

Conditional Objects Revisited: Variants and Model Translations*

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Abstract

The quality criteria of system P have been guiding qualitative uncertain reasoning now for more than two decades. Different semantical approaches have been presented to provide semantics for system P. The aim of the present paper is to investigate the semantical structures underlying system P in more detail, namely, on the level of the models. In particular, we focus on the approach via conditional objects which relies on Boolean intervals, without making any use of qualitative or quantitative information. Indeed, our studies confirm the singular position of conditional objects, but we are also able to establish semantical relationships via novel variants of model theories.

1 Introduction

Conditional expressions like in “If A then usually B ” are crucial pieces of knowledge for uncertain reasoning. The close connection between such conditionals, often written as $(B|A)$, and (nonmonotonic) inferences of the form $A \sim B$ (“From A , tentatively conclude B ”) has been intensely studied (cf., e.g., (Benferhat, Dubois, and Prade 1997)). For the formalization of uncertain reasoning the postulates of system P given in (Kraus, Lehmann, and Magidor 1990) provide what most researchers regard as a core any nonmonotonic system should satisfy; in (Hawthorne and Makinson 2007), Hawthorne and Makinson call these axioms the “industry standard” for qualitative nonmonotonic inference:

<i>Reflexivity or Inclusion</i>	:	$A \sim A$
<i>Cut</i>	:	$\frac{A \wedge B \sim C, A \sim B}{A \sim C}$
<i>Cautious Monotony</i>	:	$\frac{A \sim B, A \sim C}{A \wedge B \sim C}$
<i>Right Weakening</i>	:	$\frac{A \sim B, B \models C}{A \sim C}$
<i>Left Logical Equivalence</i>	:	$\frac{\models A \equiv B, A \sim C}{B \sim C}$
<i>Or</i>	:	$\frac{A \sim C, B \sim C}{A \vee B \sim C}$

In general, conditionals $(B|A)$ as formal representations of

default rules “If A then usually B ” and encodings of non-monotonic inferences $A \sim B$ on the syntax level, are in the focus of interest. In this paper, we will take a closer look at *conditional objects* that are presented in (Dubois and Prade 1994) together with an entailment relation modelling derivations in system P. First, we study their relationships to the different semantical structures proposed for conditional logics and modelling system P. Whereas we have equivalence on the level of the resulting inference relation (they all coincide completely with system P inferences), the models and satisfaction relation between models and sentences are inherently different. We will elaborate here whether and how translations between models can be specified that preserve the semantics of conditionals:

Given two logics L_X and L_Y that provide semantics to system P, for each model m of L_X , can we specify a model m' of L_Y such that $m \models_X (B|A)$ iff $m' \models_Y (B|A)$?

In spite of the equivalence on the inference level, our work reveals substantial semantic differences between conditional objects and most of the other logics considered here prohibiting model translations among them. On the positive side, our results will emphasize the inherent characteristics of each logic, each of them contributing in a very special way to the picture of nonmonotonic reasoning that is described by system P.

As a second contribution to the study of conditional objects, we develop two alternative variants of the logic of conditional objects, one of them having a significantly simpler model semantics, and the other one enabling a previously not possible model translation from probabilistic conditional logic.

This paper extends our work presented in (Beierle and Kern-Isberner 2009) on studying different logics via the formalism of institutions (Goguen and Burstall 1992; Goguen and Rosu 2002) that specify logical systems by means of category theory. Using simpler and more accessible notions instead, we extend the investigation to conditional objects, providing novel insights into this somewhat extraordinary conditional approach.

