple (X_i, X_j, X_k) is removed from Queue, the algorithm eventually updates the relations on the neighbouring triples (triples sharing two variables with (X_i, X_j, X_k)). If such a relation is successfully updated, the corresponding triple is sorted, in such a way to have the variable with smallest index first and the variable with greatest index last, and the sorted triple is placed in Queue (if it is not already there) since it may in turn constrain the relations on neighbouring triples: this is done by add-to-queue().

The algorithm makes use of a queue Queue. A triple may be placed in Queue if it is not already there) since it may in turn constrain the relations on neighbouring triples: this is done by add-to-queue(). The algorithm removes one variable triple from Queue at a time. When a triple (X_i, X_j, X_k) is removed from Queue, the algorithm eventually updates the relations on the neighbouring triples (triples sharing two variables with (X_i, X_j, X_k)). If such a relation is successfully updated, the corresponding triple is sorted, in such a way to have the variable with smallest index first and the variable with greatest index last, and the sorted triple is placed in Queue (if it is not already there) since it may in turn constrain the relations on neighbouring triples: this is done by add-to-queue(). The process terminates when Queue is empty.

Theorem 3 When applied to a TOCSP of size (number of variables) n, the constraint propagation algorithm runs into completion in O(n^4) time.

Proof (sketch). The number of variable triples (X_i, X_j, X_k) is O(n^3). A triple may be placed in Queue at most a constant number of times (24, which is the total number of atomic relations). Every time a triple is removed from Queue for propagation, the algorithm performs O(n) operations.

Complexity classes

Theorem 4 The propagation procedure s4c(P) achieves strong 4-consistency for the input TOCSP P.

Figure 7: A constraint propagation algorithm.

Figure 8: (I) Illustration of the proof of Theorem 5. (II) Illustration of non closure of TCH under strong 4-consistency.

Proof. A TOCSP is strongly 3-consistent (Remark 2). The algorithm clearly ensures 4-consistency, hence it ensures strong 4-consistency. ■

We refer to the subclass of all 28 entries of the four composition tables of the algebra of ternary relations as CT. We show that the closure under strong 4-consistency, CT^c of CT is tractable. We then show that the subclass PAR = \{\{ooe, ooe\}, \{cee, cee, oo\}, \{ooe, oo, ooe\}, \{ooe, eee, ooe\}\}, which expresses only information on parallel orientations, is NP-complete.

Definition 22 (s4c-closure) Let S denote a subclass of the algebra of ternary relations. The closure of S under strong 4-consistency, or s4c-closure of S, is the smallest subclass S^c of the algebra such that:

1. S \subseteq S^c;
2. \forall T_1, T_2, T_3 \in S^c(T_1, T_2, T_3 \cap T_5 \in S^c); and
3. \forall T_1, T_2, T_3 \in S^c(proj_1(T_1) = proj_1(T_2) \land proj_1(T_1) = proj_1(T_3) \land proj_1(T_2) = proj_1(T_3) \Rightarrow T_1 \land (T_1 \land T_3) \in S^c).

Theorem 5 Let P be a TOCSP expressed in TCH. If P is strongly 4-consistent then it is globally consistent.

Van Beek (1992) used the specialisation to n = 1 of Helly's convexity theorem to prove a similar result for a path consistent CSP of Allen's convex relations. The proof of Theorem 5 will use the specialisation to n = 2:

Theorem 6 (Helly's Theorem (Chvátal 1983)) Let S be a set of convex regions of the n-dimensional space IR^n. If every n + 1 elements in S have a non empty intersection then the intersection of all elements of S is non empty.

Proof of Theorem 5. Since P lies in the TCH subclass and is strongly 4-consistent, we have the following:

1. P is equivalent to its projection, say P^pr, which is a BOCSP expressed in BCH.
2. The projection P^pr is strongly 4-consistent.

