ORDER OF MAGNITUDE REASONING

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ABSTRACT

This paper presents a methodology for extending representation and reasoning in Qualitative Physics. This methodology is presently used for various applications. The qualitative modeling of a physical system is weakened by the lack of quantitative information. This may lead a qualitative analysis to ambiguity. One of the aims of this methodology is to cope with the lack of quantitative information. The main idea is to reproduce the physicist's ability to evaluate the influence of different phenomena according to their relative order of magnitude and to use this information to distinguish among radically different ways in which a physical system may behave. A formal system, FOG, is described in order to represent and structure this kind of apparently vague and intuitive knowledge so that it can be used for qualitative reasoning. The validity of FOG for an interpretation in a mathematical theory called Non-Standard Analysis is then proven. Last, it is shown how FOG structures the quantity-space.

INTRODUCTION

Qualitative Physics has had a remarkable development in the last few years. It has shown an increasing capacity to describe the qualitative behavior of physical systems. Nevertheless, the lack of quantitative information can lead a qualitative analysis to ambiguities, and the limits of qualitative simulation have recently been pointed out (Kuipers 1985). In order to overcome these difficulties, the physicist's basic approach and language can be used as guidelines. This provides us with a way to represent seemingly inaccurate and rather informal knowledge which nevertheless plays a determining role in the physicist's (or engineer's) art. This knowledge embodies concepts and rules used to qualify the relative importance of different phenomena on which the whole behavior of a physical system may depend. *This is order of magnitude reasoning*. Order of magnitude reasoning based on the technique introduced in this paper is being used to:

- build the expert system, DEDALE, for troubleshooting analog circuits [2] ,
- search for "qualitative models" by interpretation of numerical results which represent behaviors of a physical system, such as tires under stress,
- build a qualitative model of textbook macroeconomics [1].

First we go into some of the limitations of qualitative analysis methods, through a simple example of mechanics. Then we introduce the formal system FOG* designed to enable order of magnitude reasoning. We show how FOG removes ambiguity. We then demonstrate FOG's logical validity with respect to an interpretation in Non-Standard Analysis. Next we explain the relationship between this interpretation of the formal system and its practical applications. Lastly, we show how order of magnitude reasoning is related to the notion of quantity space as defined in qualitative physics (Forbus 1982). We submit that this knowledge structure plays a crucial part in identifying and differentiating between the possible ways a physical system behaves qualitatively.

I A SIMPLE EXAMPLE

Let's consider a simple example of mechanics. The impact of two masses of very different weights, M and m, coming from opposite directions, with close velocities V_i and v_i. Qualitative reasoning integrating common sense should explain what happens to such a physical system**, and for instance explain what will be the directions of the masses after impact. Following De Kleer's notation $[x]$ will be the qualitative value of quantity x, i.e. the sign of x, with possible values $\{+, 0, -\}$. Then the question is what are the values of $[V_f]$, $[v_f]$? ("f" designates a value after the impact and "i" a value before).

Before impact	$[V_i] = +$	$[v_i] = -$
Impact axis -->	M ■—>	<—■ m
After impact	$[V_f] = ?$	$[v_f] = ?$

Figure 1 : Colliding masses

[1] In French FOG stands for 'Formalisation du raisonnement sur l'Ordre de Grandeur', in English: a Formal system for Order of maGnitude reasoning.

[2] We assume the type of collision that occurs is elastic

Momentum and Energy conservation requires that, except during the shock, the following constraints are satisfied:

$$(e_1) \quad M.V + m.v = P$$
$$(e_2) \quad M.V.V + m.v.v = E$$
where P and E are constants.

A. Qualitative modeling

We are tempted to use qualitative equations to describe the behavior of this physical system. However this is constrained by the fact that in order to arrive at "qualitative differential equations" as in [3,4], equations (e_1) and (e_2) must be derived with respect to time. This cannot be done because impact causes a discontinuity of velocities V and v. For the same reason, it is not possible to work with higher order derivatives or to apply any continuity rules for velocities [5]. Nevertheless, it is still possible to arrive at qualitative equations which link the qualitative values of the velocities before and after impact. Using the classical addition of signs, denoted $\oplus$, momentum conservation (e_1) implies:

$$(1)\ [V_f] \oplus [v_f] = [V_i] \oplus [v_i]$$

Conservation of energy does not give a useful qualitative equation directly, but (e_1) and (e_2) imply:

$$(e_3) \quad V_f + V_i = v_f + v_i$$

Thus: $(2)\ [V_f] \oplus [V_i] = [v_f] \oplus [v_i]$

B. Ambiguity

Since V_i and v_i have opposite signs, $[V_i] \oplus [v_i]$ i.e $[V_i] \oplus [v_i] = ?$. Therefore the right-hand side of equation (1) is undefined. An analysis of equations (1) and (2) shows that such qualitative modeling leaves the sign of V_f and v_f unknown. Five solutions for $[V_f]$ and $[v_f]$ are equally possible (see table 1). Common sense suggests that the particularity of this situation makes it possible to remove such ambiguity. As we shall see, this can be done by applying FOG.

	$[V_f]$	$[v_f]$	$[V_i]$	$[v_i]$	$[V_f] \oplus [V_i] = [vi] \oplus [vf]$
1	+	+	+	-	+ = ?
2	0	+	+	-	+ = ?
3	-	+	+	-	? = ?
4	-	0	+	-	? = -
5	-	-	+	-	? = -

Table 1 : Possible values of $[V_f]$ and $[v_f]$

II FOG

What are the key concepts of order of magnitude reasoning? We introduce three operators Ne, Vo, Co. They are used to represent intuitive concepts:

A Ne B stands for A is negligible in relation to B.

A Vo B stands for A is close to B, ie (A - B) is negligible in relation to B.

A Co B stands for A has the same sign and order of magnitude as B. The underlying idea is that if B Ne C then A Ne C.

We now introduce the FOG formal system. The completness and minimality of FOG is not studied here. Because of the intuitive nature of the rules we won't explain them in detail. $[X]$ stands for the sign of element X.

A. The Formal System

Axiom:

A_1: A Vo A
Inference rules:

R_0: A Vo B → B Vo A
R_1: A Co B → B Co A
R_2: B Vo A → B Co A
R_3: A Vo B, B Vo C → A Vo C
R_4: A Ne B, B Ne C → A Ne C
R_5: A Co B, B Co C → A Co C
R_6: A Co B, B Vo C → A Co C
R_7: A Vo B, B Ne C → A Ne C
R_8: A Ne B, B Co C → A Ne C
R_9: A Vo B → $[A] = [B]$
R_{10}: A Co B → $[A] = [B]$
R_{11}: A Ne B → -A Ne B
R_{12}: $[A] \neq 0$, A Vo B → ¬ (A Ne B)
R_{13}: $[A] \neq 0$, A Co B → ¬ (A Ne B)
R_{14}: $[A + B] = +$, $[A] = -$ → ¬(B Ne A), $[B] = +$
R_{15}: $[A] \neq 0$, $[A] = [B]$, (A + B) Vo C → ¬(C Ne A), ¬(C Ne B)
R_{16}: $[A] = 0$, (A + B) Vo C → B Vo C
R_{17}: $[A] = [B]$, A Vo C → (A + B) Vo (C + B)
R_{18}: A Ne C, B Co D → A.B Ne C.D
R_{19}: A Ne B, C Vo D → A.C Ne B.D
R_{20}: A Ne C, B Ne D → A.B Ne C.D
R_{21}: A Co B, C Co D → A.C Co B.D
R_{22}: A Vo B, C Vo D → A.C Vo B.D
R_{23}: (A + B) Vo C, B Ne A → A Vo C
R_{24}: (A + B) Vo A → B Ne A
R_{25}: A.B Ne C.D, C Ne A, $[A] \neq 0$ → B Ne D
R_{26}: A.B Vo C.D, A Vo C, $[A] \neq 0$ → B Vo D
R_{27}: A.B Vo C.D, A Ne C, $[C] \neq 0$ → D Ne B
R_{28}: A.B Co C.D, A Ne C, $[C] \neq 0$ → D Ne B
R_{29}: $[A] = -[D.E] \neq 0$, (A + B.C) Vo D.E, B Ne D → E Ne C
R_{30}: $[A] = -[C]$, (A + B) Vo C, D Ne E → D.C Ne B.E

B. Basic Properties

If Co and Vo are both relations of equivalence, a distinction can be made when they are used in conjunction with the Ne relation: if (A + B) Vo A is true then R_{24} implies B Ne A. If instead (A + B) Co A is true, the same conclusion cannot be drawn. Co is obviously less restrictive then Vo.