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2 Background: Conditional Logic

We start with a propositional language $\mathcal{L}$, generated by a finite set Σ of atoms $a, b, c, \dots$. The formulas of $\mathcal{L}$ will be denoted by uppercase Roman letters $A, B, C, \dots$. For conciseness of notation, we will omit the logical *and*-connective, writing AB instead of $A \wedge B$, and overlining formulas will indicate negation, i.e. $\overline{A}$ means $\neg A$. Let Ω denote the set of possible worlds over $\mathcal{L}$; Ω will be taken here simply as the set of all propositional interpretations over $\mathcal{L}$ and can be identified with the set of all complete conjunctions over Σ . For $\omega \in \Omega$, $\omega \models A$ means that the propositional formula $A \in \mathcal{L}$ holds in the possible world ω . By introducing a new binary operator $|$, we obtain the set $(\mathcal{L} | \mathcal{L}) = \{(B|A) \mid A, B \in \mathcal{L}\}$ of *conditionals* over $\mathcal{L}$. $(B|A)$ formalizes “if A then (normally) B ” and establishes a plausible, probable, possible etc connection between the *antecedent* A and the *consequence* B .

A conditional $(B|A)$ is an object of a three-valued nature, partitioning the set of worlds Ω in three parts: those worlds satisfying AB , thus *verifying* the conditional, those worlds satisfying $\overline{A}B$, thus *falsifying* the conditional, and those worlds not fulfilling the premise A and so which the conditional may not be applied to at all. To give appropriate semantics to conditionals, they are usually considered within richer structures such as *epistemic states*, allowing the representation of preferences, beliefs, assumptions of an intelligent agent.

Preferential Semantics

While System P can be characterized semantically in very different ways, one of the most prominent approaches to nonmonotonic reasoning and system P is via *preferential models* (cf. e.g. (Makinson 1989)) which use a transitive and classical relation among possible worlds to express preference between them. Inference is then based upon the most preferred models. This idea of comparing worlds and considering only the “nearest” or “best” worlds is common to many approaches in conditional logics (see, e.g., the *system-of-spheres* model of Lewis (Lewis 1973)). A basic implementation can be achieved by plausibility preorders.

From this purely qualitative point of view, proper models of conditionals are provided by total preorders R (i.e. R is a total, reflexive and transitive relation) over classical propositional interpretations, or possible worlds, respectively. Possible worlds are ordered according to their *plausibility*; by convention, the least worlds are the most plausible worlds. For a preorder R , we use the infix notation $\omega_1 \preceq_R \omega_2$ instead of $(\omega_1, \omega_2) \in R$. As usual, we introduce the $\prec_R$ -relation by saying that $\omega_1 \prec_R \omega_2$ iff $\omega_1 \preceq_R \omega_2$ and not $(\omega_2 \preceq_R \omega_1)$. Furthermore, $\omega_1 \approx_R \omega_2$ means that both $\omega_1 \preceq_R \omega_2$ and $\omega_2 \preceq_R \omega_1$ hold. Each total preorder R induces a partitioning $\Omega_0, \Omega_1, \dots$ of Ω , such that all worlds in the same partitioning subset are considered equally plausible ($\omega_1 \approx_R \omega_2$ for $\omega_1, \omega_2 \in \Omega_j$), and whenever $\omega_1 \in \Omega_i$ and $\omega_2 \in \Omega_k$ with $i < k$, then $\omega_1 \prec_R \omega_2$. Moreover, a total preorder on Ω extends to a total preorder on propositional formulas A, B via: $A \preceq_R B$ iff there exists $\omega_1 \in \Omega$ with $\omega_1 \models A$ such that for all $\omega_2 \in \Omega$ with $\omega_2 \models B$, we have $\omega_1 \preceq_R \omega_2$. Again, $A \prec_R B$ means both $A \preceq_R B$ and not $B \preceq_R A$.

Given these preliminaries, the models of the logic L_K of *preferential semantics* are given by $Mod_K(\Sigma) = \{R \mid R \text{ is a total preorder on } \Omega_\Sigma\}$. The satisfaction relation defined by $R \models_K (B|A)$ iff $AB \prec_R \overline{A}B$ expresses that a conditional $(B|A)$ is satisfied (or accepted) by the plausibility preorder R iff its confirmation AB is more plausible than its refutation $\overline{A}B$.

