A THEORY OF GAME TREES*

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ABSTRACT

A theory of heuristic game tree search and evaluation functions for estimating minimax values is developed. The result is quite different from the traditional minimax approach to game playing, and it leads to product-propagation rules for backing up values when subpositions in the game are independent. In this theory Nau's paradox is avoided and deeper searching leads to better moves if one has reasonable evaluation functions.

I INTRODUCTION

For game-searching methodology, Nau (Nau, 1980) recently showed that the minimax algorithm can degrade the information provided by a static evaluation function. For the games developed by Pearl, this pathology also arises (Nau, 1981; and Pearl, 1982). Pearl (Pearl, 1982) suggested why it happens. The minimax algorithm finds the *minimax of estimates* instead of estimating a *minimax value.* He also suggested one should consider *product-propagation* rules in order to estimate a minimax value. Nau (Nau, 1983) investigated this method experimentally and found that for Pearl's game it made correct moves more often than minimaxing did. For a different class of games (Nau's games), however, each method made approximately the same number of correct moves.

This paper introduces a mathematical theory of a heuristic game tree search. For this purpose a game model and a search model are constructed. We assume that the result from the game is win or loss and that draws are not permitted. The values (1 for a win of MAX and 0 for a win of MIN) at the leaves of the corresponding game tree are assigned probabilistically. Improved *visibility* of the search can be achieved by assuring that the information given by the search of a level can be retrieved from the information obtained by the search of a deeper level. This theory applies to Pearl's game and Nau's game, but it does not strictly apply to most games.

Three particular results are derived in this theory. First, the exact way to estimate a minimax value is to find the conditional probability of a forced win, given the information from the search. Second, if this conditional probability is used to evaluate moves then the result from a deeper search is on the average more accurate. Third, if the positions on the search frontier are independent (as in Pearl's game), then this estimate is obtained by using the product-propagation rules suggested by Pearl (Pearl, 1981 and 1982).

In using minimax values as criteria of decision making, if we assume that after the next move both players make perfect plays according to the minimax values, then our estimate becomes the conditional probability of *winning the whole game,* and the theory leads to the best move in the situation where limited depth search is used to select the first move but perfect information is used for the remaining moves. Thus this theory is more realistic than the minimax theory, which should assume that perfect information is used at every move, but less realistic than a theory that recognizes that both sides mostly make their moves with imperfect information.

For more realistic game playing, the evaluation function should estimate a *win of the whole game* at a node instead of the minimax value (i.e., a forced win). The minimax theory is based on the assumption that the value of any position at each move is always the minimax value of the position. Therefore, the evaluation at a node doesn't change for each move. The function in this paper assumes that there will be a major change after the first step. On the first move, the value is estimated from a limited depth search. On all later moves, the value becomes the minimax value. In realistic game playing, the estimate (of winning the whole game) at a position should change less drastically on most moves. The value at each node is to be estimated from a search, but the search usually goes deeper at each step. In terms of how much the value of a position changes from one move to the next, the realistic situation should be intermediate between the assumptions that are implicit in the minimax theory and the assumptions in this paper.

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II AN EXAMPLE

For purposes of theoretical study, Pearl (Pearl, 1980) considered a special kind of games. A Pearl's game is represented by a complete uniform game tree with a branching factor d (≥ 2), where the terminal nodes may assume 1 (a win for MAX) or 0 (a win for MIN) with a probability of p and $1 - p$, respectively. Nau (Nau, 1983) considered a different kind of games. A Nau's game also has a complete uniform game tree. To assign LOSS or WIN at terminals, each arc of the game tree is independently given the value 1 or −1 with a probability of p and $1 - p$, respectively. Then the strength of a terminal is defined as the sum of the arc values on the path from the terminal to the root. The terminal is given WIN if its strength is positive, and LOSS otherwise.

To illustrate the idea of the new search model, let us consider an uniform binary game tree of four levels, where the terminal nodes may have independently the value 1 (win for MAX) or 0 (loss for MAX) with a probability of p $(0 < p < 1)$ and $1 - p$, respectively. The space $\Omega = \{0, 1\}^8$ of all leaf-patterns thus becomes a probability space. Assume that the heuristic search finds the number of winning leaves under the searched node. We associate each value k $(k = 0, \ldots, 8)$ at the root with the event $E_k = \{(x_1, \ldots, x_8) \in \Omega \mid \sum_{i=1}^{8} x_i = k\}$. Ω is divided into 9 different such events, which form a partition of Ω. Similarly, the searched values i and j $(0 \leq i, j \leq 4)$ at the nodes of level 2 are associated with the event

$$E_{ij} = \{(x_1, \ldots, x_8) \in \Omega \mid \sum_{k=1}^{4} x_k = i, \sum_{k=5}^{8} x_k = j\}.$$

All these sets form a finer partition of Ω: $E_k = \bigcup_{i+j=k} E_{ij}$. The events given by the search of level 3 are of the form

$$E_{i_1 i_2 i_3 i_4} = \{(x_1, \ldots, x_8) \in \Omega \mid x_1 + x_2 = i_1, x_3 + x_4 = i_2, x_5 + x_6 = i_3, x_7 + x_8 = i_4\}$$

$(0 \leq i_1, i_2, i_3, i_4 \leq 2)$ and form the finest partition of Ω: $E_{ij} = \bigcup_{i_1+i_2=i,\, i_3+i_4=j} E_{i_1 i_2 i_3 i_4}$. The relation $E_{i_1 i_2 i_3 i_4} \subset E_{ij} \subset E_k$ $(i_1 + i_2 = i, i_3 + i_4 = j$, and $i + j = k)$ means that the information from a deeper level is more accurate, and the improved visibility is thus formalized in this simple way. This formulation also holds for the general Pearl's games and Nau's games.

