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Sensitivity Analysis in Portfolio Interval Decision Analysis

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Abstract

Techniques enabling decision makers to identify a set of non-mutually exclusive projects (or alternatives) constituting a portfolio, while allowing for imprecise information with respect to projects' benefits, costs, and overall resource constraints, have emerged as an area of great applicability. To reach applicability, reasonable and computationally meaningful decision evaluation methods are needed. In this paper, we propose an embedded form of sensitivity analysis for portfolio interval decision analysis building upon the concept of interval contraction. Both *a priori* sensitivity analysis and *a posteriori* sensitivity analysis for portfolio interval decision analysis are supported by the approach.

Introduction

The evaluation of portfolio decision problems is an important feature of contemporary decision analysis frameworks since in many real-life decision situations, the available decision alternatives are not mutually exclusive. Portfolio decision analysis (PDA) then assists the decision maker in the selection of a portfolio of projects (or decision alternatives) while considering the individual projects' performance/benefits, and where the selection of projects to consider is constrained by some resource constraint such as a given budget. In many decision analysis applications, the benefits are derived from some aggregation of the project's performance on a set of evaluation criteria, i.e. it is a multi-criteria decision analysis problem. Still, imprecision in decision data is just as prominent for portfolio problems as for traditional decision analysis applications while the decision theoretic principles of project discrimination do remain, cf., e.g., (Salo, Keisler, and Morton 2011).

A few different approaches to the evaluation of portfolio decision analysis problems have been suggested recently. The PROBE method presented in (Lourenço, Morton, and Bana e Costa 2012) uses an optimization approach and handles incomplete information with respect to project costs, project utilities under each criterion, and criteria weights. Although PROBE supports imprecise statements, it models and solves a portfolio problem using crisp numbers derived from the statements and conducts a portfolio robustness evaluation based upon the imprecision provided. The Robust Portfolio Modeling (RPM) method presented in (Liesiö, Mild, and Salo 2007; 2008) also uses an optimization approach but consider imprecision with respect to costs, utilities, and weights in the portfolio generation.

Lourenço, Morton, and Bana e Costa (2012) provide a useful categorization of sensitivity analysis approaches in decision analysis. They distinguish between *a priori* and *a posteriori* approaches, where an *a priori* sensitivity analysis should account for the imprecision prior to portfolio generation, and an *a posteriori* sensitivity analysis is conducted once the portfolio is generated, typically by considering the impact of the difference between extreme values of the input statements. Further, Lourenço, Bana e Costa, and Morton (2008) claim that no portfolio selection software conducts an *a posteriori* sensitivity analysis that takes into account imprecision with respect to more than one input type at a time. Given this categorization, RPM uses an *a priori* approach, while PROBE uses an *a posteriori* approach (Lourenço, Morton, and Bana e Costa 2012).

Furthermore, Kleinmuntz (2007) states that sensitivity analysis of portfolios can be conducted by "forcing" a project to be included in the portfolio, so that if the portfolio has a fixed resource constraint, this will result in the exclusion of one or more of the portfolio projects. The projects that are excluded in order to include a less preferable project are thereby identified. Another approach is to vary the resource constraint, having the underlying assumption that the resource constraint often may change in real-life situations.

Another recent approach to PDA is the PDA extension of the DELTA method (Fasth and Larsson 2012). The DELTA framework for interval decision analysis enables the decision maker to enter numerically imprecise information in the modeling and analysis of a decision problem. Decision evaluation is carried out by means of investigating minimum and maximum differences in expected utility between alternatives. A central concept is that an alternative A_i δ-dominates alternative A_j if the minimum difference in expected utility between the two alternatives is greater than zero. However, such dominance may not exist, yielding a partial order of the alternatives which could be considered as insufficient for decision purposes. In such cases, the alternatives can be further evaluated in an embedded sensitivity analysis by contracting the input intervals until δ-dominance is reached. The level of contraction needed in order to reach dominance is then viewed as a measure of robustness, see e.g., (Danielson and Ekenberg 1998; Danielson 2009).