Conditional Objects

Another semantics for system P has been proposed in (Dubois and Prade 1994; Benferhat, Dubois, and Prade 1997) by using conditional objects, i.e. propositional conditionals of the form $(B|A)$. Based on the three-valued conditional semantics of de Finetti (DeFinetti 1974) Dubois and Prade define an entailment $\models_{DP}$ relation between conditional objects by:

$$(B|A) \models_{DP} (D|C) \text{ iff } AB \models CD \text{ and } A \Rightarrow B \models C \Rightarrow D. \quad (1)$$

and interpret conditional objects by *Boolean intervals*

$$[L, U] := \{C \in \mathcal{L} \mid L \models C \text{ and } C \models U\} \quad (2)$$

(over Σ) where $L, U \in \mathcal{L}$ with $L \models U$. Note that a Boolean interval is the set of all propositional formulas “between” two formulas L and U serving as the lower and upper bound of the interval; e.g., $[L, U]$ and $[L \vee L, U \wedge (U \vee \top)]$ are not distinguished as they are exactly the same Boolean interval. Thus, the models of the logic L_{CO} of *conditional objects* are given by:

$$Mod_{CO}(\Sigma) = \{[L, U] \mid [L, U] \text{ is a Bool. interval over } \Sigma\}.$$

The satisfaction relation for L_{CO} is defined by:

$$[L, U] \models_{CO} (B|A) \text{ iff } L \models AB \text{ and } U \models A \Rightarrow B, \quad (3)$$

which is equivalent to

$$[L, U] \models_{CO} (B|A) \text{ iff } L \models AB \text{ and } \overline{A}B \models \overline{U}. \quad (4)$$

It is easily shown that this provides a model theory for $\models_{DP}$, i.e. we have:

$$(B|A) \models_{DP} (D|C) \text{ iff } Mod_{CO}((B|A)) \subseteq Mod_{CO}((D|C)).$$

In order to establish the full characterization of system P inferences via conditional objects, so-called *quasi-conjunctions* of conditional objects have to be considered to allow for the study of inferences from a set of conditional objects. As in this paper, we focus mainly on the satisfaction relation between models and sentences and not on the full entailment relation between sets of sentences, we need not go into further details on quasi-conjunctions here.

3 Model Translations

The semantical structures discussed in the previous section have been proposed from different points of view and are quite diverse. We will now analyze whether and how they can be translated to each other while respecting the corresponding satisfaction relations. More precisely, for any two different conditional logics L_X and L_Y as defined above, we say that a function $\beta : Mod_X(\Sigma) \rightarrow Mod_Y(\Sigma)$ is a *model*

translation from $L_{\mathcal{X}}$ to $L_{\mathcal{Y}}$, denoted by $\beta : L_{\mathcal{X}} \rightarrow L_{\mathcal{Y}}$, iff for all $m \in \text{Mod}_{\mathcal{X}}(\Sigma)$ and all conditionals $(B|A)$ the following *satisfaction condition* holds:

$$m \models_{\mathcal{X}} (B|A) \text{ iff } \beta(m) \models_{\mathcal{Y}} (B|A) \quad (5)$$

As an additional point of reference for semantical structures that makes use of the full range of probability distributions we will also consider the logic L_C of probabilistic conditionals, using *quantitative* conditionals $(B|A)[x]$ with $x \in [0, 1]$ as sentences, standard probability distributions $P : \Omega_{\Sigma} \rightarrow [0, 1]$ as models, and the satisfaction relation given by $P \models_C (B|A)[x]$ iff $P(A) > 0$ and $P(B|A) = \frac{P(AB)}{P(A)} = x$. The logic L_C is one of the most expressive frameworks to study conditionals, including both classical implications (via conditionals with probability 1) and a full numerical scale to express any proportional relationship between 0 and 1. Correspondences between material implications and conditionals are not easily established in general, but the most obvious way for doing so is to associate the quantified conditional $(B|A)[1]$ and the unquantified conditional $(B|A)$, with one another. Therefore, in extending (5) to this case, we say that a function β mapping probability distributions over Ω_{Σ} to the models of one of the logics $L_{\mathcal{Y}}$ defined before, is a model translation $\beta : L_C \rightarrow L_{\mathcal{Y}}$ iff