So the problem becomes that of showing that P^pr is globally consistent. For this purpose, we suppose that the instantiation (X_{i1}, X_{i2}, ..., X_{ik}) = (z_1, z_2, ..., z_k), k \geq 4, is a solution to a k-variable sub-CSP, say S, of P^pr whose variables

\[ \begin{align*}
X_{i_1} &= z_1 \\
X_{i_j} &= z_j \\
X_{i_k} &= z_k
\end{align*} \]
are \(X_{i1}, X_{i2}, \ldots, X_{ik}\). We need to prove that the partial solution can be extended to any \((k + 1)\)st variable, say \(X_{i(k+1)}\), of \(P^{pr}\).\(^5\) This is equivalent to showing that the following sectors have a non empty intersection (see Figure 8(1) for illustration): 
\[
\text{sect}(z_1, P^{pr}_{i(jk+1)}), \text{sect}(z_2, P^{pr}_{i(jk+1)}), \ldots, \text{sect}(z_k, P^{pr}_{i(jk+1)}).
\]

Since the \(P^{pr}_{i(jk+1)}, j = 1 \ldots k\), belong to \(BCH\), each of these sectors is:

1. a convex subset of the plane; or
2. almost equal to the surface of circle \((1)\) and those verifying condition (2). We assume, without loss of generality, that the first \(m\) verify condition (1), and the last \(k - m\) verify condition (2). We write the intersection of the sectors as \(I = I_1 \cap I_2\), with 
\[
I_1 = \bigcap_{j=1}^m \text{sect}(z_j, P^{pr}_{i(jk+1)}),
\]
\[
I_2 = \bigcap_{j=m+1}^k \text{sect}(z_j, P^{pr}_{i(jk+1)}).
\]

Due to strong 4-consistency, every three of these sectors have a non empty intersection. If any of the sectors is a radius (the corresponding relation is either \(e\) or \(o\)) then the whole intersection must be equal to that radius since the sector intersects with every other two.

We now need to show that when no sector reduces to a radius, the intersection is still non empty:

**Case 1:** \(m=k\). This means that all sectors are convex. Since every three of them have a non empty intersection, Helly's theorem immediately implies that the intersection of all sectors is non empty.

**Case 2:** \(m=0\). This means that no sector is convex; which in turn implies that each sector is such that its topological closure covers the whole surface of \(C_{0,1}\). Hence, for all \(j = 1 \ldots k\):

1. the sector \(\text{sect}(z_j, P^{pr}_{i(jk+1)})\) is equal to the whole surface of \(C_{0,1}\) minus the centre (the relation \(P_{i(jk+1)}\) is equal to \(B/IN\)); or
2. the sector \(\text{sect}(z_j, P^{pr}_{i(jk+1)})\) is equal to the whole surface of \(C_{0,1}\) minus one or two radii (the relation \(P_{i(jk+1)}\) is equal to \(\{e, l, r\}, \{l, o, r\}, \{l, r\}\)).

So the intersection of all sectors is equal to the whole surface of \(C_{0,1}\) minus a finite number (at most 2\(k\)) of radii. Since the surface is of dimension 2 and a radius is of dimension 1, the intersection must be non empty (of dimension 2).

**Case 3:** \(0 < m < k\). This means that some sectors (at least one) are convex, the others (at least one) are such that their topological closures cover the whole surface of \(C_{0,1}\). The intersection \(I_1\) is non empty due to Helly's theorem, since every three appearing in it have a non empty intersection:

**Subcase 3.1:** \(I_1\) is a single radius, say \(r\). Since the sectors appearing in \(I_1\) are less than \(\pi\), there must exist two sectors, say \(s_1\) and \(s_2\), appearing in \(I_1\) such that their intersection is \(r\). Since, due to strong 4-consistency, \(s_1\) and \(s_2\) together with any sector appearing in \(I_2\) form a non empty intersection, the whole intersection, i.e., \(I\), must be equal to \(r\).

**Subcase 3.2:** \(I_1\) is a 2-dimensional (convex) sector. It should be clear that the intersection \(I_2\) is the whole surface of \(C_{0,1}\) minus a finite number of radii (at most 2\((k - m)\) radii). Since a finite union of radii is of dimension 0 or 1, and that the intersection \(I_1\) is of dimension 2, the whole intersection \(I\) must be non empty (of dimension 2).

