# The Use of Continuity in a Qualitative Physics

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## ABSTRACT

The ability to reason about a series of complex events over time is essential in analyzing physical systems. This paper discusses the role of continuity in qualitative physics and its application in a system for analyzing the behavior of Digital MOS circuits that exhibit analog behavior. The discussion begins with a brief overview of the reasoning steps necessary to perform a qualitative simulation using Temporal Qualitative (TQ) Analysis. The discussion then focuses in on the use of continuity and the relationship between quantities and their higher order derivatives in describing how physical quantities change over time.

## INTRODUCTION

The ability to reason about behavior at the qualitative level is essential to perform such tasks as designing, modeling, analyzing and trouble-shooting physical systems. One objective of a qualitative physics is to provide a theory for this type of reasoning. Over the last few years a framework for a qualitative physics has been evolving which includes mechanisms for both device centered (de Kleer and Brown, 1984) and process centered ontologies (Forbus, 1983), through the use of a qualitative algebra for expressing physical interactions. This paper examines the role of continuity in reasoning about change, drawing from a few simple theorems of calculus relevant to a qualitative physics. The discussion begins with a brief overview of the reasoning steps necessary in performing a qualitative simulation using Temporal Qualitative (TQ) Analysis, a system for analyzing the large signal behavior of MOS circuits. The discussion then focuses on the use of continuity and the relationship between quantities and their derivatives in describing the behavior of physical quantities over time.

## TEMPORAL QUALITATIVE ANALYSIS

Temporal Qualitative Analysis describes the causal qualitative behavior of a circuit in response to an input over time, where time is viewed as a set of intervals in which devices move through different operating regions. The qualitative reasoning process, modeled by TQ Analysis, is best illustrated by a simple example. Figure 1 shows a parallel RC circuit which exhibits the following behavior:

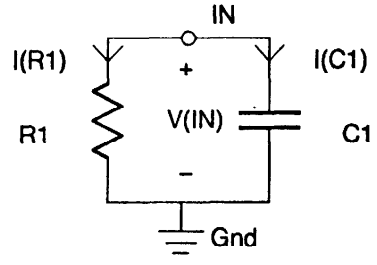


Figure 1 : RC Circuit

> Assume that at instant $t1$ the voltage across the capacitor ($V_{IN}$) is positive. This causes the voltage across the resistor to be positive, producing a positive current through the resistor, which begins to discharge the capacitor and decrease ($V_{IN}$). $V_{IN}$ decreases for an interval of time and eventually reaches zero. [1] At this point the current stops flowing and the circuit has reached a steady state at zero volts.

This description is marked by a series of events such as $V_{IN}$ being initially positive or $V_{IN}$ moving to zero, which break the description into a series of time intervals. Two types of reasoning are required to analyze the circuit during each interval.

One type of reasoning involves determining the instantaneous response of the circuit to a set of *primary causes* which mark the event; for example, "a positive voltage across the resistor, produces a positive current through the resistor . . ." The mechanism corresponding to this type of reasoning in TQ Analysis is Causal Propagation.

The second type of reasoning determines the long term effects of these qualitative inputs; for example, "$V_{IN}$ decreases for an interval of time and eventually reaches zero." This type of reasoning is modeled by Transition Analysis.

## REPRESENTATION

To provide a mechanism for analyzing circuits, a representation for the circuit and its resulting behavior is needed. Quantitatively, a circuit is represented as a network of devices. The functionality of each type of device is described by a device model and the interactions

[1] Since $V_{IN}$ is a decaying exponential, it is positive for $t < \infty$ and reaches zero at $\infty$.

between devices are described by a set of network laws. A device model consists of a set of algebraic relations between state variables associated with the device's terminals (e.g., current, voltage, charge and their derivatives). The relevant equations constraining the circuit's behavior in the above example are:

$$\begin{aligned} V_{IN} &= I_R R && \text{Resistor Model} \\ I_C &= C\tfrac{dV_{IN}}{dt} && \text{Capacitor Model} \\ I_R &= -I_C && \text{Kirchoff's Current Law} \end{aligned}$$

The behavior of the overall circuit is inferred from the network laws and device models and is expressed as a function of time. The behavior of $V_{IN}$ in the RC circuit is:

$$V_{IN} = V_{initial} e^{-\frac{t}{RC}} \text{ for } t > 0$$

Qualitatively, the space of values which a quantity of interest can take on is broken into a set of open intervals or regions separated by a set of boundaries. Time is represented as a sequence of open intervals, separated by instants, and the circuit's state variables are represented by their sign, using zero as a boundary between positive and negative. (The sign of a quantity $(X)$ is denoted $[X]$.)