$$m \models_C (B|A)[1] \text{ iff } \beta(m) \models_{\mathcal{Y}} (B|A) \quad (5')$$

for all $m \in \text{Mod}_C(\Sigma)$ and all conditionals $(B|A)$. (In fact, both (5) and (5') are simplified version of the satisfaction condition for institution morphisms (Goguen and Rosu 2002) that substantially generalize our notion of model translation.)

We first investigate the relationship of conditional objects to the purely qualitative logic L_K .

Proposition 1 *There is no model translation between L_{CO} and L_K in either direction.*

Proof. Assume that there were a model translation $\beta : L_{CO} \rightarrow L_K$. Then for each $[L, U] \in \text{Mod}_{CO}(\Sigma)$, $R_{[L, U]} := \beta([L, U])$ is a total pre-order on $\mathcal{L}$. Due to the satisfaction condition (5), for all sentences conditionals $(B|A)$ it must hold that $[L, U] \models_{CO, \Sigma} (B|A)$ iff $R_{[L, U]} \models_{K, \Sigma} (B|A)$, which is equivalent to $(L \models AB \text{ and } A\bar{B} \models \bar{U})$ iff $(AB \prec_{R_{[L, U]}} A\bar{B})$. W.l.o.g., we may assume that $|\Sigma| \geq 2$ and L is consistent and has at least two (propositional) models. Consider any two different models $\omega_1, \omega_2 \in \Omega_{\Sigma}$, and the conditional $(\omega_1 | \omega_1 \vee \omega_2)$. Since $(\omega_1 \vee \omega_2) \omega_1 \equiv \omega_1$ and $(\omega_1 \vee \omega_2) \bar{\omega}_1 \equiv \omega_2$, from (5) we would obtain $\omega_1 \prec_{R_{[L, U]}} \omega_2$ iff $(L \models \omega_1 \text{ and } \omega_2 \models \bar{U})$, where the right hand side is a contradiction to presupposing that L has at least two models since $L \models \omega_1$ implies that ω_1 is the only model of L . So, for all $\omega_1, \omega_2 \in \Omega_{\Sigma}$, we must have $\omega_1 \approx_{R_{[L, U]}} \omega_2$, hence $R_{[L, U]}$ does not satisfy any non-trivial conditional (i.e. having a precondition other than $\top$). However, we have $[L, U] \models_{CO} (L|U \Rightarrow L)$ which is in conflict to the satisfaction condition. A similar argumentation shows the other direction. $\square$

Using the extended notion of model translation as in (5'), it turns out that conditional objects are also isolated from probabilistic conditionals.

