Intensional Combination of Rankings for OCF-Networks

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Abstract

Similar to Bayesian networks, so-called OCF-networks combine structural information encoded in a directed graph with qualitative information expressed by ranking degrees of (conditional) formulas. The benefits of such techniques are twofold: First, the high complexity of the semantical ranking functions approach is reduced substantially, and second, global ranks are obtained from local information. However, in many practical applications, even the local rankings are only available in parts, or not exactly in the format that is needed. In this paper, we apply inductive reasoning methods like system Z^+ or c-representations, to fill up missing values in the local conditional tables. This allows the user to specify knowledge for such OCF-networks in its most appropriate and reliable form and leave the technical details to an inference engine.

1 Introduction

Uncertain and defeasible reasoning is often crucially based on appropriate semantical frameworks like, e.g., probability theory that allow for a rich and meaningful representation of the problem domain under consideration while leaving enough semantical room for handling exceptions and non-monotonic phenomena. One of these frameworks is provided by the theory of *ordinal conditional functions (OCF)* (Spohn 2012), also called *ranking functions*, that assign a degree of disbelief to any possible world, Ranking functions have become increasingly popular within the last decade, as they are basically qualitative and more easily understandable than probabilities but share some nice features with probabilities. Most importantly, they provide proper interpretations for (meaningful, non-material) conditionals $(B|A)$ – *If A then plausibly B* encoding a plausible relationship between their antecedents A and consequents B.

However, the drawback of most semantical approaches is their high complexity as query answering and inference procedures have to take (basically) all models into account. To make local computations considering only a subset of all variables possible, graphical structures like Bayesian networks (Pearl 1989) have proved extremely useful. Usually, they come along with a causal interpretation considering the parents of a variable as its (common) causes. For OCFs, a similar type of networks has been proposed in (Goldszmidt

and Pearl 1996; Benferhat and Tabia 2010). In these approaches, analogous to Bayesian networks, OCFs are factorized according to the structure of a graph, and local ranking tables involving only a few nodes serve to construct a global ranking function. Still, as in Bayesian networks, the local ranking tables need full information of how plausible a literal is given all configurations of the parents of the respective variable. In many cases, only partial information is available here, typically, the user is only aware of the plausibility of a variable given each cause separately and cannot say much about cases when some configuration of causes is present. To fill up information, often external combination rules like naive Bayes (Castillo, Gutiérrez, and Hadi 1997) are applied which, however, do not take the semantical structure of the problem under consideration into account.

In this paper, we propose methods to combine partial ranking information in an intensional way in order to come up with full local ranking tables so that the OCF-network and hence the induced global ranking function can be completely specified. The basic idea is to apply inductive conditional reasoning mechanisms like system Z+ (Goldszmidt and Pearl 1991) and c-representations (Kern-Isberner 2004) locally to find appropriate (complete) rankings for the respective subgraph, i.e., a node and its parents, and extract from this semantical information the missing values in the local table of the child node. Similar approaches have been presented for Bayesian networks by making use of the principle of maximum entropy (Lukasiewicz 2000; Paris 2005; Schramm and Fronhöfer 2005). Indeed, the maximum entropy distribution is a probabilistic c-representation for the given knowledge base (Kern-Isberner 2004), and for the OCF framework, inferences based on c-representations have also proved to satisfy all major properties of nonmonotonic reasoning (Kern-Isberner 2001). Therefore, we employ high quality semantical methods to exploit the given partial information in an optimal way.

The rest of this paper is organized as follows: After short preliminaries in section 2 we recall ranking functions in section 3. Then, in section 4, we introduce the systems to be used for inductive conditional reasoning, namely System Z^+ and c-representations. In section 5 we elaborate on the concept of networks for ranking functions. We discuss why local ranks may not be available in the needed format for all vertices in the network and show how to solve this problem with the

presented approaches in section 6. Finally, we conclude in section 7.

2 Preliminaries

Let $\Sigma = \{V_1, \ldots, V_n\}$ be a set of propositional atoms with domains $dom(V_i) = \{v_i, \overline{v}_i\}$ representing V_i in its positive resp. negated form for every $V_i \in \Sigma$; for a specific outcome of V_i, we write $\dot{v}_i \in \{v_i, \overline{v}_i\}$. A *literal* is a positive or negative atom. The set of formulas $\mathfrak{L}$ over Σ joined with the symbols for tautology ($\top$) and contradiction ($\bot$), with the connectives $\wedge$ (*and*), $\vee$ (*or*) and $\neg$ (*not*) shall be defined in the usual way. For $A, B \in \mathfrak{L}$, we will usually omit the connective $\wedge$ and write AB instead of $A \wedge B$ as well as indicate negation by overlining, i.e., $\overline{A}$ means $\neg A$. The symbol $\Rightarrow$ is used for the material implication, i.e., $A \Rightarrow B$ is semantically equivalent to $\overline{A} \vee B$.

Interpretations, or *possible worlds*, are also defined in the usual way; the set of all possible worlds is denoted by Ω. We often use the 1-1 association between worlds and *complete conjunctions*, i.e., conjunctions of literals where every variable $V_i \in \Sigma$ appears exactly once.