State variables are then combined into a set of relations using a qualitative algebra consisting of addition, subtraction and multiplication on signs. For example, the sum of two negative numbers is negative $((-)+(-)=-)$, while the sum of a positive and a negative number is unknown $((+)+(-)=?)$(de Kleer, 1979),(Forbus, 1983). The qualitative equivalent of the above models and laws are:

$$\begin{aligned} [V_{IN}] &= [I_R] && \text{Resistor Model} \\ [I_C] &= [\tfrac{dV_{IN}}{dt}] && \text{Capacitor Model} \\ [I_R] &= -[I_C] && \text{Kirchoff's Current Law} \end{aligned}$$

An analogous set of equations may also be created for the first and higher order derivatives of current and voltage. The number of higher order derivatives used in the analysis depends on the level of detail of behavior which must be observed in the particular analysis task. For the analysis of performance MOS circuits we have found it adequate to examine first and second derivatives, making it possible to recognize minimums, maximums and inflection points in the circuit behavior.[2] For simplicity, we only keep track of quantities and their first derivatives in the RC example.

The circuit's overall behavior, in response to a set of inputs is described by a sequence of intervals and the qualitative values of the circuit's state variables for each interval. During an interval each quantity of interest remains within a single qualitative region (e.g., "the voltage is positive" or "the mosfet is in saturation during the interval"). The end of the interval and the beginning of the next is marked by one or more quantities transitioning between qualitative regions.

[2]This differs from earlier qualitative reasoning systems which focused only on first derivatives (de Kleer, 1979).

## CAUSAL PROPAGATION

Causal Propagation occurs at the start of a time interval when a set of qualitative inputs (referred to as *primary causes*) are propagated forward, using the device models and network laws, to determine their instantaneous effect on other circuit quantities. This may be viewed as a qualitative small signal analysis. [3] In the RC explanation, it is given that $(V_{IN})$ is positive at instant $t1$. Using $V_{IN}$ as the primary cause, Causal Propagation produces the following result (where "$A \rightarrow B$" reads "$A$ causes $B$"):

$$\begin{aligned} [V_{IN}] &= +\text{Given} \\ [I_{R1}] &= +\text{Resistor Model} \\ [I_{C1}] &= -\text{Kirchoff's Current Law} \\ [\tfrac{dV_{IN}}{dt}] &= -\text{Capacitor Model} \\ [\tfrac{dI_{R1}}{dt}] &= -\text{Resistor Model} \\ [\tfrac{dI_{C1}}{dt}] &= +\text{Kirchoff's Current Law} \end{aligned}$$

## TRANSITION ANALYSIS

Causal Propagation predicts the instantaneous response of the circuit, but does not describe how quantities change over time. Transition Analysis determines whether or not a quantity transitions between two qualitative regions (e.g., moving from positive to zero or saturation to cutoff) at the end of a time interval, and may be viewed as a qualitative large signal analysis.[4]

Transition Analysis is broken into two steps: *Transition Recognition* and *Transition Ordering*. Transition Recognition determines whether or not a quantity is moving towards another qualitative region or boundary (e.g., the positive charge on the capacitor is decreasing towards zero, or a mosfet is moving from the boundary between $ON$ and $OFF$ to the region $ON$). Transition Recognition often determines that more than one quantity is moving towards another region or boundary. Transition Ordering determines which subset of these quantities will *transition* into a new region or boundary first, marking the end of that interval. Although this article only discusses transitions across zero, the mechanism described here is easily extended to recognize transitions across boundaries other than zero (e.g., transitions between device operating regions) and is described in (Williams, 1984).