Proposition 2 *There is no model translation $\beta : L_C \rightarrow L_{CO}$.*

Proof. Assume there were such a model translation. Let $P \in \text{Mod}_C(\Sigma)$ be a probability distribution, and set $[L_P, U_P] := \beta(P) \in \text{Mod}_{CO}(\Sigma)$. Then, the satisfaction condition (5') says that, for all conditionals $(B|A)$ it must hold that $P \models_C (B|A)[1]$ iff $[L_P, U_P] \models_{CO} (B|A)$, which is equivalent to $(P(A) > 0 \text{ and } P(A\bar{B}) = 0)$ iff $(L_P \models AB \text{ and } A\bar{B} \models \bar{U}_P)$. Assume $|\Sigma| \geq 2$ and the existence of two different propositional models ω_1, ω_2 such that $P(\omega_1) > 0$ and $P(\omega_2) > 0$. Let $C := \bigvee_{\omega: P(\omega) > 0} \omega$. Then $P \models_C (\omega_1 | \bar{C} \vee \omega_1)[1]$ and $P \models_C (\omega_2 | \bar{C} \vee \omega_2)[1]$, but at least one of $[L_P, U_P] \models_{CO} (\omega_1 | \bar{C} \vee \omega_1)$ and $[L_P, U_P] \models_{CO} (\omega_2 | \bar{C} \vee \omega_2)$ must be false, since $L_P \models \omega_1$ and $L_P \models \omega_2$ can not hold simultaneously. $\square$

4 Variants of Conditional Objects

In the previous section, we have shown that L_{CO} is isolated from the logics L_K and L_C w.r.t. model translations. The main reason is that the semantics of L_{CO} is based on the entailment relation $\models$ between propositions which is a partial order, and from this, no total preorder on possible worlds can be derived. In particular, no information on the models of L can be derived from $[L, U]$. Moreover, although a Boolean interval $[L, U]$ seems to suggest a partitioning of sentences in three sets (“before” L , “between” L and U , “beyond” U), it does not specify anything about $\bar{L}U$. In general, the models $[L, U]$ together with the satisfaction relation (3) provide a too coarsely grained semantics for conditionals that can not comply with the main feature of qualitative logics based on plausibility, namely, that only some models of the premise (the most plausible ones) are able to establish a conditional relationship. Based on our findings, we propose two alternatives:

Modify the class of models: Instead of considering arbitrary Boolean intervals, restrict the class of models to *basic intervals* where the lower bound has exactly one model. This yields the logic L_{CO1} with

$$\text{Mod}_{CO1}(\Sigma) = \{[\omega, U] \mid \omega \in \Omega_{\Sigma}, U \in \mathcal{L}, \omega \models U\}.$$

Modify the satisfaction relation: Instead of requiring $L \models AB$ as in the satisfaction relation (3) for L_{CO} , relax this condition for consistent lower bounds L by requiring $LAB \not\models \perp$, yielding L_{CO2} with

$$[L, U] \models_{CO2} (B|A) \text{ iff } LAB \not\models \perp \text{ and } A\bar{B} \models \bar{U}.$$

Note that applying the modification of the satisfaction relation as in L_{CO2} also to L_{CO1} does not lead to any change on L_{CO1} since for any world ω , $\omega AB \not\models \perp$ is equivalent to $\omega \models AB$. For the induced entailment relations $\models_{CO1}$ and $\models_{CO2}$ it is straightforward to check that they are equivalent

to both $\models_{\mathcal{CO}}$ and $\models_{DP}$:

$$(B|A) \models_{\mathcal{CO}_1} (D|C) \text{ iff } \text{Mod}_{\mathcal{CO}_1}((B|A)) \subseteq \text{Mod}_{\mathcal{CO}_1}((D|C)) \quad (6)$$

$$(B|A) \models_{\mathcal{CO}_2} (D|C) \text{ iff } \text{Mod}_{\mathcal{CO}_2}((B|A)) \subseteq \text{Mod}_{\mathcal{CO}_2}((D|C)) \quad (7)$$

Proposition 3 *For any consistent $(B|A)$ (i.e. $AB \neq \perp$) we have: $(B|A) \models_{\mathcal{CO}_1} (D|C)$ iff $(B|A) \models_{\mathcal{CO}_2} (D|C)$ iff $(B|A) \models_{\mathcal{CO}} (D|C)$ iff $(B|A) \models_{DP} (D|C)$.*

