Modeling periodic functions for time series analysis

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Abstract

Time series problems involve analysis of periodic functions for predicting the future. A flexible regression method should be able to dynamically select the appropriate model to fit the available data. In this paper, we present a function approximation scheme that can be used for modeling periodic functions using a series of orthogonal polynomials, named Chebychev polynomials. In our approach, we obtain an estimate of the error due to neglecting higher order polynomials and thus can flexibly select a polynomial model of the proper order. We also show that this approximation approach is stable in the presence of noise.

Keywords: Function approximation,Orthogonal polynomials

Introduction

Analysis of time series data plays an important role in finance and marketing. A model derived from past data can help to predict future behavior of the phenomenon under study. For example , models predicting demand for product can be used to direct capital allocation. For analysis of time series data, we, in general, start with a predetermined model and try to tune its parameters to fit the data. For example, we may choose to fit a linear model to the data. for a flexible approximation scheme, however, a model should be chosen dynamically, based on data. In this paper, a function approximation scheme is presented, which can be used for modeling periodic functions. We have used a series of orthogonal polynomials, called Chebychev polynomials for approximating time series data. We know that any function can be represented with an infinite series of Chebychev polynomials. But we cannot work with an infinite series and use only a finite number of terms to derive an approximation. This method gives an approximate model for the data using n Chebychev polynomials where the number of polynomials can be dynamically decided from the data. The approximated model with n Chebychev polynomials gives the least squared error with n polynomials.

Furthermore, our algorithm allows incremental development of the model, i.e. with each piece of data, the model can be improved. We have also provided the convergence proof of the algorithm, which says that under infinite sampling this method is guaranteed to give accurate model and the level of accuracy is limited only by error due to truncation of the series.

Chebychev polynomials

Chebychev polynomials are a family of orthogonal polynomials (Geronimus 1961). Any function $f(x)$ may be approximated by a weighted sum of these polynomial functions with an appropriate selection of the coefficients.

$$f(x) = \frac{a_0}{2} + \sum_{i=1}^{\infty} a_i * T_i(x)$$

where

$$T_n(x) = \cos(n * \cos^{-1}(x))$$

and

$$a_i = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x) * T_i(x)}{\sqrt{1-x^2}} dx.$$

Working with an infinite series is not feasible in practice. We can, however, truncate the above series and still obtain an approximation, $\hat{f}(x)$, of the function (Gerald & Wheatly 1992). The Chebychev polynomials converges faster than the Taylor series for the same function (Gerald & Wheatly 1992). Let us assume we are using only the first n terms of the series. For a rapidly converging series, the error due to truncation is approximately given by the first term of the remainder, i.e., $a_n T_n(x)$. We have chosen Chebychev polynomials for function approximation because truncation points can be chosen to provide approximate error bounds.

It should be noted that the expansion of a function $f(x)$ with n Chebychev polynomials, gives the least squared polynomial approximation with the n polynomials (Foxand & Parker 1968).

To produce a viable mechanism for building a model based on Chebychev polynomials would require the development of an algorithm for calculating the polynomial coefficients. We would also need to prove that this algorithmic updates would result in a convergence of the approximated function to the actual time function

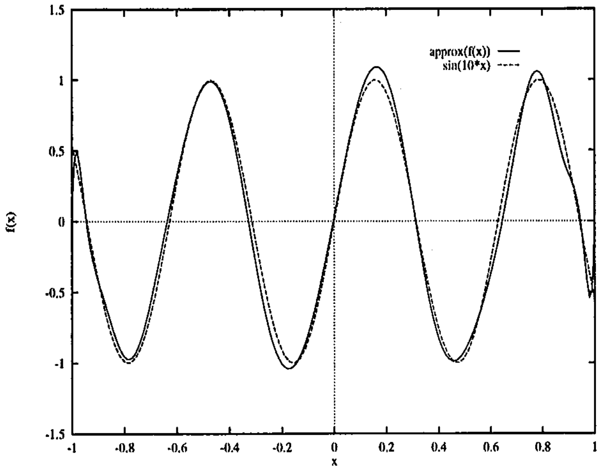

Figure 1: Approximating $f(x) = sin(10*x)$ for Number of Chebychev Polynomials used = 20 and number of data points = 1000.