III PROBABILISTIC MODELS

Consider a game tree T with k leaves and h levels. Without loss of generality, we assume that all game trees discussed in this paper have their leaves on the same level. All leaf-patterns $\omega = (x_1, \ldots, x_k)$, $x_i = 0$ or 1, form the space $\Omega = \{0, 1\}^k$. If the values on the leaves are assigned according to a probability measure P on Ω w.r.t. the total Borel field (Chung, 1974) F (i.e., the collection of all subsets of Ω), then we say that these games are in a probabilistic model:

Definition 1. *A probabilistic game model for a game tree T with h levels and k leaves is a pair (Ω, P), where $\Omega = \{0, 1\}^k$ and P is a probability measure on Ω w.r.t. the total Borel field.*

Definition 2. *Let (Ω, P) be a probabilistic game model for a game tree with h levels and k leaves. Then a search S on this model is probabilistic if S consists of an increasing sequence of Borel fields*

$$F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_h = F, \qquad (3.1)$$

where $F_0 = \{\phi, \Omega\}$ is the trivial Borel field and each F_i $(1 \leq i \leq h)$ is generated by a partition of Ω. For each leaf-pattern $\omega \in \Omega$, the search S at level i determines the event E of the partition generating F_i such that $\omega \in E \in F_i$, $i = 1, \ldots, h$.

The increasing sequence (3.1) is the improved visibility of the search. In our example, F_1 is generated by all events E_k, F_2 by all E_{ij}'s, and all $E_{i_1 i_2 i_3 i_4}$'s generate F_3. For a general Pearl's game or a Nau's game (Nau, 1983), the heuristic search that finds the number of 1's on the leaves under the searched node is similarly probabilistic.

In a probabilistic game model (Ω, P) with a probabilistic search (3.1), we define the estimation of the minimax value M (a random variable on Ω) at any fixed node of the game tree as follows.

Definition 3. *For each i, $0 \leq i \leq h$, the conditional expectation of the minimax value M w.r.t. F_i, $M_i = E(M \mid F_i)$, is called the i-th evaluation function of M.*

Given an event E of level i, the value of M_i is $M_i(\omega) = P(M=1 \mid E)$ for all $\omega \in E$, which is just the conditional probability of a forced win, given E.

From the increasing sequence (3.1), we know that the sequence $\{M_i, F_i\}$ forms a martingale (Chung, 1974): $M_i = E(M_j \mid F_i)$ for $0 \leq i < j \leq h$. In words, M_i is an average of M_j if $i < j$. From this property the following theorem is derived (Tzeng and Purdom, 1982):

Theorem 1. *Let i and j* ($0 \le i \le j \le h$) *be any two integers. For any two reals x and y* ($0 \le x, y \le 1$), *we have* $P(M=1 \mid M_i=x, M_j=y) = y$ *if* $P(M_i=x, M_j=y) \neq 0$. *In particular, we have* $P(M=1 \mid M_i=x) = x$ *if* $P(M_i=x) \neq 0$.

This theorem shows that given the estimations of two levels, the deeper one has all the information and, therefore, the other estimation can be dispensed with.

IV DECISION MAKING

Let MAX move from a node A with n different children $B_1, \ldots, B_n$. Let T be the current game tree with A as the root. Suppose that the game is in a probabilistic game model (Ω, P) with a search (3.1). Let $M^{(i)}$ ($1 \le i \le n$) be the minimax value at the node B_i. Relative to the search of level j ($1 \le j \le h$), consider the estimation of $M^{(i)}$ for each i ($1 \le i \le n$): $M_j^{(i)} = E(M^{(i)} \mid F_j)$. If a move is said to be correct if and only if a node with minimax value 1 is chosen, then we have the following main result:

Theorem 2. *A decision making method that depends on the search of a fixed level j, which chooses the node with the largest estimation* $M_j^{(i)}$ ($1 \le i \le n$), *will be improved, if j is increased.*