Rule R_9, and R_{10}, imply that FOG can work with the qualitative values of quantities. Thus relations in FOG contain both information on the signs, and on the relative order of magnitude of the quantities. We call these relations "order of magnitude equations".

One should notice that there is no rule that concludes (B Vo D) from (A Vo C) and (A + B) Vo (C + D). In fact, the orders of magnitude of B and D may occasionally be concealed by those of A and C. This last remark shows that this calculus is not as simple as it may look at first glance.

III BACK TO THE EXAMPLE

A. Qualitative Constraints

(3) $(MV_f + mv_f)$ Vo $(MV_i + mv_i)$
(4) $(MV_fV_f + mv_fv_f)$ Vo $(MV_iV_i + mv_iv_i)$
(5) $[V_i] = +$
(6) $[v_i] = -$
(7) $[M] = +$
(8) $[m] = +$
(9) m Ne M
(10) V_i Vo $-v_i$*

B. Firing Rules of FOG

- R_{19} to (9, 10) → (11) $-mv_i$ Ne MV_i
- R_{11} to (11) → (12) mv_i Ne MV_i
- R_{19} to (11, 10) → (13) mv_iv_i Ne MV_iV_i
- R_{23} to (3, 12) → (14) $(MV_f + mv_f)$ Vo MV_i
- R_{23} to (4, 13) → (15) $(MV_fV_f + mv_fv_f)$ Vo MV_iV_i
- R_9 to (14, 5) → (16) $[MV_f + mv_f] = +$
  - Hypothesis: $[Vf] = -$

    R_{14} to (16) → (17) $[vf] = +$

    R_{29} to (17, 14 , 5, 9) → (18) V_i Ne v_f

    R_{30} to (15, 18) → (19) MV_iV_i Ne mv_fv_f

    R_{15} to (15) → (20) $\neg(MV_iV_i)$ Ne mv_fv_f

    (19, 20) → Contradiction
  - Hypothesis: $[Vf] = 0$

    R_{16} to (15) → (21) mv_fv_f Vo MV_iV_i

    R_{16} to (14) → (22) mv_f Vo MV_i

    R_{27} to (22,9) → (23) V_i Ne v_f

    R_{26} to (21,22) → (24) V_i Vo v_f

    (23,24) → Contradiction
- → $[V_f] = +$

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[3] If instead of asserting V_i Vo $-v_i$, we weaken this this assertion to V_i Co $-v_i$, the same conclusion for the signs of velocities can be drawn, and the final result is v_f Co v_i instead of v_f Vo $3V_i$

C. Results

Ambiguity for the qualitative value of V_f is removed. $[V_f] = -$ and $[V_f] = 0$ lead to contradiction. Thus, the only right solution in table 1 is:

$$[V_f] = + \text{ and } [v_f] = +$$

Furthermore the complete analysis [10] of the case also implies: V_f *Vo* V_i and v_f *Vo* $(V_i + V_i + V_i)$, which means that the velocity of the larger mass remains about the same after impact, and that the smaller mass resumes with a velocity close to three times the velocity of the larger mass.

D. Comparing the Results with a Reasoning by Analogy

Mass m is negligible as compared to M, so everything happens as if mass m were hitting a wall (mass M). If the frame of reference is mass M, the velocity of mass m before impact is close to $(-2V_i)$. Mass m rebounds at a velocity of $2V_i$. Since mass M's velocity is already V_i, after impact the velocity of mass m is $2V_i + V_i$.

It should be noted that this reasoning implicitly uses the steps proven with FOG For example, the sentence "everything happens as if mass m were hitting a wall" is equivalent to "the momentum and energy of M remains unchanged". And these conclusions are obtained when using FOG [10] that infers:

$$M\,V_f \text{ Vo } M\,V_i \quad \text{and} \quad M\,V_f\,V_f \text{ Vo } M\,V_i\,V_i$$

E. The Added Information Derived from FOG

Analysing another simple case will help illustrate the rewards of using order of magnitude reasoning, and the limitations of focusing only on the sign of quantities.