The intersection of all sectors is non empty in all cases. The partial solution can hence be extended to variable \(X_{i(k+1)}\) (which can be instantiated with any orientation in the intersection of the \(k\) sectors).

It follows from Theorems 3, 4 and 5 that if the \(TCH\) subclass is closed under strong 4-consistency, it must be tractable. Unfortunately, as illustrated by the following example, \(TCH\) is not so closed.

**Example 2** The \(BCOSP\) depicted in Figure 2(II) can be represented as the projectable \(TOCSP\) \(P\) whose matrix representation verifies: 
\[
P_{123} = III, P_{124} = \Pi(l, l, r, \{l, r\}), P_{134} = P_{234} = \Pi(l, l, \{l, r\}).
\]

Applying the propagation algorithm to \(P\) leaves unchanged 
\[
P_{123}, P_{124}, P_{234}, but transforms \(P_{124}\) into the relation \(\{III, l\{l, r\}, l\{l, r\}\}\), which is not projectable: this is done by the operation 
\[
P_{124} := P_{124} \cap (P_{123} \otimes P_{234}).
\]

Enumerating \(CT^c\) leads to 49 relations (including the empty relation), all of which are \{convex,holed\} relations. Therefore:

**Corollary 1** (tractability of \(CT^c\)) The subclass \(CT^c\) is tractable.

**Proof.** Immediate from Theorems 3, 4 and 5. ■

The enumeration of \(CT^c\) is given in Figure 9.

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\(^5\)Since the \(TOCSP\) \(P\) is projectable, any solution to any sub-CSP of the projection \(P^{pr}\) is solution to the corresponding sub-CSP of \(P\). This would not be necessarily the case if \(P\) were not projectable.
Example 3 Transforming the BOCSP of the 'Indian tent' into a TOCSP, say $P'$, leads to $P_{123}' = rrr, P_{124}' = rrl, P_{134}' = rll, P_{234}' = rlr$. $P'$ lies in $CT^3$, hence the propagation algorithm must detect its inconsistency. Indeed, the operation $P_{124}' := P_{124}' \cap (P_{123}' \otimes_3 P_{134}')$ leads to the empty relation, since $rrr \otimes_3 rll = rll$.

We now show NP-completeness of PAR.

Theorem 7 (NP-completeness of PAR) The subclass PAR is NP-complete.

Proof. The subclass PAR belongs to NP, since solving a TOCSP of atomic relations is polynomial (Corollary 1). We need to prove that there exists a (deterministic) polynomial transformation of an NP-complete problem (we consider 3-SAT: a SAT problem of which every clause contains exactly three literals) into a TOCSP of atomic relations is polynomial (Corollary 11). We need to prove that every clause contains exactly three literals) into a TOCSP, say $P_{34}$, leads to $P_{34}$ consistent. Indeed, the operation $P_{34} := P_{34} \cap (P_{34} \otimes_3 P_{34})$ leads to the empty relation, since $rrr \otimes_3 rll = rll$.

$M$ assigns the value true to literal $\ell$, the value opposite to that of $X_0$ otherwise. For all $(\ell_1 \lor \ell_2) \in \text{BinCl}(S)$, $X(\ell_1 \lor \ell_2)$ is assigned the same value as $X_0$ if either $X(\ell_1)$ or $X(\ell_2)$ is assigned the same value as $X_0$, the opposite value otherwise. On the other hand, any solution to $P_S$ can be mapped to a model of $S$ by assigning to every literal $\ell$ the value true if and only if the variable $X(\ell)$ is assigned the same value as $X_0$.