## TRANSITION RECOGNITION

The basic assumption underlying Transition Recognition and Transition Ordering is:

[3]Causal Propagation is similar to de Kleer's Incremental Qualitative Analysis (de Kleer, 1979) except that the quantities being propagated are not restricted to first derivatives, but may include quantities and higher order derivatives.

[4]Alternative approaches to describe the behavior of quantities across qualitative region boundaries have been proposed by (de Kleer and Brown, 1984), (Forbus, 1983) and (Kuipers, 1982a).

*The behavior of real physical systems is continuous.*[5]

More precisely, it is the functions which describe a physical system that are continuous. There are a number of simple theorems of calculus which describe the behavior of continuous functions over time intervals. In this section we discuss the intuition which these theorems provide in determining how quantities move between and within qualitative regions. These theorems are then used to derive two rules about qualitative quantities: the *Continuity Rule* and the *Integration Rule*. The first rule requires that a quantity is continuous over the interval of interest, while the second assumes that a quantity is both continuous and differentiable.[6]

### The Intermediate Value Theorem

Inorder to describe the behavior of some quantity over time, a set of rules is needed for determining how a quantity changes from one interval or instant to the next. If, for example, a quantity is positive during some interval of time, will it be positive, zero or negative during the next time interval? The *Intermediate Value Theorem* states that: [7]

> If $f$ is continuous on the closed interval $[a, b]$ and if $l$ is any number between $f(a)$ and $f(b)$, then there is at least one point $X$ in $[a, b]$ for which $f(X) = l$. (Loomis, 1977)

Intuitively, this means that a continuous quantity will *always* cross a boundary when moving from one qualitative open region to another. Thus each state variable must cross zero when moving between the positive and negative regions. In the above example, the positive quantity may be positive or zero during the next time interval, however, it cannot be negative.

### State Variables and Time

By assuming that quantities are continuous and by using the results of the Intermediate Value Theorem, a relationship can be drawn between the representations for state variables and time. Recall that the representation for time consists of a series of *instants* separated by *open intervals*. An instant marks a quantity moving from an open region to a boundary or from a boundary to an open region. Also, recall that the range of a state variable is represented by the open regions *positive* $(0, \infty)$ and *negative* $(-\infty, 0)$ separated by the boundary *zero*, which we denote $+$, $-$ and 0, respectively. If some quantity $(Q)$ is positive at some time instant $t1$ $(Q@t1 = \epsilon$ where $\epsilon > 0)$, then there exists some finite open interval $(\epsilon, 0)$ separating the value of $Q$ from zero (any two distinct points are separated by an open interval).

If we assume that $Q$ is described by a continuous function of time, then it will take some finite interval of time $\{(t1, t2)$ where $t1 \neq t2\}$ to move from $\epsilon$ to 0, traversing the interval $(\epsilon, 0)$. Similarly, it will take a finite interval of time to move from 0 to some positive value $\epsilon$. Furthermore, we can say that a quantity moving from 0 to $\epsilon$ will leave zero at the *beginning* of an open interval of time, arriving at $\epsilon$ at the end of the interval. Conversely, a quantity moving from $\epsilon$ to 0 will leave $\epsilon$ at the beginning of an open interval and arrive at 0 at the *end* of the open interval.

Another way of viewing this is that a quantity will move through an open region during an open interval of time, and a quantity will remain on a boundary for some closed interval of time (possibly for only an instant). This notion of continuity is captured with the following rule:

**Continuity Rule**

1. If some quantity $Q$ is positive (negative) during an instant, it will remain positive (negative) for some open interval of time immediately following that instant.
2. If some quantity $Q$ is zero during some open interval of time, it will remain zero during the instant following the open interval.

Returning to the RC example, we deduced by Causal Propagation that all of the circuit's state variables were positive or negative during instant $t1$. Using the first part of the Continuity Rule, we predict that each state variable must remain positive or negative during the open interval immediately following $t1$ (interval $I2$). They may, however, transition to zero at the instant following $I2$.