Take the following case:	The results in this case are:
$[V_i] = +$	$[V_f] = ?$
m Vo M	$[v_f] = +$
v_i Ne V_i	V_f Ne V_i
$[v_i] = -$	$[v_f] = +$
	v_f Vo V_i

With a qualitative analysis restricted to signs, ambiguity would remain for both velocities V_f and v_f. With FOG the sign of V_f remains ambiguous, but a qualitative property is obtained: V_f Ne V_i is provided, and compared to the velocity of mass m, mass M remains steady after impact. So the main phenomenon is derived by FOG, namely that there is a transfer of velocity, momentum and energy from mass M to m.

These two cases show that information relative to order of magnitude structures the behavior of the physical system. For more complicated systems, it is often essential for the practitioner to use this order of magnitude knowledge to deduce the different possible behaviors of the physical system.

An interesting question is whether it is preferable to solve the problem symbolically and then use order of magnitude considerations. A first remark is that in some cases the model can only be described in terms of order of magnitude [1,2]. Secondly if we look at the resolution of the simple example above, using initially order of magnitude reasoning produces inferences that at each step can be interpreted in terms of velocity, momentum, or energy. We expect the more complex the system the greater the gain, by using order of magnitude reasoning as early as possible in the analysis, for the resolution and for the explanation.

IV VALIDITY OF FOG IN NON-STANDARD ANALYSIS

Let's give a justification for the use of FOG from a logical point of view. Under the name of Non-Standard Analysis, A. Robinson introduces [11] a calculus on infinitesimal. In essence and with a gross simplification he describes a way to introduce a halo around quantities. This suggest that Non-Standard Analysis might be a good tool to validate FOG.

A. A Quick Glimpse on Non-Standard Analysis

Field K of Non-Standard Analysis, noted N.S.A., is a totally ordered non archimedean* field [9]. The field R of real numbers is imbedded in K.

Let F be the ring of finite elements of K, I the set of infinitesimals. Then $R \cap I = \{0\}$, and I is an ideal of F. In particular the sum of two infinitesimals is an infinitesimal and the product of an infinitesimal and a finite element is an infinitesimal.

Positive infinitesimals are smaller than any strictly positive real number.

B. Definitions For The Qualitative Operators

Let A, B, be elements of K:

- A Vo B iff o ε I, A = B.(1 + o).
  One could be tempted to use the definition of a halo to define A Vo B, i.e A Vo B iff (A - B) ε I. But with this definition A Vo B would not imply $[A] = [B]$, and FOG would lose it's capacity to remove ambiguity.
- A Ne B iff o ε I, A = B.o
- A Co B iff O ε F-I, A = B.O

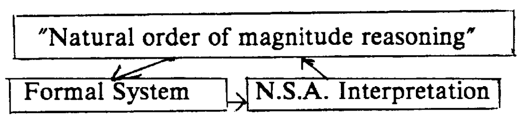
C. Validity of the Inference Rules

All the rules of FOG are valid for this interpretation [10]. Let us demonstrate, for example, the validity of some rules.

- R_1 A Vo B, B Vo C → A Vo C
  A = B(1 + o_1), B = C (1 + o_2), with o_1 and o_2 elements of I. Hence A = C (1 + o_1 + o_2 + $o_1.o_2$). Since I is stable by addition and multiplication, definition (d1) shows that A Vo C.
- R_{23} (A + B) Vo C, B Ne A → A Vo C
  According to R_1, (A + B) Vo C → C Vo (A + B), i.e. C = (A + B) (1 + o_1), B Ne A, gives B = A .o_2, with o_1 and o_2 elements of I. Hence C = A (1 + o_1 + o_2 + $o_1.o_2$). The stability of I for addition and multiplication still results in C Vo A, in replying R_1 we finally get A Vo C.
- R_{27} A.B Vo C.D, A Ne C, $[C] \neq 0$ → D Ne B
  A = C .o_1, by applying R_1 we get: C.D Vo A B, i.e. C.D = A.B (1 + o_2), with o_1 and o_2 elements of I. Hence D = B.o_1 .(1 + o_2), by using the properties of stability of I, we get: D Ne B.
- R_{11} A Co B, B Ne C → A Ne C
  A = B O_1 and B = C o_2, with O_1 element of F-I, o_2 element of I. Hence $O_1.o_2$ is an element of I and A Ne C.