A solution search algorithm

Since the constraint propagation procedure $s_{4c}$ of Figure 7 is complete for the subclass of atomic ternary relations (Corollary 1), it is immediate that a general TOCSP can be solved using a solution search algorithm such as the one in Figure 10, which is similar to the one provided by Ladkin and Reinefeld (1992) for temporal interval networks, except that:

1. it instantiates triples of variables at each node of the search tree, instead of pairs of variables; and
2. it makes use of the procedure $s_{4c}$, which achieves strong 4-consistency, in the preprocessing step and as the filtering method during the search, instead of a path consistency procedure.

The other details are similar to those of Ladkin and Reinefeld’s algorithm.

Related work

Representing a panorama

In (Levitt & Lawton 1990), Levitt and Lawton discussed QUALNAV, a qualitative landmark navigation system for mobile robots. One feature of the system is the representation of the information about the order of landmarks as seen by the visual sensor of a mobile robot. Such information provides the panorama of the robot with respect to the visible landmarks.

Figure 11 illustrates the panorama of an object $S$ with respect to five reference objects (landmarks) $A, B, C, D, E$ in Schlieder’s system (Schlieder 1993) (page 527). The panorama is described by the total cyclic order of the five directed straight lines $(SA), (SB), (SC), (SD), (SE)$, and the lines which are opposite to them, namely $(Sa), (Sb), (Sc), (Sd), (Se)$.
Figure 11: The panorama of a location.

Figure 12: Frank’s system of cardinal directions.

By using the algebra of binary relations, only the five straight lines joining $S$ to the landmarks are needed to describe the panorama: \{$(SB)r(SA)$, $(SC)r(SB)$, $(SD)r(SB)$, $(SE)r(SC)$, $(SE)r(SA)$\}; using the algebra of ternary relations, the description can be given as a 2-relation set: \{rll($(SA)$, $(SB)$, $(SE)$), rrr($(SB)$, $(SA)$)\}.

Schlieder’s system seems to make an implicit assumption, which is that the object to be located (i.e., $S$) supposed not to be on any of the straight lines joining pairs of the reference objects. The use of the algebra of binary relations rules out the assumption (the relations $e(qual)$ and $o(pposite)$ can be used to describe object being on a straight line joining two reference objects). Note that Schlieder does not describe how to reason about a panorama description.

Sector models for reasoning about orientations

These models use a partition of the plane into sectors determined by straight lines passing through the reference object, say $S$. The sectors are generally equal, and the granularity of a sector model is determined by the number of sectors, therefore by the number of straight lines ($n$ straight lines determine $2n$ sectors). Determining the relation of another object relative to the reference object becomes then the matter of giving the sector to which the object belongs.

Suppose that we consider a model with $2n$ sectors, determined by $n$ (directed) straight lines $\ell_1, \ldots, \ell_n$ which we shall refer to as reference lines. We can assume without loss of generality that (the orientations of) the reference lines verify: $\ell_i$ is to right of $\ell_j$ (i.e., $(\ell_i r \ell_j)$), for all $i \in \{2, \ldots, n\}$, for all $j \in \{1, \ldots, i - 1\}$. We refer to the sector determined by $\ell_i$ and $\ell_{i+1}$, $i = 1 \ldots n - 1$, as $s_i$, to the sector determined by $\ell_n$ and the directed line opposite to $\ell_1$ as $s_n$. For each sector $s_i$, $i = 1 \ldots n$, the opposite sector will be referred to as $s_{n+i}$.

1. The reference lines $\ell_1, \ldots, \ell_4$ are as indicated in the figure.

2. The sectors $s_1, \ldots, s_2 \times 4$ are North, North-East, East, South-East, South, South-West, West, and North-West, respectively.

Hernández’s (Hernández 1991) sector models can also benefit from this representation.

Suppose that a description is provided, consisting of qualitative positions of objects relative to the reference object $S$. $S$ may be a robot for which the current panorama has to be given; the description may consist of sentences such as “landmark 1 is North-East, and landmark 2 South of the robot”. We refer to such a description as a sector description of a configuration.