A model ω of a propositional formula $A \in \mathfrak{L}$ is a possible world that satisfies A, written as $\omega \models A$. The set of all models of A is denoted by $Mod(A)$. A formula A is *consistent* if $Mod(A) \neq \varnothing$. For formulas $A, B \in \mathfrak{L}$, A entails B, written as $A \models B$, iff $Mod(A) \subseteq Mod(B)$, i.e., iff for all $\omega \in \Omega$, $\omega \models A$ implies $\omega \models B$. For sets of formulas $\mathcal{A} \subseteq \mathfrak{L}$ we have $Mod(\mathcal{A}) = \bigcap_{A \in \mathcal{A}} Mod(A)$.

A *conditional* $(B|A)$ with $A, B \in \mathfrak{L}$ encodes a defeasible rule "if A then *usually* B" with the trivalent evaluation $[\![(B|A)]\!]_\omega = true$ iff $\omega \models AB$ (verification), $[\![(B|A)]\!]_\omega = false$ iff $\omega \models A\overline{B}$ (falsification) and $[\![(B|A)]\!]_\omega = undefined$ iff $\omega \models \overline{A}$ (non-applicability) (De Finetti 1974; Kern-Isberner 2001). The language of all conditionals over $\mathfrak{L}$ is denoted by $(\mathfrak{L} | \mathfrak{L})$. Let $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\} \subseteq (\mathfrak{L} | \mathfrak{L})$ be a finite set of conditionals. A conditional $(B|A)$ is *tolerated* by Δ iff there is a world $\omega \in \Omega$ such that $\omega \models AB$ and $\omega \models A_i \Rightarrow B_i$ for every $1 \leq i \leq n$. Δ is *consistent* iff for every nonempty subset $\Delta' \subseteq \Delta$ there is a conditional $(B|A) \in \Delta'$ that is tolerated by Δ' (Goldszmidt and Pearl 1996). We will call such a consistent Δ a *knowledge base* and it shall represent the knowledge an agent uses as a base for reasoning. Note that the consistency of Δ implies that all propositional formulas occurring in Δ are consistent.

3 Ranking Functions (OCF)

An *ordinal conditional function* (OCF, (Spohn 2012)), also called *ranking function*, is a function $\kappa : \Omega \to \mathbb{N}_0^\infty$ with $\kappa^{-1}(0) \neq \varnothing$ which maps each world $\omega \in \Omega$ to a degree of implausibility $\kappa(\omega)$, i.e., if for two possible worlds $\omega, \omega' \in \Omega$ it holds that $\kappa(\omega) < \kappa(\omega')$ then ω' is believed to be less plausible than ω. Ranks of formulas $A \in \mathfrak{L}$ are calculated as $\kappa(A) = \min\{\kappa(\omega) \,|\, \omega \models A\}$. For conditionals $(B|A)$ a rank is defined via $\kappa(B|A) = \kappa(AB) - \kappa(A)$, and $\kappa \models (B|A)$ if and only if $\kappa(AB) < \kappa(A\overline{B})$, i.e., iff AB is more plausible than $A\overline{B}$. If $\kappa \models (B|A)$ we call κ a (ranking) model of $(B|A)$.

$\kappa(\omega)$	worlds		ω	$\kappa(\omega)$		ω	$\kappa(\omega)$
$\kappa(\omega) = 4$	$p\overline{b}f$		pbf	2		$\overline{p}bf$	0
$\kappa(\omega) = 2$	pbf $p\overline{b}\overline{f}$		$pb\overline{f}$	1		$\overline{p}b\overline{f}$	1
$\kappa(\omega) = 1$	$pb\overline{f}$ $\overline{p}b\overline{f}$		$p\overline{b}f$	4		$\overline{p}\overline{b}f$	0
$\kappa(\omega) = 0$	$\overline{p}bf$ $\overline{p}\overline{b}f$ $\overline{p}\overline{b}\overline{f}$		$p\overline{b}\overline{f}$	2		$\overline{p}\overline{b}\overline{f}$	0

Figure 1: OCF admissible to the penguin example given as worlds stacked by their plausibility and in tabular form.

The OCF approach is an "order of magnitude abstraction of probabilities" (Goldszmidt and Pearl 1996, §1) that provides full semantics for conditionals. It does not rely on extensional combination rules of values from a fixed scale of certainty degrees, like, e.g., (Osseiran 2001).

Definition 1 (Admissibility of ranking functions) *A ranking function κ is Δ-admissible iff $\kappa \models (B|A)$ for all $(B|A) \in \Delta$.*

Figure 1 is an example for an OCF which is admissible to the knowledge base $\Delta = \{(f|b), (\overline{f}|p), (b|p)\}$, the well-known *penguin-example* encoding the rules "*b*irds usually *f*ly", "*p*enguins usually do not *f*ly" and "*p*enguins usually are *b*irds".