The PDA extension of the DELTA method has an approach similar to that of PROBE. The extension enables the possibility to evaluate the so-called "strength" between two portfolios, i.e. obtaining the minimum and maximum difference in expected utility between Pareto optimal portfolios (Fasth and Larsson 2012). However, this PDA extension does not per se support sensitivity analysis of portfolios, which is the concern in this paper. This paper thus presents how the DELTA framework and its embedded sensitivity analysis based upon the concept of interval contraction can be applied to portfolio decision analysis. The concept of contraction is employed while investigating the extreme values with respect to projects' expected utilities and costs constitute the basis for the *a priori* sensitivity analysis, as it is part of the portfolio generation. The concept of contractions is later also used *a posteriori* to the portfolio generation, in order to evaluate how the imprecision with respect to costs and utilities of so called borderline projects affects the choice of a portfolio.

Interval Decision Analysis

In interval decision analysis, numerically imprecise information is modeled by means of interval statements (range constraints) complemented with comparative statements (Danielson 2009). Conforming to the DELTA framework, we consider three different types of variables; probability variables, utility variables, and weight variables. These variables are subject to linear constraints collected in three different constraint sets; a probability base **P**, a utility base **U**, and a weight base **W**. A constraint in a base is either a range constraint or a comparative statement. A comparative statement is of the form *"consequence A has a greater probability of occurring than consequence B"*, which corresponds to the inequality $p(A) \geq p(B)$. An interval statement simply states that the variable (probability, weight or utility) is bounded from below and above. For example, *"consequence A has a probability of occurring that lies between probability a and b"*, which corresponds to the two inequalities $p(A) \geq a$, and $p(A) \leq b$. In addition to the bases (or constraint sets), a focal point for each variable is stipulated. The focal point for a variable x_i is a point consistent with the constraints affecting x_i, i.e. it lies within the solution set, and it represents the most reliable point within this set. The focal point for a variable x_i is either given by the decision maker or suggested as the center of mass point of the polytope spanned by the constraints, projected to x_i. Thus, there is an underlying assumption that there is less belief in points close to the outer endpoints of the intervals than in points closer to center of mass. This assumption is supported by investigations in the effects of assuming a second-order distribution over the intervals, see, e.g., (Sundgren, Danielson, and Ekenberg 2009). The probability (and weight) variables for a set of consequences or criteria must sum up to 1, i.e., $\sum_k p_{ik} = 1$, there are no such requirements for utility values (Danielson et al. 2007).

The information collected in the bases is captured in an information frame which constitue the formal representation of a decision problem, $\langle\{A_1, \ldots, A_m\}, \mathbf{P}, \mathbf{U}, \mathbf{W}\rangle$, where $A_i = \{C_{i1}, \ldots, C_{ih_i}\}$ is the set of consequences belonging to an alternative A_i. In the rest of the paper, p_{ik} denotes the probability of consequence k given alternative A_i, and v_{ik_l} the utility of that consequence under the l:th criterion. The expected utility of alternative A_i under criterion l is calculated according to (1), the global utility of A_i given all criteria is then given from (2) and the difference between the expected utility of two alternatives according to (3) (Larsson et al. 2005).

$$E_l(A_i) = \sum_k p_{ik} v_{ik_l} \tag{1}$$

$$V(A_i) = \sum_l w_l \cdot E_l(A_i) \tag{2}$$

$$V(A_i) - V(A_j) = \sum_l w_l \cdot \big(E_l(A_i) - E_l(A_j)\big) \tag{3}$$

Since the utilities and weights are interval-valued and we are studying δ-dominance, we are interested in obtaining $\max(V(A_i) - V(A_j))$ and $\min(V(A_i) - V(A_j))$, see (Larsson et al. 2005) for how this is treated within the framework. If there is no δ-dominance, the decision problem is further investigated by contracting the intervals towards the more representative focal point values. Let **X** denote a base which includes the variables $x_i, \ldots, x_n$ having focal points $\overline{k} = (k_1, \ldots, k_n)$. Let $x_i \in [a_i, b_i]$, $\pi \in [0, 1]$ be a real number, and let $\{\pi_i \in [0, 1] : i = 1, \ldots, n\}$ be a set of real numbers. The π-contraction of **X** is then conducted by including the statements in (4) in **X**. Once this is done, we may investigate whether $\max(V(A_i) - V(A_j))$ and $\min(V(A_i) - V(A_j))$ and whether dominance hold or not (Danielson 2009).

$$\{x_i \in [a_i + \pi \cdot \pi_i \cdot (k_i - a_i), b_i - \pi \cdot \pi_i \cdot (b_i - k_i)] \\ : i = 1, \ldots, n\} \tag{4}$$

Typically, the level of contraction is indicated as a percentage, so that for a 100% contraction all intervals are reduced to the focal point so that the alternatives are at least weakly ordered. The amount of contraction required to obtain $\min(V(A_i) - V(A_j)) > 0$ is referred to as the *level of intersection*.