### Mean Value Theorem

In addition to looking at the continuity of quantities, information can also be derived by looking at the relationship between quantities and their derivatives. The following two corollaries of the *Mean Value Theorem* (Thomas, 1968) are of particular interest to TQ Analysis:

1. If a function $(f)$ has a derivative which is equal to zero for all values of $x$ in an interval $(a, b)$, then the function is constant throughout the interval.
2. Let $f$ be continuous on $[a, b]$ and differentiable on $(a, b)$. If $f'(x)$ is positive throughout $(a, b)$, then $f$ is an increasing function on $[a, b]$, and if $f'(x)$ is negative throughout $(a, b)$, then $f$ is decreasing on $[a, b]$.

By combining these two corollaries with the Intermediate Value Theorem, the behavior of a state variable is described over an interval (instant) in terms of its value during the previous instant (interval) and its derivative. At the qualitative level, this is similar to integration and is captured by the following rule:

**Qualitative Integration Rule**

**Transitions to Zero**

1. If a quantity is positive and decreasing (negative and increasing) over an open time interval, then it will move towards zero during that interval and possibly transition to zero at the end of the

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[5]Continuity: "The function $f$ is continuous if a small change in $x$ produces only a small change in $f(x)$, and if we can keep the change in $f(x)$ as small as we wish by holding the change in $x$ sufficiently small." (Loomis, 1977)

[6]Even when a circuit's behavior is modeled by a discontinuous function, the discontinuities are isolated at a few places and the rest of the function behaves continuously (e.g., a step is only discontinuous at one point). If the point at which a quantity is discontinuous can be identified, Transition Analysis can deal with it simply by not applying those rules which depend on continuity to the particular quantity at that point in time.

[7]The notation $(a, b)$ denotes the open interval between $a$ and $b$, while $[a, b]$ denotes the closed interval between $a$ and $b$ inclusive.

interval.

2. If a quantity is positive but not decreasing (negative and not increasing) over an open time interval, then it cannot transition to zero and will remain positive (negative) during the following instant.

**Transitions Off Zero**

3. If a quantity is increasing (decreasing) during some open time interval and was zero during the previous instant, then it will be positive (negative) during the interval.
4. If a quantity is constant during some open time interval and was zero during the previous instant, then it will be zero during that interval.

It is interesting to note that, while in the first two parts of the rule the derivative of the quantity affects how it behaves during the following instant, in the last two parts the derivative of a quantity affects that quantity during the same interval. For example, suppose that a quantity ($Q$) is resting at zero at some instant ($t1$) (i.e., $[Q]@t1 = 0$ and $[\frac{dQ}{dt}]@t1 = 0$). If $\frac{dQ}{dt}$ becomes positive for the next open interval ($I2$), then it will cause $Q$ to increase during that interval and become positive. Furthermore, $Q$ moves off zero instantaneously, thus $Q$ is also positive during $I2$. In the above case, the causal relationship between a quantity and its derivative is similar to that between two different quantities related by a qualitative expression (e.g., in a resistor a change in current instantaneously causes a change in voltage).

If we are interested in analyzing a system which includes a number of higher order derivatives, then the Integration Rule may also be applied between each derivative and the next higher order derivative. For example, suppose the system being analyzed involves the position ($x$), velocity ($v$) and acceleration ($a$) of a mass (where $\frac{dv}{dt} = a$) and that all three quantities are constant at some instant ($t1$). If $a$ becomes positive for the next open interval ($I2$), then it will cause an increase in $v$, making it positive for $I2$. Similarly, positive $v$ causes an increase in $x$, making it positive for $I2$. Thus the Integration Rule uses the relation between each quantity and its derivative to locally propagate the effects of changes along a chain from higher order derivatives down towards the lower order derivatives.