Thus, neither the restriction to more atomic models ($\models_{\mathcal{CO}_1}$) nor the slight generalization in the satisfaction relation ($\models_{\mathcal{CO}_2}$) makes a difference at the level of the entailment relation for consistent conditionals. Thus, using quasi-conjunctions as in (Dubois and Prade 1994), also $L_{\mathcal{CO}_1}$ and $L_{\mathcal{CO}_2}$ yield inference relations modelling system P, where, in particular, $L_{\mathcal{CO}_1}$ has significantly simpler and fewer models than the original conditional objects (Benferhat, Dubois, and Prade 1997). For instance, for any $U \equiv \omega_1 \vee \dots \vee \omega_n$ with pairwise distinct $\omega_i \in \Omega$, $\text{Mod}_{\mathcal{CO}}(\Sigma)$ contains exactly 2^n different Boolean intervals $[L, U]$ with upper bound U , while in $\text{Mod}_{\mathcal{CO}_1}(\Sigma)$ there are only n different basic intervals $[\omega_i, U]$ with upper bound U .

W.r.t. the relationship to the logic L_K there is no change when moving from $L_{\mathcal{CO}}$ to $L_{\mathcal{CO}_1}$ or $L_{\mathcal{CO}_2}$.

Proposition 4 *For $i = 1, 2$, there are no model translations between $L_{\mathcal{CO}_i}$ and L_K in either direction.*

When relating the probabilistic models of L_C to conditional objects in $L_{\mathcal{CO}_1}$ and $L_{\mathcal{CO}_2}$, a difference to $L_{\mathcal{CO}}$ surfaces: The modified satisfaction relation enables a model translation from probabilistic conditional logic to the logic of conditional objects as given by $L_{\mathcal{CO}_2}$.

Proposition 5 (a) *There is no model translation $\beta : L_C \rightarrow L_{\mathcal{CO}_1}$. (b) $\beta : L_C \rightarrow L_{\mathcal{CO}_2}$ is a model translation iff $\beta(P) = [C_P, C_P]$ with $C_P = \bigvee_{\omega: P(\omega) > 0} \omega$ for all $P \in \text{Mod}_C(\Sigma)$.*

Proof. (a) follows as in Prop. 2. For (b), the satisfaction condition (5') holds because $[C_P, C_P] \models_{\mathcal{CO}_2} (B|A)$ iff $C_P AB \neq \perp$ and $A\bar{B} \models \bar{C}_P$ iff $\exists \omega \models AB$ such that $P(\omega) > 0$ and $P(A\bar{B}) = 0$ iff $P(AB) > 0$ and $P(A\bar{B}) = 0$ iff $P(A) > 0$ and $P(A\bar{B}) = 0$ iff $P(A) > 0$ and $P(B|A) = 1$ iff $P \models_C (B|A)[1]$. By exploiting (5') it can also be shown that β is uniquely determined. $\square$

The only model translations among the three logics having conditional objects as models are the two model injections from $L_{\mathcal{CO}_1}$ to $L_{\mathcal{CO}}$ and to $L_{\mathcal{CO}_2}$.

5 Conclusions and Further Work

In this paper, we focussed on conditional objects as syntactic representations of nonmonotonic inferences. As the entailment relation between conditional objects basically encodes crucial properties of uncertain reasoning, namely system P (Benferhat, Dubois, and Prade 1997), the semantics of conditional objects is of primary interest for uncertain reasoning in general. We studied relationships between conditional objects and a preferential semantics of system P, and also

to the full probabilistic framework, on the level of the models. These investigations emphasized the singular position of conditional objects, their semantics is substantially different from other approaches based on qualitative or probabilistic scales. We also presented two novel variants of model theories for conditional objects and could thereby succeed in linking conditional objects at least to the probabilistic semantics of conditionals.

In (Beierle and Kern-Isberner 2012) we extend our study of conditional objects in a more general setting. Our investigations make subtle and substantial differences between various semantical approaches to system P explicit, but also show similarities that may help to bridge gaps between different formalisms, with the aim of providing a more coherent picture of uncertain reasoning according to system P.

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