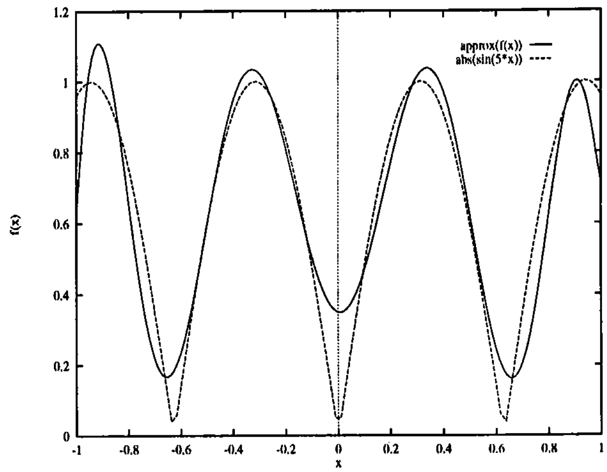

Figure 2: Approximating $f(x) = |sin(5*x)|$ for Number of Chebychev Polynomials used = 10 and number of data points = 1000.

underlying the sampled data. We provide such an algorithm and the associated convergence theorem in the next section.

An algorithm to get the temporal model from time series data

Let $f(x)$ be the target function of x and $\hat{f}(x)$ be its approximation based on the set of samples $S = \{S_j\}$, where $S_j = \langle x_j, v_{xj} \rangle \forall j = 1, 2, \ldots, k$ and k = Number of instances and $v_{x_j} = f(x_j)$. We may have to change the scale for the values x_j, so that all the values are in the range $[-1:1]$; then we get the approximated function in the range $[-1:1]$, which we may need to scale back to get the desired value.

Let n be the number of Chebychev polynomials we are going to use for the purpose of learning. Let $T_i(x) \quad i \in [0..n]$ be the Chebychev polynomials. The steps of the algorithm are:

1. Initialize $C_i = 0, \quad \forall i = 0, 1, 2, \ldots, n$
2. For all j do
3. $\forall i = 0, 1, 2, \ldots, n$

$$C_i \leftarrow C_i + \frac{T_i(x) * v_{x_j}}{\sqrt{1 - x^2}}$$

4. End for
5. Set

$$\hat{f}(x) = K * (\frac{C_0}{2} + \sum_{i=1}^{n} C_i * T_i(x))$$

where $K = \psi(k)$, is function of number of interactions.

Theorem 1 Under infinite sampling the algorithm can approximate the actual function.

Proof: See Appendix.

Approximations of periodic functions

We know that a periodic function $f(x)$ can in general be expressed as

$$f(x) \;=\; f(x + T * n),$$

where T is the period and $n \in [0, 1, 2, \ldots]$. So the nature of the function in a single period and the length of the period is enough for the prediction of behavior for unknown x values. From our approximation scheme we get these two pieces of information and can successfully predict the behavior of the functional values for unseen situations. We have taken some typical periodic functions and tested the algorithm for those functions:

$f(x) = sin(x * 10)$, and $f(x) = |sin(x * 5)|$.

We know that the error due to truncation is given by the first term in the series and thus we can flexibly decide the number of terms to be chosen for approximation. The actual functions and the polynomial approximations obtained for these two functions are presented in Figures 1 and in Figures 2. We have given each data point sequentially into the algorithm and the resulting functions generated are found to be fairly good approximations of the underlying functions. Thus we can get a good measure of the period and nature of the function in a period. We can use this informations to get the functioal values in unseen points.

We have also shown the nature of improvement of the model with more data points for the $f(x) = |sin(x * 5)|$ in Figures 3. We have calculated error as the average of the difference of approximate functional values and the accurate values, over a set of sampled points in the interval $[-1:1]$. It shows that the approximation accuracy increases rapidly with number of points.

Effects of Noise: We have also tested the approximation scheme under noisy data, where noise is added

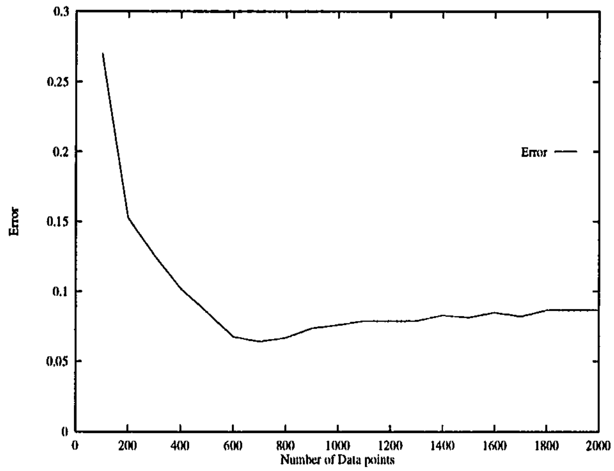

Figure 3: Nature of decreasing error in approximating $f(x) = |sin(x * 5)|$ Number of Chebychev Polynomials used is 11 and sampling resolution is 1000.