Portfolio Interval Decision Analysis

In the PDA extension of the above approach to interval decision analysis, the projects are viewed as alternatives. Each project/alternative has an associated cost. Numerically imprecise information regarding costs is modeled as interval statements, such as *"the cost of project A_i lies within an interval ranging from a to b"*, resulting in the inequalities $c_{A_i} \geq a$, and $c_{A_i} \leq b$. A focal point for each cost is expressed, and all cost constraints together with the overall resource constraint B are included in a cost constraint set **C**.

The information frame is then equipped with this additional base **C** representing the statements regarding project costs and the resource limit. The extended information frame for portfolio decision analysis $\langle \{A_i, \ldots, A_m\}, \mathbf{P}, \mathbf{U}, \mathbf{W}, \mathbf{C} \rangle$, contains the consequence set for each project A_i, the probability base **P**, the the utility base **U**, the weight base **W**, and the cost base **C**.

Similar to both PROBE and RPM, the total cost of a portfolio is the sum of the included projects' costs, $\sum_{A_i \in \mathcal{P}} c_{A_i}$, where c_{A_i} denotes the cost of project A_i in portfolio $\mathcal{P}$. The difference in cost between two portfolios is given from $\sum_{\mathcal{P}} c_{A_i} - \sum_{\mathcal{P}'} c_{A_j}$.

The expected utility of a portfolio can be seen as the summation of the expected utilities of the included projects/alternatives. Thus, instead of investigating the difference in expected utility between two alternatives (3) we focus on the difference in expected utility between two project portfolios (5). In (5), $\mathcal{P}$ and $\mathcal{P}'$ denote two portfolios compared.

$$\begin{aligned} \sum_{A_i \in \{\mathcal{P} \setminus \mathcal{P}'\}} V(A_i) - \sum_{A_j \in \{\mathcal{P}' \setminus \mathcal{P}\}} V(A_j) = \\ \sum_{A_i \in \{\mathcal{P} \setminus \mathcal{P}'\}} \left(\sum_l w_l E_l(A_i) \right) - \\ \sum_{A_j \in \{\mathcal{P}' \setminus \mathcal{P}\}} \left(\sum_l w_l E_l(A_j) \right) \end{aligned} \tag{5}$$

In this setting, an efficient portfolio is created by solving a knapsack optimization problem (Martello and Toth 1990). Since the utility of a project $V(A_j)$ is interval-valued, it is reasonable to delimit the range and use a conservative approach to portfolio creation. Let ${}^{L}V(A_j)$ denote the lower utility bound and let ${}^{U}c_{A_j}$ denote the upper bound for the cost with respect to the j:th project. B denotes the resource constraint of the portfolio. The decision variable x_j is set to 1 if the project is included in the portfolio and set to 0 if the project is excluded. A minimax portfolio is then created by solving the knapsack problem (6).

$$\begin{aligned} &\text{maximize } \sum_{j=1}^{m} {}^{L}V(A_j) \cdot x_j \\ &\text{subject to } \sum_{j=1}^{m} {}^{U}c_{A_j} \cdot x_j \leq B \\ &x_j \in \{0, 1\}, \;\; j = 1, \ldots, m \end{aligned} \tag{6}$$

If the minimax approach is deemed as too conservative, a neutral portfolio could be generated by using focal point values ${}^{F}V(A_j)$ and ${}^{F}c_{A_j}$ when solving the knapsack problem in (6). Similar to both PROBE and RPM, the dominance relation between two portfolios is a combination of the difference in utility and the difference in cost. Portfolio $\mathcal{P}$ dominates $\mathcal{P}'$ if and only if $\mathcal{P}$ has a lower cost,

$$\min \left(\sum_{A_i \in \{\mathcal{P} \setminus \mathcal{P}'\}} c_{A_i} - \sum_{A_j \in \{\mathcal{P}' \setminus \mathcal{P}\}} c_{A_j} \right) < 0,$$

and a higher utility,

$$\min \left(\sum_{A_i \in \{\mathcal{P} \setminus \mathcal{P}'\}} V(A_i) - \sum_{A_j \in \{\mathcal{P}' \setminus \mathcal{P}\}} V(A_j) \right) > 0.$$

If several portfolios are non-dominated, we can utilize methods for both *a priori* and *a posteriori* methods for sensitivity analysis to get a better understanding of the decision problem and make a well informed decision in the discrimination of portfolios.