Definition 2 (Firmness) *A formula A is believed in an OCF κ with firmness m, $m \in \mathbb{N}, m \geq 1$, (Spohn 2012), in symbols $\kappa \models A[m]$, iff $\kappa(\overline{A}) \geq m$, the same applies for conditionals: a conditional $(B|A)$ is believed with firmness m ($\kappa \models (B|A)[m]$), iff $\kappa(\overline{B}|A) \geq m$.*

So $(B|A)$ is believed in κ iff $\kappa(B|A) < \kappa(\overline{B}|A)$, hence if $\kappa(AB) < \kappa(A\overline{B})$ and, following the above definition, believed with *firmness m* if $\kappa(AB) + m \leq \kappa(A\overline{B})$.

Note that $\kappa \models A[m]$ iff $\kappa \models (A|\top)[m]$, so (plausible) formulas can be considered as a special case of conditionals. Hence, we will focus on conditional knowledge bases in this paper, keeping in mind that such knowledge bases may also contain plausible propositions. Moreover, we presuppose $m \geq 1$ in this paper since $\kappa \models (B|A)[m]$ should imply in particular $\kappa \models (B|A)$. Nevertheless, the case $m = 0$ is interesting but requires further considerations as we might have $\kappa(AB) = \kappa(A\overline{B})$, or $\kappa \models (\overline{B}|A)$. In order to keep the technical details as clear and simple as possible, we leave the case $m = 0$ for future work.

A ranking function is *admissible with respect to a knowledge base $\mathcal{R} = \{(B_1|A_1)[m_1], \ldots, (B_n|A_n)[m_n]\}$* of conditionals annotated with a firmness values ($\kappa \models \mathcal{R}$) iff $\kappa \models (B|A)[m]$ for every $(B_i|A_i)[m_i] \in \mathcal{R}$.

For the networks to be considered in this paper, we will need a notion of independence regarding ranking functions. We recall the results of (Spohn 2012) on this behalf.

Let **A**,**B**,**C** be disjoint sets of variables. **A** is κ-*independent* of **B** given **C**, written $\mathbf{A} \perp\!\!\!\perp_\kappa \mathbf{B} \mid \mathbf{C}$ iff $\kappa(\mathbf{ab}|\mathbf{c}) = \kappa(\mathbf{a}|\mathbf{c}) + \kappa(\mathbf{b}|\mathbf{c})$ – which is equivalet to postulating $\kappa(\mathbf{a}|\mathbf{bc}) = \kappa(\mathbf{a}|\mathbf{c})$ – for all complete conjunctions **a**,**b**,**c** built over **A**,**B**,**C**, resp. For notational convenience,

we will also write, e.g., $\kappa(\mathbf{A}|\mathbf{B})$ in equations holding for all complete conjunctions $\mathbf{a}, \mathbf{b}$ built over sets of variables $\mathbf{A}, \mathbf{B}$.

4 Inductive conditional reasoning

Let $\mathcal{R} = \{(B_1|A_1)[m_1], \ldots, (B_n|A_n)[m_n]\}$ be an annotated knowledge base. Taking all admissible ranking functions into account yields quite a weak inference from $\mathcal{R}$. A popular approach to obtain informative inferences from $\mathcal{R}$ is realised by selecting a "best" ranking model of $\mathcal{R}$ that can be used for further inferences.

In the following, we recall two approaches to obtain such a "best" ranking function for inductive model-based inference, namely System Z^+ and c-representations.

A well known approach to compute a ranking function given a firmness-annotated knowledge base $\mathcal{R} = \{(B_1|A_1)[m_1], \ldots, (B_n|A_n)[m_n]\}$ is System Z^+ (Goldszmidt and Pearl 1991). Different from System Z (Pearl 1990), no layer model of conditionals is calculated but the conditional's firmness is taken into account when computing a conditional's Z-value in an iterated process. We start by determining the set of conditionals Δ_0 which are tolerated by the whole knowledge base. According to section 2, Δ_0 consists of all conditionals $(B|A)[m] \in \Delta$ with the property that there is a world ω such that $\omega \models AB$ and $\omega \models (A_i \Rightarrow B_i)$ for each $(B_i|A_i)[m] \in \mathcal{R}$ These conditionals get a Z-value identically to their firmness, i.e., $Z(B_i|A_i) = m_i$ for all $(B_i|A_i) \in \Delta_0$. We set up a set $\mathcal{RZ}$ to $\mathcal{RZ} = \Delta_0$. In the iteration step we start with an index variable $j = 0$ set up a set of worlds Ω_j which solely falsify conditionals in $\mathcal{RZ}$ and verify at least one conditional outside of $\mathcal{RZ}$. These worlds get a temporally Z-value assigned which is

$$\kappa_Z^*(\omega) = \max_{(B_i|A_i) \in \mathcal{RZ}} \{Z(B_i|A_i) | \omega \models A_i \overline{B}_i\} + 1.$$

From Ω_j we take a world ω^* with the smallest κ_Z^*-value, that is, a world ω^* such that ω^* is a world with $\kappa_Z^*(\omega^*) = \min_{\omega \in \Omega_j} \{\kappa_Z^*(\omega)\}$. Conditionals $(B_i|A_i) \notin \mathcal{RZ}$ which are verified by the world ω^* are given a Z-value of $Z(B_i|A_i) = \kappa_z^*(\omega^*) + m_i$ and added to $\mathcal{RZ}$, j is incremented and the iteration starts again until $\mathcal{RZ} = \Delta$.