As we have seen above, the Integration Rule describes the direction a quantity is moving with respect to zero (e.g., towards or away from zero). If a quantity is zero and increasing or decreasing during the next interval, then the quantity must transition from zero. If, however, a quantity ($A$) is moving towards zero for some interval of time, it may or may not reach zero by the end of the interval. Suppose some other quantity ($B$) reaches zero first and $B$ causes $\frac{dA}{dt}$ to become zero, then $A$ will not reach zero. Thus we need a mechanism for determining which quantity or set of quantities will reach zero first during an open interval of time.

## TRANSITION ORDERING

As a result of Transition Recognition we have divided the set of all quantities into 1) those which may transition (they are moving towards zero) 2) those which can't transition (they are not moving towards zero) and 3) those whose status is unknown (their direction is unknown).

Next we want to determine which subsets of these quantities can transition without leading to 1) quantities which are inconsistent with the set of qualitative relations (e.g., $[A] = +$ and $[B] = 0$ when $[A] = [B]$) and 2) quantities which violate the Intermediate Value Theorem and thus are discontinuous (e.g., $Q$ is caused to jump from $+$ to $-$ without crossing 0).

The simplest solution to this is to enumerate all sets of possible transitions and test each for the above two criteria. However, the number of sets of possible transitions grows exponentially with the number of quantities which can transition, thus this solution becomes intractable for large systems. (de Kleer and Bobrow, 1984) use a similar approach, but only need to consider the transitions of the independent state variables.

Instead, Transition Ordering uses 1) the direction each quantity is moving with respect to zero, and 2) the qualitative relations between these quantities as a set of constraints, to determine which quantities can transition first and still satisfy the criterion of consistency and continuity. If in the worst case, every qualitative relation is used during Transition Ordering, then this solution grows linearly with the number of relations in the system.

If the derivative of a non-zero quantity ($Q$) is unknown, then its direction cannot be determined by Transition Recognition. In this case a qualitative relation associated with $Q$, along with the directions of the other quantities involved in that relation can sometimes be used to determine $Q$'s direction.

The qualitative relations used in modeling devices consists of equality, negation, addition and multiplication. Thus for each of these operations Transition Ordering contains a set of rules which place constraints on the direction (e.g., toward zero) and transition status (e.g., can't transition) of each quantity involved in the operation. The next section provides a few examples of these rules for each type of operation. A complete list of Transition Ordering rules is presented in (Williams, 1984).

### Transition Ordering Rules

If the signs of two continuous quantities are equivalent (i.e., $A = kB$, where $k$ is a positive constant) over the open interval of interest and the following instant, then we know that 1) they are moving in the same direction, and 2) if one of the quantities transitions to zero then the other quantity must transition at the same time. This may be viewed simply as a consistency check on equality. The above rule also holds for negation (i.e., $A = -kB$), since negating a quantity does not change its direction with respect to zero.

The case where a quantity is the sum or difference of two other continuous quantities is more interesting. For example, assume that quantities $A$ and $C$ are moving towards zero and $B$ is constant, where $C = k_1A + k_2B$. If $A$, $B$ and $C$ are positive, then $A$ will transition to zero before $C$ and $C$ can be eliminated from the list of potential

transitions.[8] On the other hand, if $B$ is negative, then $C$ will transition before $A$, and finally, if $B$ is zero, then $A$ and $C$ will transition at the same time (since $C = k_1 A$). Also, consider the case where $A$ and $C$ are positive and $B$ is negative but the direction of $C$ is unknown. If $B$ is known to be constant and $A$ is moving towards zero, then $C$ must also be moving towards zero and will reach zero before A.

Finally, for multiplication (e.g., $A \times B = kC$) we know that, if $A$ and/or $B$ transitions to zero, then $C$ will transition to zero at the same time; otherwise, neither $A$ nor $B$ is transitioning and $C$ won't transition.

Thus, Transition Ordering 1) factors the quantities into sets which transition at the same time and 2) creates an ordering between these sets according to which transitions precede other transitions.

### Applying the Transition Ordering Rules

Transition Ordering rules are applied using a constraint propagation mechanism similar to the one used in propagating qualitative values. If as the result of applying these inference rules it is determined that 1) all the remaining potential transitions will occur at the same time, and 2) the direction of these quantities is known to be toward zero, then the transitions occur at the end of the current interval. Otherwise, an ordering may be externally provided for the remaining potential transitions, or the system can try each of the remaining sets of possible transitions. More quantitative techniques which help resolve the remaining sets of possible transitions are currently being explored.