A sector description can be translated into a $BOCSP$ $P$ in the following natural way. $P$ includes all the relations described above on pairs of the reference lines. For each sentence such as the one above, the relations $(X_{r(\ell_1)} r \ell_2)$, $(X_{r(\ell_2)} l \ell_3)$, $(X_{r(\ell_3)} r \ell_4)$, and $(X_{r(\ell_4)} l \ell_1)$ are added to $P$. $X_{r(\ell_1)}$, for instance, stands for the orientation of the directed straight line joining the reference object ‘robot’ to landmark 1.

An important point to notice, which is not hard to show, is that a sector description is consistent if and only if the corresponding translation into a $BOCSP$ is consistent. The ‘only-if’ is trivial. The ‘if’ can be shown by exploiting the fact that if the $BOCSP$ is consistent then any solution to it can be mapped into a solution of the sector description by appropriate rotations of the values assigned to the variables which bring the reference lines to the desired positions.

Reasoning about 2D segments

In his paper “Reasoning About Ordering”, Schlieder (1995) presented a set of line segment relations. These relations are based on the cyclic ordering of endpoints of the segments involved. We believe that reasoning about 2D segments should combine at least orientational and topological information. Orientational information would be information about cyclic ordering of the orientations of the directed lines supporting the segments; on the other hand, topological information would be the description of the relative positions of the segments’ endpoints. For instance, using the algebra of binary relations on 2D orientations, as defined in this work, we could define an algebra of 2D segments, which would have the following segment relations (given a segment $s$, we denote by $s_l$ and $s_r$ its left and right endpoints, respectively, i.e., $s$ is the segment $[s_l, s_r]$; by $s$,
the orientation of the directed line \((s_1s_r)\) supporting segment \(s\):

1. If the orientations \(z_{s_1}\) and \(z_s\) are equal, the endpoints of \(s_2\) are:
   (a) both to the left of the directed line supporting segment \(s_1\) (one relation);
   (b) both on the line supporting segment \(s_1\) (13 relations: see Allen's (1983) temporal interval algebra); or
   (c) both to the right of the directed line supporting segment \(s_1\) (one relation).

2. If \(z_{s_2}\) is to the left of \(z_{s_1}\), this leads to 25 segment relations, which are obtained as follows:
   (a) The endpoints of \(s_1\) partition the directed line supporting the segment into five regions: the region strictly to the left of the left endpoint, the region consisting of the left endpoint, the region strictly between the two endpoints, the region consisting of the right endpoint, and the region strictly to the right of the right endpoint. Similarly, the endpoints of \(s_2\) partition the directed line supporting the segment into five regions.
   (b) The lines supporting the segments \(s_1\) and \(s_2\) are intersecting, and the intersecting point is in either of the five regions of the first line, and in either of the five regions of the second line. This gives the 25 segment relations.

3. If \(z_{s_1}\) and \(z_{s_2}\) are opposite to each other, we get 15 relations in a similar manner as in point 1. above.

4. If \(z_{s_2}\) is to the right of \(z_{s_1}\), we get 25 relations in a similar manner as in point 2. above.

Therefore, the total number of segment relations would be 80.

**Summary and future work**

We have provided a refinement of the theory of cyclic ordering of 2D orientations, known as CYCORD theory (Megiddo 1976; Röhrig 1994; 1997). The refinement has led to an algebra of ternary relations, for which we have given a constraint propagation algorithm and shown several complexity results.

We have discussed briefly how this work relates to some others in the literature; in particular, the discussion has highlighted the following: (1) Existing systems for reasoning about 2D orientations are covered by the presented approach (CYCORDs (Megiddo 1976; Röhrig 1994; 1997) and sector models (Frank 1992; Hernández 1991)); (2) The presented approach seems more adequate than the one in (Schlieder 1993) for the representation of a panorama.

There has been much work on Allen's interval algebra (Allen 1983). For instance, Nebel and Bürckert (1995) have shown that the ORD-Horn subclass of the algebra was the unique maximal tractable subclass containing all 13 atomic relations. Most of this work could be adapted for the two algebras of 2D orientations we have defined.

Finally, a calculus of 3D orientations, similar to what we have presented for 2D orientations, might be developed.

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**References**