V HOW TO USE FOG

Getting back to the practical aspect, it is interesting to complete the path in the diagram below:



In concrete terms we must go from Robinson's infinitesimals to sufficiently small reals. To do this one can associate with infinitesimals, sequences of real numbers tending towards 0 [7]. Algebraic computing in N.S.A. then becomes the study of limits in the world of real numbers. Thus the following result completes the path.

Let us consider a formal deduction using the rules of FOG a finite number of times. Given neighborhoods I_1 of 0, I_2 of 1 and I_3 a finite interval containing I_2 with $I_1 \cap I_3 = O$ and defining A Ne B iff A/B ε I_1, A Vo B iff A/B ε I_2, A Co B iff A/B ε I_3, all results derived from applying FOG will hold, provided that the initial intervals allowed for the use of Ne, Vo, Co are "tight" enough compared to I_1, $I_2 I_3$.

The above result does not specify the size of these intervals but confirms their existence. If we reason without specifying these ranges, we have a purely symbolic qualitative reasoning. In practice, such symbolic reasoning is applied either because the data available is not accurate enough to use quantitative methods, or because qualitative reasoning has been deliberately chosen. Even in the case of a pure symbolic reasoning with order of magnitude knowledge, we can extract an interesting explanation of the behavior of a system. This is the case for example in the macroeconomic model [1].

* A field is archimedean if for every strictly positive element x of the field, and for every element y of the field, there exists an integer n such that n.x > y.

For certain applications, we must be able to determine whether or not we are within ranges for which this reasoning is acceptable for real numbers. In this case, using order of magnitude reasoning for a given application requires the specific expertise of the system builder. Deciding to use premise A Ne B is only of interest with respect to a given situation, and an expert is capable of deciding which qualitative relations are suited to the system. For instance, as far as the DEDALE expert system is concerned, the choice of initial relations requires expertise. The system user will consider the expertise used to specify the acceptable ranges as given initial knowledge when solving a particular case.

VI FOG AND THE QUANTITY SPACE

FOG's contribution can be understood through what is referred to in qualitative physics as the quantity space [6]. The notion of quantity space is used to define "landmarks" for values of qualitative variables. The basic structuring of the quantity space is to locate an element $[x]$ in relations to fields $[x] = +$ and $[x] = -$. Here the landmark considered is the value 0, but there may also be other landmarks. Landmark "L", for example, then separates the quantity space according to the sign of $[X - L]$. The set of landmarks defines a partial order on the quantity space [6].

FOG provides the quantity space with qualitative landmarks and a structure for the regions defined by these landmarks:

- The equivalence relations, Co and Vo, define regions within this space containing elements which have the same order of magnitude.
- The Ne relation sets up a hierarchy between these regions, in other words a scale of comparison for this space.
- The rule: A Ne B $\rightarrow$ (A + B) Vo B, shows that for orders of magnitude which differ, the regions are stable with respect to addition.
- The rule: A Ne B and C Co D $\rightarrow$ A.C Ne B.D, means that the hierarchy between two regions is maintained, when multiplying their elements by elements which are on comparable scales.

The way in which FOG applies these characteristics to the quantity space makes it possible to express what it means to detect a contradiction or to make a hypothesis concerning orders of magnitude.

- A contradiction is detected if regions defined by classes of equivalence associated with relations Vo and Co, do not follow the hierarchy between the classes imposed by the Ne relation.
- Making a hypothesis concerning the order of magnitude comparing two elements means imposing an additional relation between their classes. This may involve merging them or establishing a hierarchy between them.

CONCLUSION

The aim of order of magnitude reasoning is to provide a level of description, eliminating secondary aspects and showing the main properties of a system. This implies a quantity space with the added structure derived from the use of the operators Ne, Vo, Co. This allows the introduction of common sense knowledge, and simplifies the representation of complex systems. FOG handles order of magnitude reasoning through symbolic computation. Thus, the formal system FOG creates a framework to represent this category of qualitative knowledge. This representation belongs to the scientist's traditional and intuitive way of reasoning. Experience in applying FOG to Macroeconomics [1] indicates that it should have a wide range of applications.

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