Proof. Given the information of levels j_1 and j_2 ($1 \le j_1 < j_2 \le h$), suppose that for an arbitrary fixed game in this model the node B_{k_1} is chosen relative to level j_1 and B_{k_2} is chosen relative to level j_2. Furthermore, for this fixed game let $M_{j_1}^{(k_1)} = x_1$, $M_{j_2}^{(k_1)} = x_2$, $M_{j_1}^{(k_2)} = y_1$, $M_{j_2}^{(k_2)} = y_2$. Then $x_1 \ge y_1$ and $x_2 \le y_2$. From Theorem 1 we have $P(M^{(k_1)}=1 \mid M_{j_1}^{(k_1)}=x_1, M_{j_2}^{(k_1)}=x_2) = x_2$ and $P(M^{(k_2)}=1 \mid M_{j_1}^{(k_2)}=y_1, M_{j_2}^{(k_2)}=y_2) = y_2$. Thus, the conditional probability that the chosen node relative to level j_2 is a forced win is always greater than or equal to the conditional probability that the chosen node relative to level j_1 is a forced win. Since it is a sum of all such conditional probabilities, the probability that the chosen node relative to the level j_2 is a forced win is therefore greater than or equal to the probability that the chosen node relative to the level j_1 is a forced win. If $x_2 < y_2$ for at least one game in this model, then the decision making of the deeper level is strictly improved. Q.E.D.

Note that the estimation $M_j^{(i)}$ ($1 \le j \le h$) depends on the search of the whole tree T instead of the subtree T_i under the node B_i. But if the searched moves are independent as in Pearl's games, then each estimation depends on the corresponding subtree only (Tzeng and Purdom, 1982).

V PRODUCT-PROPAGATION RULES

Consider the backing up process on our example. Let M be the minimax value at the root, M_1 and M_2 the minimax values at the left son and the right son of the root, respectively. Given the searched event E_{ij} of level 2, the estimation of M is

$$P(M = 1 \mid E_{ij}) = P(M = 1 \mid \textstyle\sum_{k=1}^{4} x_k = i, \sum_{k=5}^{8} x_k = j).$$

If the root is a MIN node, then it can be proved that

$$P(M = 1 \mid \textstyle\sum_{k=1}^{4} x_k = i, \sum_{k=5}^{8} x_k = j) = P(M_1 = 1 \mid \sum_{k=1}^{4} x_k = i) P(M_2 = 1 \mid \sum_{k=5}^{8} x_k = j). \quad (5.1)$$

If the root is a MAX node, then the corresponding equation becomes

$$P(M = 1 \mid \textstyle\sum_{k=1}^{4} x_k = i, \sum_{k=5}^{8} x_k = j) = 1 - (1 - P(M_1 = 1 \mid \sum_{k=1}^{4} x_k = i))(1 - P(M_2 = 1 \mid \sum_{k=5}^{8} x_k = j)). \quad (5.2)$$

The values $P(M_1 = 1 \mid \sum_{k=1}^{4} x_k = i)$ and $P(M_2 = 1 \mid \sum_{k=5}^{8} x_k = j)$ are the estimates at the two children of the root. The rules (5.1) and (5.2), called the *product-propagation* rules, hold because of the independence of the values at the leaves. This process also applies to general Pearl's games for the search of each level. For more general independent cases, this result can be formalized and proved (Tzeng and Purdom, 1982).

In Nau's games, nodes on the same level are generally dependent and thus these rules are not applicable. For the general dependent case, the evaluation function exists so far only theoretically. Practical methods of finding its values should depend on the dependence of the searched nodes and should be studied for each individual case. The product-propagation method is only for the independent case, and if it is applied to general games, some unexpected features like Nau's pathology are also possible.

CONCLUSIONS

The research reported here illustrates the importance of search uncertainty and search visibility in developing a realistic mathematical model of heuristic search in game trees. In the presence of uncertainty, minimaxing is not the optimum method to combine the values obtained from the search.

REFERENCES

[1] Chung, K. L., *A Course in Probability Theory.* New York, Academic Press 1974.

[2] Knuth, D. E. and R. N. Moore, "An Analysis of Alpha-Beta Pruning," *Artificial Intelligence* **6** (1975) 293-326.

[3] Nau, D. S., "Decision Quality as a Function of Search Depth of Game Tree." Technical Report TR-866, Computer Science Department, University of Maryland, 1980.

[4] Nau, D. S., "Pearl's Game is Pathological." Technical Report TR-999, Computer Science Department, University of Maryland, January 1981.

[5] Nau, D. S., "Pathology on Game Trees Revisited, and an Alternative to Minimaxing," *Artificial Intelligence* **21** (1983) 221-244.

[6] Nilsson, N., *Principles of Artificial Intelligence.* Palo Alto, CA. Tioga Publishing Company, 1980.

[7] Pearl, J., "Asymptotic Properties of Minimal trees and Game-Searching Procedures," *Artificial Intelligence* **14** (1980) 113-138.

[8] Pearl, J., "Heuristic Search Theory: Survey of Recent Results." UCLA-ENG-CSL-8017, June 1980, *Proc. 7th Int. Joint Conf. on AI,* August 1981.

[9] Pearl, J., "On the Nature of Pathology in Game Searching." UCLA-ENG-CSL-8217, January 1982.

[10] Tzeng, C.-H. and Purdom, P. W., "A Theory of the Heuristic Game Tree Search." Technical Report No. 135, Department of Computer Science, Indiana University, December 1982.