Sensitivity Analysis of Portfolios

A portfolio is a set of projects, where each project is represented similarly to how an alternative is represented in the traditional DELTA approach to interval decision analysis with the exception of the additional cost base **C**. The concept of interval contraction can therefore be applied by contracting the utility, probability, weight and cost bases for each project in a portfolio. Of interest is that the contractions can be used both in the generation of portfolios (*a priori*) and after the portfolio has been generated (*a posteriori*). This approach to sensitivity analysis of portfolios consists of three steps:

1. A priori sensitivity analysis.
2. A posteriori sensitivity analysis.
   (a) Sensitivity analysis of portfolio utilities.
   (b) Sensitivity analysis of portfolio costs.

A Priori

The concept of contraction can be used prior to the portfolio generation, in order to investigate how the imprecise information affects the portfolio composition. This is conduced by stepwise contracting the bases in the information frame, and for each contraction step generate a new portfolio by solving (6). The *a priori* sensitivity analysis consists of the following steps:

1. Set the step size τ to 0.
2. Contract the probability base **P**, the utility base **U**, the weight base **W**, and the cost base **C** by τ.
3. Minimize the expected utility and the maximize the cost for all projects.
4. Generate a minimax portfolio.
5. Add α to τ.
6. Return to step 2 while $\tau \leq 1$.

A Posteriori

The set of portfolios generated in the a priori sensitivity analysis can then be analyzed in *a posteriori* manner. The RPM concept of *core index* is a useful index here. RPM evaluates the robustness of a project by the degree of inclusion in the set of all non-dominated portfolios, on a scale from 0 to 1. An exterior project receives a core index of 0, a core project a core index of 1, and projects with a core index between 0 and 1 are the so-called "borderline" projects (Liesiö, Mild, and Salo 2007; 2008). The core projects and the exterior projects are excluded from further evaluation since the decision of including or excluding the projects is trivial. The remaining borderline projects are further analyzed, for both imprecise project utilities and project costs.

PROBE allow for uncertain project cost information in the *a posteriori* sensitivity analysis, doing this by adding uncertainty with respect to the cost information to one or more projects in the form of intervals. We take the opposite approach, i.e. we let uncertainty with respect to the cost be introduced from the beginning in order to investigate the cost stability of two portfolios. We then remove the uncertain cost for all projects except for one in order to investigate how one project affects the stability. This approach is also taken for portfolio utilities.

Portfolio Utilities. The portfolio utility is evaluated by two approaches; i) contracting the utility, probability and weight bases from 0% to 100% and for each contraction step, calculate the minimum and maximum difference in expected utility between two portfolios; ii) contracting the utility and probability base from 0% to 100% for one borderline project, fixing the weight variables at their focal points, and for each contraction step calculate the minimum and maximum difference in expected utility of two portfolios. The other projects' utilities, probabilities and weights are locked at their focal points. The level of intersection is then evaluated.

The first approach consists of the following steps:

1. Set the step size τ to 0.
2. Contract the probability base **P**, the utility base **U**, and weight base **W** by τ.
3. Calculate the minimum and maximum difference in expected utility between the portfolios.
4. Add α to τ.
5. Return to step 2 while $\tau \leq 1$.

The second approach consists of the following steps:

1. Set the step size τ to 0.
2. Contract the probability base **P**, the utility base **U** for one borderline project by τ, and fix the weights at the focal point.
3. For all other projects, fix the utilities and weights at the focal points.
4. Calculate the minimum and maximum difference in expected utility between the portfolios.
5. Add α to τ.
6. Return to step 2 while $\tau \leq 1$.

Portfolio Costs. The difference in portfolio cost between two portfolios is evaluated by two approaches; i) contracting the cost bases from 0% to 100% and for each contraction step, calculate the minimum and maximum difference in expected cost between two portfolios; ii) contracting the cost base from 0% to 100% for one borderline project and for each contraction step calculate the minimum and maximum difference in expected cost between two portfolios. The other projects costs are locked at their focal points. The level of intersection is then evaluated.

The first approach consists of the following steps:

1. Set the step size τ to 0.
2. Contract the cost base **C** by τ.
3. Calculate the minimum and maximum difference in expected cost between the portfolios.
4. Add α to τ.
5. Return to step 2 while $\tau \leq 1$.