If the original knowledge base $\mathcal{R}$ is consistent we will find a world ω with $\omega \models AB$ for every conditional $(B|A)[m] \in \mathcal{R}$. This world either does not falsify any conditional $(B_i|A_i)[m] \in \mathcal{R}$, then $(B|A)$ is an element of Δ_0, or there is a conditional $(D|C)[n] \in \mathcal{R}$ with $\omega \models C\overline{D}$, but then, ω is chosen as one of the worlds Ω_j at a time after $(D|C)$ was added to $\mathcal{RZ}$.

By this we get an associated Z-value $Z(B_i|A_i)$ for all the conditionals in $\mathcal{R}$ and from these values we obtain a ranking function κ_Z which is defined as

$$\kappa_Z(\omega) = \begin{cases} 0 \text{ iff } \omega \text{ does not falsify any } (B_i|A_i) \\ \max_{\omega \models (A_i \overline{B}_i)} \{Z(B_i|A_i)\} \text{ otherwise.} \end{cases} \quad (1)$$

Note that, differently from the original approach (Goldszmidt and Pearl 1991), for $\kappa \models (B|A)[m]$ we require $\kappa(AB) + m \leq \kappa(A\overline{B})$, but presuppose $m > 0$ and

Table 1: Ranking function κ_Δ^c obtained with minimal c-representation and (without explicit demonstration of the algorithm) SystemZ$^+$ for the penguin-example.

ω	pbf	$pb\overline{f}$	$p\overline{b}f$	$p\overline{b}\overline{f}$	$\overline{p}bf$	$\overline{p}b\overline{f}$	$\overline{p}\overline{b}f$	$\overline{p}\overline{b}\overline{f}$
$\kappa_\Delta^c(\omega)$	3	1	6	3	0	1	0	0
$\kappa_Z(\omega)$	4	1	7	7	0	1	0	0

therefore set $\kappa_Z(\omega) = \max_{\omega \models (A_i \overline{B}_i)} \{Z(B_i|A_i)\}$ instead of $\kappa_Z(\omega) = \max_{\omega \models (A_i \overline{B}_i)} \{Z(B_i|A_i)\} + 1$.

The framework of c-representations (Kern-Isberner 2001) generates ranking functions κ_Δ^c for knowledge bases $\mathcal{R}$ that are $\mathcal{R}$-admissible and are based on the conditionals in the knowledge base and their structure, solely. In this section, we will recall this approach briefly.

Definition 3 (c-Representation)

A c-representation (Kern-Isberner 2001) of a knowledge base $\mathcal{R} = \{(B_1|A_1)[m_1], \ldots, (B_n|A_n)[m_n]\}$ is defined as an OCF of the form

$$\kappa_\mathcal{R}^c(\omega) = \sum_{i=1, \, \omega \models A_i \overline{B}_i}^{n} \kappa_i^-, \qquad \kappa_i^- \in \mathbb{N}_0$$

where the values κ_i^- are penalty points for falsifying conditionals and have to be chosen to make $\kappa_\mathcal{R}^c$ $\mathcal{R}$-admissible, i.e. for all $1 \leq i \leq n$ it holds that $\kappa_\mathcal{R}^c \models (B_i|A_i)[m_i]$ which is the case iff (Kern-Isberner 2004), (cf. definition 2):

$$\kappa_i^- \geq m_i + \min_{\omega \models A_i B_i} \left\{ \sum_{\substack{i \neq j \\ \omega \models A_j \overline{B}_j}} \kappa_j^- \right\} - \min_{\omega \models A_i \overline{B}_i} \left\{ \sum_{\substack{i \neq j \\ \omega \models A_j \overline{B}_j}} \kappa_j^- \right\} \quad (2)$$

A minimal c-representation is obtained by choosing κ_i^- minimally for all $1 \leq i \leq n$. Note that there may be several different (minimal) c-representations for a knowledge base.

Example 1 (c-represented penguins) *We use the penguin example to illustrate how this framework works, so let $\mathcal{R} = \{(f|b)[1], (\overline{f}|p)[2], (b|p)[2]\}$. For the κ_i^- values of a c-representation we get, according to inequality (2),*

$$\kappa_1^- \geq 1 + \min\{\kappa_2^-, 0\} - \min\{0\} = 1$$
$$\kappa_2^- \geq 2 + \min\{\kappa_1^-, \kappa_3^-\} - \min\{0\} = 2 + \min\{\kappa_1^-, \kappa_3^-\}$$
$$\kappa_3^- \geq 2 + \min\{\kappa_1^-, \kappa_2^-\} - \min\{0\} = 2 + \min\{\kappa_1^-, \kappa_2^-\}.$$

This leads to a minimal c-representation for Δ with $\kappa_1^- = 1$, $\kappa_2^- = \kappa_3^- = 3$ and the resulting ranking function κ_Δ^c shown in table 1.