### RC Example Revisited

Returning to the RC circuit, we have deduced thus far that the capacitor has a positive voltage across it and is discharging through the resistor. Next it must be determined whether or not any quantities will transition to zero at the end of interval $I2$. By applying the Integration Rule to $[V_{IN}] = +$ and $[\frac{dV_{IN}}{dt}] = -$, we know that $V_{IN}$ is moving towards zero. Using a similar argument, we determine that $[I_{R1}]$ and $[I_{C1}]$ are also moving towards zero.

The direction of $[\frac{dV_{IN}}{dt}]$, $[\frac{dI_{R1}}{dt}]$ and $[\frac{dI_{C1}}{dt}]$, however, cannot be determined using the Integration Rule, since their derivatives are unknown. The direction of each of these quantities can be determined using the Transition Ordering rule for equivalences described above. For example, we know that $[\frac{dV_{IN}}{dt}]$ is moving towards zero, since $[I_{C1}]$ is moving towards zero and $[I_{C1}] = [\frac{dV_{IN}}{dt}]$ from the capacitor model. In addition, it is deduced from KCL and the resistor model, which are both equivalences, that $[\frac{dI_{R1}}{dt}]$ and $[\frac{dI_{C1}}{dt}]$ are also moving towards zero.

Finally, since all of the quantities are qualitatively equivalent, they will all transition to zero at the same time. Since no other potential transitions exist, each of these quantities will transition to zero at the end of interval $I2$. Thus the voltage, currents and their derivatives are zero at the next instant.

Both Causal Propagation and Transition Analysis have been implemented and used to correctly predict the behavior of many RLC and mosfet circuits such as high and low pass filters, oscillators and bootstrap circuits. Temporal Qualitative Analysis is currently being extended to incorporate more quantitative information, allowing it to make more precise predictions about complex physical systems. In addition TQ Analysis is being incorporated into a system for designing and debugging high performance MOS circuits.

## SUMMARY

Two components of Temporal Qualitative Analysis have been discussed: Causal Propagation determines the incremental response of a system to a change in an input or its higher order derivative, while Transition Analysis determines the long term effect of these changes. By assuming that physical quantities are modeled by continuous functions, we have been able to develop a few rules to determine how state variables move between qualitative regions. These rules capture one's intuitive notion of continuity and integration.


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## REFERENCES


[1] de Kleer, J. and Brown, J.S., "A Qualitative Physics Based on Confluences," to appear in *Artificial Intelligence.*

[2] de Kleer, J. and Bobrow, D., "Qualitative Reasoning with Higher Order Derivatives," in *Proceedings of the National Conference on Artificial Intelligence*, Austin, Texas, August, 1984.

[3] de Kleer, J., "Causal and Teleological Reasoning in Circuit Recognition," TR-529, MIT Artificial Intelligence Laboratory, Camb., Massachusetts, September 1979.

[4] Forbus, K.D., "Qualitative Process Theory," AIM-664A, MIT Artificial Intelligence Laboratory, Cambridge, Massachusetts., May 1983.

[6] Kuipers, B., "Getting the Envisionment Right," in *Proceedings of the National Conference on Artificial Intelligence*, Pittsburgh, Penn., August, 1982, pp. 209–212.

[7] Loomis, L., *Calculus*, Addison-Wesley, Reading, Massachusetts, 1977.

[8] Thomas, G., *Calculus and Analytic Geometry*, Addison-Wesley, Reading, Massachusetts, 1968.

[9] Williams, B.C., "Qualitative Analysis of MOS Circuits," to appear in *Artificial Intelligence.*


[8] If instead we had said that $C$ transitioned to zero first then $A$ would have to jump from plus to minus without crossing zero (i.e., $[A] = [C] - [B] = (0) - (-) = -$). This violates the Intermediate Value Theorem and, therefore, cannot occur.