The second approach consists of the following steps:

1. Set the step size τ to 0.
2. Contract the cost base **C** for one borderline project by τ.
3. For all other projects, lock the costs at the focal points.
4. Calculate the minimum and maximum difference in expected cost between the portfolios.
5. Add α to τ.
6. Return to step 2 while $\tau \leq 1$.

The first approach (i) analyses the stability of two portfolios, both for the portfolio utility and the portfolio cost. The second approach (ii) investigates how the utilities and cost of one borderline project affect the choice of portfolio.

Example

In the following example portfolio problem, we demonstrate the contraction based approach to *a priori* and *a posteriori* portfolio sensitivity analysis. The problem at hand consists of six projects, where each project is associated with a cost, and is evaluated against four criteria. The utilities, weights, and project costs are interval-valued, and where a focal point denotes the most reliable value within each interval, see Table 1, 2 and 3.

Project	Cost		
	L	F	U
P01	1.20	1.40	1.70
P02	1.20	1.35	1.60
P03	1.30	1.50	1.70
P04	1.20	1.45	1.60
P05	0.70	0.90	1.10
P06	1.30	1.60	1.80

Table 1: The lower bound, focal point, and upper bound for the cost of each of the six projects P01 to P06.

A Priori

The a priori analysis analysed how uncertain project utilities and cost affected the portfolio composition. The utility, cost and weight bases were contracted stepwise by 20%, within an interval ranging from 0% to 100%. In each step, the minimum difference in utility (Table 4), and the maximum difference in cost Table (5) was calculated for each project, problem (6) was thereafter solved with the projects' minimum expected utilities, the projects' maximum cost values, and a budget constraint of 5.

Project	Criterion 1			Criterion 2		
	L	F	U	L	F	U
P01	40	85	95	45	80	95
P02	40	80	95	45	85	95
P03	50	65	85	50	70	90
P04	55	65	80	55	75	80
P05	55	70	85	50	75	85
P06	5	35	50	10	40	55
Weight	15%	20%	25%	25%	30%	35%

Table 2: The lower bound, focal point, and upper bound for utility variables under criterion 1 and criterion 2 and the lower bound, focal point, and upper bound for criteria weight variables.

Project	Criterion 3			Criterion 4		
	L	F	U	L	F	U
P01	35	85	95	40	85	90
P02	40	80	90	45	75	85
P03	55	70	90	55	65	90
P04	45	70	80	55	65	80
P05	50	80	95	45	65	75
P06	15	40	50	10	30	45
Weight	15%	20%	25%	25%	30%	35%

Table 3: The lower bound, focal point, and upper bound for utility variables under criterion 3 and criterion 4 and the lower bound, focal point, and upper bound for criteria weight variables.

The process resulted in six portfolios of which two were unique, see Table 6. The first portfolio $\mathcal{P}$ was generated at contraction level 0%, 20%, and 40% and included the projects P03, P04, and P05. The second portfolio $\mathcal{P}'$ was generated at contraction level 60%, 80%, and 100% and included project P01, P02, and P05. The core project P05 (included in all portfolios for all contraction levels) and the exterior project P06 (not included in any portfolio for all contraction levels) were excluded from further evaluation since the decision of including or excluding the portfolio was deceive. The borderline projects P01, P02, P03 and P04 (included one of the two portfolios) was further evaluated in the a posteriori approach.

Project	Contracted Minimum Expected Utility					
	0%	20%	40%	60%	80%	100%
P01	40.0	48.82	57.58	66.24	74.86	83.5
P02	42.5	50.08	57.62	65.08	72.52	80.0
P03	52.0	55.18	58.32	61.38	64.42	67.5
P04	52.5	55.84	59.16	62.46	65.70	69.0
P05	49.0	53.60	58.20	62.76	67.34	72.0
P06	9.5	14.76	20.04	25.34	30.66	36

Table 4: The minimum expected utility for each of the six projects at six contraction levels.

A Posteriori

The a posteriori sensitivity analysis was performed by means of analysing upon portfolio utilities as well as analysing upon portfolio costs.

Project	Contracted Maximum Project Cost					
	0%	20%	40%	60%	80%	100%
P01	1.70	1.64	1.58	1.52	1.46	1.40
P02	1.60	1.55	1.50	1.45	1.40	1.35
P03	1.70	1.66	1.62	1.58	1.54	1.50
P04	1.60	1.57	1.54	1.51	1.48	1.45
P05	1.10	1.06	1.02	0.98	0.94	0.90
P06	1.80	1.76	1.72	1.68	1.64	1.60

Table 5: The maximum project cost for each of the six projects at six contraction levels.