5 OCF Networks

In this section, we elaborate on the concept of networks for OCFs. First approaches that make crucial use of the idea of causality have been presented in (Goldszmidt and Pearl 1996; Benferhat and Tabia 2010). However, like in Bayesian networks, causal interpretations are not mandatory for such

networks although they support appropriate modellings of the problem domain. More importantly, it is the idea of conditional independence that provides the basis for factorizing OCFs, i.e., for local representations of global ranking functions. So, we prefer to develop the approach of OCF networks in full analogy to Bayesian networks (as far as possible), making assumptions underlying the works (Goldszmidt and Pearl 1996; Benferhat and Tabia 2010) explicit.

Let $\Gamma = \langle \mathcal{V}, \mathcal{E} \rangle$ be a directed, acyclic graph (DAG) with a set of vertices $\mathcal{V} = \{V_1, \ldots, V_n\}$ and a set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. We define the *parents* of a vertex V, $pa(V)$, as the direct predecessors of V (i.e., $pa(V) = \{V'|(V',V) \in \mathcal{E}\}$) and the *descendants* of V, $desc(V)$, as the set of vertices V' for which a path from V to V' exists in $\mathcal{E}$. The set of *non-descendants* of V is $nd(V) = \mathcal{V} \setminus (desc(V) \cup \{V\} \cup pa(V))$.

To connect a DAG with ranking information we define an OCF-Network as follows:

Definition 4 (OCF-Network) *A directed, acyclic graph* $\Gamma = \langle \Sigma, \mathcal{E}, \{\kappa_V\}_{V \in \Sigma} \rangle$ *over a set of propositional atoms* Σ *is an* OCF-network *if each vertex* $V \in \Sigma$ *is annotated with a table of local rankings* $\kappa_V(V|pa(V))$ *with (local) ranking values specified for every configuration of* V *and* $pa(V)$, *such that* $\min_{\dot{v}}\{\kappa(\dot{v}|pa(V))\} = 0$ *for every configuration of* $pa(V)$. *The local rankings must be* normalized, *i.e.,*

$$\min_{\omega} \sum_{V \in \Sigma} \kappa_V(V(\omega)|pa(V)(\omega)) = 0, \qquad (3)$$

where $V(\omega)$ *resp.* $pa(V)(\omega)$ *indicates the outcome* $v \in dom(V)$ *with* $\omega \models v$ *resp. the configuration* p *of the variables in* $pa(V)$ *with* $\omega \models p$.

The local ranking information in Γ can be used to define a global ranking function κ over Σ by applying the idea of stratification (Goldszmidt and Pearl 1996): A ranking function κ is *stratified* relative to an OCF-network Γ iff

$$\kappa(\omega) = \sum_{V \in \Sigma} \kappa_V(V(\omega)|pa(V)(\omega)), \qquad (4)$$

for every world ω. With this stratifiction, given the tables of local rankings, we can generate a stratified OCF by formula (4). Condition (3) ensures that κ is indeed an OCF.

Example 2 *As an illustration we use the penguin example already presented in example 1 with a graph set up according to (Goldszmidt and Pearl 1996) and local conditional ranking values calculated as conditional ranks from the ranking function given in example 1 shown in figure 2, i.e,* $\kappa_P(B|P) = \kappa(B|P)$, $\kappa_F(F|BP) = \kappa(F|BP)$ *and* $\kappa_P(P) = \kappa(P)$.

Conversely, given a DAG Γ with vertices Σ and an OCF κ over Σ such that each vertex $V \in \Sigma$ is κ-independent of its non-descendants given its parents, we obtain a stratification of κ relative to Γ. This is stated in the following proposition; the proof is straightforward.
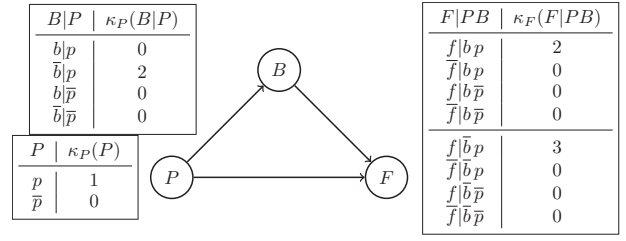
$B\|P$	$\kappa_P(B\|P)$
$b\|p$	0
$\bar{b}\|p$	2
$b\|\bar{p}$	0
$\bar{b}\|\bar{p}$	0

P	$\kappa_P(P)$
p	1
$\bar{p}$	0

$F\|PB$	$\kappa_F(F\|PB)$
$f\|b\,p$	2
$\bar{f}\|b\,p$	0
$f\|b\,\bar{p}$	0
$\bar{f}\|b\,\bar{p}$	0
$f\|\bar{b}\,p$	3
$\bar{f}\|\bar{b}\,p$	0
$f\|\bar{b}\,\bar{p}$	0
$\bar{f}\|\bar{b}\,\bar{p}$	0

Figure 2: Network of the penguin-example.

Proposition 1 *Let* Σ *be a propositional alphabet and* $\Gamma = \langle \Sigma, \mathcal{E} \rangle$ *be a DAG. Let* $\Sigma = \{V_1, \ldots, V_n\}$ *be enumerated such that for each* $V_i \in \Sigma$ *we have* $pa(V_i) \subseteq \{V_1, \ldots, V_{i-1}\}$. *Let* κ *be an OCF over* Σ *such that* $V \perp\!\!\!\perp_\kappa nd(V) \mid pa(V)$ *for all* $V \in \Sigma$. *Then it holds that*

$$\kappa(V_1, \ldots, V_n) = \sum_{i=1}^{n} \kappa(V_i|pa(V_i)). \qquad (5)$$

Hence, a ranking function that implements the conditional independence assumptions of a network Γ can be stratified relative to Γ.