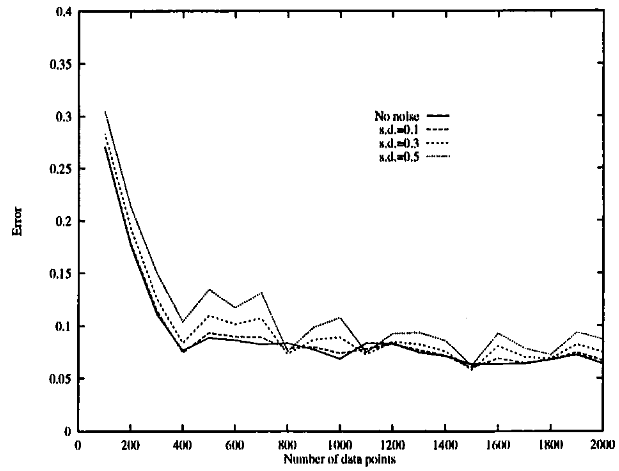

Figure 4: Nature of decreasing error in approximating $f(x) = sin(x * 10)$ under noise Number of Chebychev Polynomials used is 20 and sampling resolution is 1000.

to each data point. Here we assume that the random noise follow a Normal distribution.

We have run experiments with random noise following Normal distribution($\mu = 0, \sigma$) where $\sigma = 0$, $\sigma = 0.1$, $\sigma = 0.3$, $\sigma = 0.5$ values are applied where value of $f(x)$ is in the range $[-1 : 1]$. We have investigated the nature of decreasing error with more data points and see that the process is resilient to noise as average error do not increase rapidly with noise shown in Figures 4.

Conclusion

We have presented an algorithm that can build a model from the time series data where this model is not a predetermined one. Number of Chebychev polynomials is determined automatically, as the first term in the remainder gives the measure of the error for approximation. So, we can suitably select the model for better approximation. Moreover, we have tested this approach with noisy data and see that its performance doesn't decrease rapidly with noise and hence stable against noise.

Appendix

Proof of Theorem 1

We know, that any function can be represented as a combination of Chebychev polynomials as,

$$f(x) = \frac{a_0}{2} + \sum_{i=1}^{\infty} a_i * T_i(x)$$

Next, we are going to prove that the function $\hat{f}(x)$ tends to f(x) as more and more samples of the function, of greater resolution is received. We have

$$C_i = \frac{\phi}{\pi} * \sum_{j=1}^{k} \frac{(f(x_j))}{\sqrt{1 - x_j^2}} * T_i(x_j)$$

where ϕ is a function of no of interactions and can be given by $\phi = \frac{C}{k}$. So,we get,

$$C_i = \frac{\phi}{\pi} * \sum_{j=1}^{k} \frac{f(x_j)}{\sqrt{1 - x_j^2}} * T_i(x_j)$$

Now, in case of infinite sampling and infinite resolution, we have,

$$k \to \infty \Rightarrow \phi \to 0$$

So for $\forall i = 0, 1, 2, \ldots n$ i.e. the coefficient becomes,

$$C_i = \lim_{\phi \to 0} \frac{\phi}{\pi} * \{\sum_{j=1}^{k} \frac{f(x_j)}{\sqrt{1 - x_j^2}} * T_i(x_j)$$

$$C_0 = \lim_{\phi \to 0} \frac{1}{\pi} * \sum_{j=1}^{k} \frac{f(x_j)}{\sqrt{1 - x_j^2}} * T_i(x_j) * \phi$$

$$C_i = \frac{\phi}{\pi} * \int_x \frac{f(x)}{\sqrt{1 - x^2}} * T_i(x) dx$$

As, the x range is [-1:1] and in case of infinite sampling and resolution we have information at all the values of x, the integral limits becomes,

$$C_i = \frac{1}{\pi} * \int_{-1}^{1} \frac{f(x)}{\sqrt{1 - x^2}} * T_0(x) dx$$

$$\Rightarrow C_i = \frac{a_i}{2}$$

Thus, we show that the learned coefficients approximate the actual polynomial coefficients for infinite sampling.

References

Foxand, L., and Parker, I. 1968. *Chebyshev Polynomials in Numerical Analysis.* New York, NY: Oxford.

Gerald, C. F., and Wheatly, P. O. 1992. *Applied Numerical Analysis.* Menlo Park, CA: Addison-Wesley Publishing Company.

Geronimus, L. Y. 1961. *Orthogonal Polynomials.* New York, NY: Consultants Bureau.