Project	Contraction Levels					
	0%	20%	40%	60%	80%	100%
P01	-	-	-	X	X	X
P02	-	-	-	X	X	X
P03	X	X	X	-	-	-
P04	X	X	X	-	-	-
P05	X	X	X	X	X	X
P06	-	-	-	-	-	-

Table 6: The portfolio of projects for each of the six contraction levels. Two unique portfolios can be distinguished, the first portfolio $\mathcal{P}$ contained the projects P03, P04 and P05. The second portfolio $\mathcal{P}'$ contained the projects P01, P02 and P05.

Sensitivity Analysis of Portfolio Utilities. The evaluation was conducted by stepwise contract the weight and utility bases and in each step calculate the minimum and maximum difference in expected utility between the two portfolios. The focal point based difference in expected utility between portfolio $\mathcal{P}$ and $\mathcal{P}'$ was -27.00. The maximum difference in utility was 86.25 and the minimum difference was -80. The expected utility of portfolio $\mathcal{P}$ and $\mathcal{P}'$ intersected at 76.14%, and $\mathcal{P}'$ was thereafter the dominating portfolio according to δ-dominance.

However, it was not clear how the individual borderline projects' utility intervals affected the choice of portfolio. To investigate this, we fixed the utility and weight variables at their respective focal points for all projects except for the evaluated project, for which only the weight variables were fixed at the focal point. The utility base was then stepwise contracted, and the minimum and maximum difference in utility was calculated in each step. Project P03 and P04 did not affect the choice of portfolio since there were no intersection between the portfolios, i.e., the maximum difference was negative. However, when the utilities were varied for project P01, the maximum difference in utility was 16, the minimum difference -37, and intersected in 37.21%. A negative outcome of P01 could thereby change the dominating portfolio to $\mathcal{P}$. The same reasoning applies to project P02, in which the maximum difference in expected utility was 10, the minimum difference -38, and the intersection 27.03%. See the results in Table 7.

Sensitivity Analysis of Portfolio Costs. In order to conduct a similar analysis of project costs such that $\min(c_{P0i} -$

Project	Max	Min	Intersection
P01	16.0	-37.0	37.21%
P02	10.0	-38.0	27.03%
P03	-5.5	-42.0	-
P04	-16.0	-43.0	-

Table 7: The minimum and maximum difference in portfolio utility when all utility and weight variables are locked at their focal point, except for the one project being evaluated for which they are allowed to vary within its interval bounds.

c_{P0j}) shall be positive to claim δ-dominance, we model a positive cost as a negative income. The cost base was stepwise contracted, and the minimum and maximum difference in expected cost between the two portfolios was calculated in each step. The focal point based difference in cost between portfolio $\mathcal{P}$ and $\mathcal{P}'$ was -0.20, the maximum difference 0.8 and the minimum difference -0.9. The portfolios' costs intersected at an intersection level of 80%, and $\mathcal{P}'$ was thereafter the dominating portfolio with respect to cost.

In order to evaluate how individual project costs affected the portfolio choice, we fixed all cost variables at their respective focal points, except for the cost of the evaluated project. The cost base was then stepwise contracted and the minimum and maximum difference in expected value was calculated in each step. As seen in Table 8 the individual cost outcome of project P01, P02 and P04 could result in another dominating portfolio, since the portfolios intersect. Only project P03 did not affect the portfolio choice.

Project	Max	Min	Intersection
P01	0.10	-0.40	33.33%
P02	0.05	-0.35	20.00%
P03	0.00	-0.40	0.00%
P04	0.05	-0.35	20.00%

Table 8: The minimum and maximum difference in cost when all cost variables are locked at their respective focal point, except for the one project being evaluated for which cost is allowed to vary within its interval bounds.

Conclusions

This paper presents how the concept of interval contraction can be used for sensitivity analysis in portfolio decision analysis. The *a priori* sensitivity analysis enables the decision maker to investigate how imprecision in project utilities and project costs affects the portfolio composition. In the *a posteriori* sensitivity analysis, the contraction approach enables the decision maker to evaluate the stability of two portfolios and exploit the intersection level as a measure of robustness. Further work includes an implementation of the suggested approach in a decision software tool and a validation of the approach in a case study.

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