Note that OCF-networks differ from Ceteris-Paribus-Networks (Boutilier et al. 2004), since there is no notion of stratification in CP-networks.

6 Intensional Combination

OCF networks and stratifications are most valuable concepts for practical applications of the ranking framework as they help to cut down the complexity of full semantical information. However, one often has also to struggle with the problem of incomplete information, i.e., only some (conditional) relationships between variables can be expressed with sufficient reliability. Typically, experts are quite certain about stating relationships between variables and each of its causes, or between variables and special configurations of its parents. In these cases we first have to fill in missing values in the local ranking tables by somehow exploiting the partial explicit information before being able to apply the OCF networks approach.

Hence we aim at calculating the local table of ranking values $\kappa_V(V|pa(V))$ exploiting the available knowledge as well as possible and use inductive inference mechanisms like c-representations and System Z$^+$ on local knowledge. From these local ranking functions, we can easily read the missing tabular values for V and fill up the complete local tables. In this way, given values are combined in an intensional way, i.e., based on local semantical information, as opposed to using extensional combination functions like min or noisy-or.

More precisely, the procedure for filling in missing values in the ranking tables is as follows:

Let a DAG $\Gamma = \langle \Sigma, \mathcal{E} \rangle$ over Σ be given, and for each $V \in \Sigma$, let $\mathcal{R}_V$ be a local conditional knowledge base containing statements of the form $(\dot{v}|A)[m]$ where A is a formula involving only the parents of V. For example, $\mathcal{R}$ might have the form $\mathcal{R}_V = \{(\dot{v}|\dot{v}_i)[m_{\dot{v}_i}]|V_i \in pa(V)\}$.

In cases where $\mathcal{R}_V$ is not a complete conditional ranking table, do the following:

1. Consider $\mathcal{R}_V$ as a knowledge base over $\Sigma' = \{V\} \cup pa(V)$.

2. Compute an OCF κ_V over Σ' from $\mathcal{R}_V$ by using an inductive conditional reasoning method, like system Z+ or c-representations (cf. section 4).

3. Compute from κ_V complete ranking tables $\kappa_V(V|pa(V))$ for every configuration of V and $pa(V)$.

If System Z^+ or c-representations are used to complete local tables, then it can be shown that $\langle \Sigma, \mathcal{E}, \{\kappa_V\}_{V \in \Sigma} \rangle$ is an OCF-network.

Example 3 *As an illustration, we modify an example from (Goldszmidt and Pearl 1996; Benferhat and Tabia 2010): A car starts $(S = s)$ if the battery is charged $(B = b)$ and the fuel tank is full $(F = f)$. If either the battery is discharged $(B = \bar{b})$ or the fuel tank is empty $(F = \bar{f})$, the car does not start $(S = \bar{s})$; additionally, if, for some reason, the headlights have been left on overnight $(H = h)$, the battery is discharged. We assume to know that it is very implausible to have left the headlights on $(\kappa_H(h) = 15)$ and usually the tank is not empty $(\kappa_F(\bar{f}) = 10)$. We also know that if the headlights have been left on, the battery is plausibly discharged $(\kappa_B(b|h) = 4)$ but if the headlights have been switched off, the battery usually is charged $(\kappa_B(\bar{b}|\bar{h}) = 8)$. Unfortunately, we do not know the ranking values at vertex S. However, we know it is very implausible for a car with an empty battery to start, i.e. $\kappa_S(s|\bar{b}) = 12$, and even less plausible that a car without any fuel will start, i.e. $\kappa_S(s|\bar{f}) = 15$, so the knowledge base of our concern is $\mathcal{R}_S = \{r_1 = (\bar{s}|\bar{b})[12], r_2 = (\bar{s}|\bar{f})[15]\}$. The OCF network to this situation is shown in figure 3. In this situation, we search for a local ranking function on S, B, F from which we can obtain the missing ranks of vertex S given all its parents. This can be achieved by using inductive conditional reasoning, i.e., by applying the methods presented in sec. 4.*

First, we apply System Z^+. We compute the partition Δ_0 of tolerated conditionals for the approach of System Z^+ and we find that all conditionals belong to Δ_0. Therefore we can assign to each conditional the Z-value of its firmness in $\mathcal{R}$ and get $Z(r_1) = 12$ and $Z(r_2) = 15$. We then set up a table indicating verification/falsification of the conditionals in $\mathcal{R}_S$ for each configuration of the local variables B, F and S, and associate with them the ranks according to equation (1). So we obtain the local OCF $\kappa_S^Z(BFS)$ shown in table 2.

With the verification/falsification rows of tablee 2 we can set up the inequations needed to calculate the c-representation of the knowledge base according to equation (2). Here we obtain

$$\kappa_1^- \geq 12 + \min\{0, \kappa_2^-\} - \min\{0, \kappa_2^-\} = 12$$
$$\kappa_2^- \geq 15 + \min\{0, \kappa_1^-\} - \min\{0, \kappa_1^-\} = 15$$

and in this way the minimal c-representation $\kappa_S^c(BFS)$ given in the rightmost column of table 2.

We notice that here by using the minimal c-representation we are able to distinguish between the configurations $b\,\bar{f}\,s$
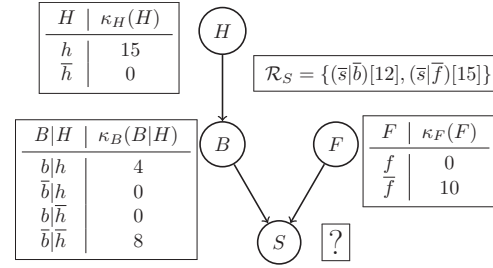

Figure 3: Problem description of the car example: local ranking values $\kappa_S(S|BF)$ for S are unknown but partial information on the rankings of S is given according to $\mathcal{R}_S$.

Table 2: Verification/falsification behaviour of configurations given the local car start knowledge base $\mathcal{R}_S$ and local ranks calculated using System Z^+ (κ_S^Z) and minimal c-representation (κ_S^c).

BFS	verifies	falsifies	$\kappa_S^Z(BFS)$	$\kappa_S^c(BFS)$
$b\,f\,s$	—	—	0	0
$b\,f\,\bar{s}$	—	—	0	0
$b\,\bar{f}\,s$	—	r_2	15	15
$b\,\bar{f}\,\bar{s}$	r_2	—	0	0
$\bar{b}\,f\,s$	—	r_1	12	12
$\bar{b}\,f\,\bar{s}$	r_1	—	0	0
$\bar{b}\,\bar{f}\,s$	—	r_1, r_2	15	27
$\bar{b}\,\bar{f}\,\bar{s}$	r_1, r_2	—	0	0

and $\bar{b}\,\bar{f}\,s$ whereas this is not the case if we use the values calculated with System Z^+.

For these values we set up the local conditional ranking values for $S|BF$ by calculating $\kappa_S^Z(\dot{s}|\dot{b}\dot{f}) = \kappa_S^Z(\dot{s}\dot{b}\dot{f}) - \kappa_S^Z(\dot{b}\dot{f})$ for the System Z^+ approach and $\kappa_S^c(\dot{s}|\dot{b}\dot{f}) = \kappa_S^c(\dot{s}\dot{b}\dot{f}) - \kappa_S^c(\dot{b}\dot{f})$ using c-representations, respectively, which we list in figure 4. Note that in this example we have coincidentally $\kappa_S^Z(\dot{b}\dot{f}) = 0$ and $\kappa_S^c(\dot{b}\dot{f}) = 0$. With these local ranking tables we can construct the two (global) ranking functions κ_Z (with System Z^+) and κ_c (with c-representations) by means of stratification given in table 3.

In general, c-representations process conditional dependencies more accurately, as can be seen from table 2 when comparing the respective values for $\bar{b}\,\bar{f}\,s$. This leads to locally establishing the conditionals $(b|\bar{f}s)$ for the c-representations but not for System Z^+. In the global picture, the difference becomes even more apparent when considering the conditionals $(\dot{b}|h\bar{f}s)$ – if the fuel tank is empty and the headlights have been left on but the car starts, is it plausible that the battery is charged? Obviously yes, but only the c-representation approves this, i.e., $\kappa_c \models (b|h\bar{f}s)$ while System Z^+ plainly rejects this by $\kappa_Z \models (\bar{b}|h\bar{f}s)$.

In the following, as an add-on, we will verify whether the conditional ranks specified in the local ranking tables also hold for the global functions κ_Z and κ_c. We do this exemplarily for the "expert knowledge" $\mathcal{R}_S$ used for the
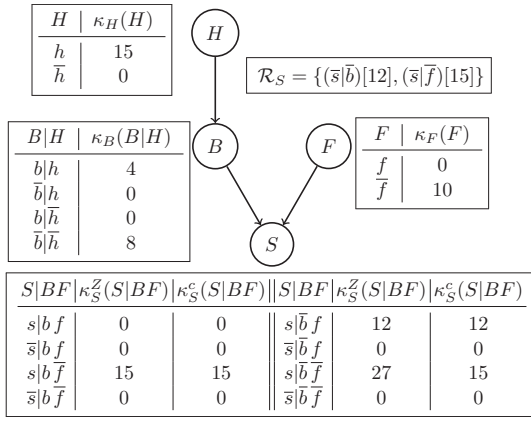
Figure 4: Solution for the car starting example with local OCF at S calculated with c-representations and System Z^+.

Table 3: Ranking functions κ_Z based on System Z^+ and κ_c based on c-representations obtained through stratification for the car-starting-example.

ω	$h\,b\,f\,s$	$h\,b\,f\,\overline{s}$	$h\,b\,\overline{f}\,s$	$h\,b\,\overline{f}\,\overline{s}$	$h\,\overline{b}\,f\,s$	$h\,\overline{b}\,f\,\overline{s}$	$h\,\overline{b}\,\overline{f}\,s$	$h\,\overline{b}\,\overline{f}\,\overline{s}$
$\kappa_Z(\omega)$	19	19	44	29	27	15	40	25
$\kappa_c(\omega)$	19	19	44	29	27	15	52	25

ω	$\overline{h}\,b\,f\,s$	$\overline{h}\,b\,f\,\overline{s}$	$\overline{h}\,b\,\overline{f}\,s$	$\overline{h}\,b\,\overline{f}\,\overline{s}$	$\overline{h}\,\overline{b}\,f\,s$	$\overline{h}\,\overline{b}\,f\,\overline{s}$	$\overline{h}\,\overline{b}\,\overline{f}\,s$	$\overline{h}\,\overline{b}\,\overline{f}\,\overline{s}$
$\kappa_Z(\omega)$	0	0	25	10	20	8	33	18
$\kappa_c(\omega)$	0	0	25	10	20	8	45	18

generation of the local ranks.

$$\kappa_Z(s|\overline{b}) = \kappa(\overline{b}s) - \kappa(\overline{b}) = 12 - 0 = 12 \geq 12$$
$$\kappa_Z(s|\overline{f}) = \kappa(\overline{f}s) - \kappa(\overline{f}) = 15 - 0 = 15 \geq 15$$
$$\kappa_c(s|\overline{b}) = \kappa(\overline{b}s) - \kappa(\overline{b}) = 12 - 0 = 12 \geq 12$$
$$\kappa_c(s|\overline{f}) = \kappa(\overline{f}s) - \kappa(\overline{f}) = 15 - 0 = 15 \geq 15$$

So in this example, global conditional rankings coincide with the locally specified ranks.

7 Conclusion

In this paper, we showed that for OCF networks with missing local ranking values inductive reasoning approaches like System Z^+ as well as c-representations are capable of generating complete local ranking tables. This helps us accomplishing the goal to allow the user of an OCF network based system to specify her knowledge in an appropriate way and still rely on network techniques, leaving the technical details regarding local tables to the mentioned inference mechanisms. As part of our ongoing work, we explore these ideas for efficient implementations of OCF based knowledge representation.

References

Benferhat, S., and Tabia, K. 2010. Belief change in ocf-based networks in presence of sequences of observations and interventions: Application to alert correlation. In Zhang, B.-T., and Orgun, M., eds., *PRICAI 2010: Trends in Artificial Intelligence*, volume 6230 of *LNCS*. Springer. 14–26.

Boutilier, C.; Brafman, R. I.; Domshlak, C.; Hoos, H. H.; and Poole, D. 2004. CP-nets: A Tool for Representing and Reasoning with Conditional Ceteris Paribus Preference Statements. *J. of Artif. Intell. Research* Volume 21:135–191.

Castillo, E.; Gutiérrez, J.; and Hadi, A. 1997. *Expert Systems and Probabilistic Network Models*. Monographs in Computer Science Series. Springer Verlag.

De Finetti, B. 1974. *Theory of Probability*, volume 1,2. New York, USA: John Wiley and Sons.

Goldszmidt, M., and Pearl, J. 1991. System-Z+: a formalism for reasoning with variable-strength defaults. In *Proc. 9th Nat. conf. on AI*, AAAI'91, 399–404. AAAI Press.

Goldszmidt, M., and Pearl, J. 1996. Qualitative probabilities for default reasoning, belief revision, and causal modeling. *Artif. Intell.* 84(1-2):57–112.

Kern-Isberner, G. 2001. *Conditionals in Nonmonotonic Reasoning and Belief Revision – Considering Conditionals as Agents*. Number 2087 in LNCS. Springer.

Kern-Isberner, G. 2004. A thorough axiomatization of a principle of conditional preservation in belief revision. *AMAI* 40:127–164. 10.1023/A:1026110129951.

Lukasiewicz, T. 2000. Credal networks under maximum entropy. In *Proc. 16th conf. on Uncert. in AI*, UAI'00, 363–370. San Francisco, CA, USA: Morgan Kaufmann.

Osseiran, A. C. 2001. Qualitative bayesian networks. *Inf. Sci.* 131(1-4):87–106.

Paris, J. B. 2005. On filling-in missing conditional probabilities in causal networks. *INT J UNCERTAIN FUZZ* 13(3):263–280.

Pearl, J. 1989. *Probabilistic reasoning in intelligent systems – networks of plausible inference*. Morgan Kaufmann series in representation and reasoning. Morgan Kaufmann.

Pearl, J. 1990. System Z: A natural ordering of defaults with tractable applications to nonmonotonic reasoning. In *Proc. 3rd conf. on Theor. asp. of reasoning about knowledge*, TARK '90, 121–135. San Francisco, CA, USA: Morgan Kaufmann Publishers Inc.

Schramm, M., and Fronhöfer, B. 2005. Completing incomplete bayesian networks. In Kern-Isberner, G.; Rödder, W.; and Kulmann, F., eds., *Conditionals, Information, and Inference*, volume 3301 of *LNCS*. Springer. 200–218.

Spohn, W. 2012. *The Laws of Belief: Ranking Theory and Its Philosophical Applications*. Oxford University Press.