Models of Axioms for Time Intervals

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Abstract

James Allen and Pat Hayes have considered axioms expressed in first-order logic for relations between time intervals *[AllHay85, AllHay87.1, AllHay87.2]*. One important consequence of the results in this paper is that their theory is decidable *[Lad87.4]*. In this paper, we characterise all the models of the theory, and of an important subtheory. A model is isomorphic to an interval structure *INT(S)* over some unbounded linear order S, and conversely, *INT(S)*, for an arbitrary unbounded linear order S, is a model. The models of the subtheory are similar, but with an arbitrary number of copies of each interval (conversely, all structures of this form are models). We also show that one of the original axioms is redundant, and we exhibit an additional axiom which makes the Allen-Hayes theory complete and countably categorical, with all countable models isomorphic to *INT(Q)*, the theory of intervals with rational endpoints, if this is desired. These results enable us to directly compare the Allen-Hayes theory with the theory of Ladkin and Maddux *[LadMad87.1]*, and of van Benthem *[vBen83]*.

1 Introduction

The Interval Calculus

The representation of time by means of intervals rather than points has a history in philosophical studies of time (*[Ham71, vBen83, Hum78, Dow79, Rop79, New80]*).
James Allen defined a calculus of time intervals in *[All83]*, as a representation of temporal knowledge that could be used in AI. We call this the *Interval Calculus*. Allen investigated constraint satisfaction in the Interval Calculus, and use of the Calculus for representing time in the context of planning *[All84, AllKau85, PelAll86]*. Allen and Pat Hayes in *[AllHay85, AllHay87.1, AllHay87.2]* reformulated the calculus as a formal theory in first-order logic. Our interest in this representation of time stems from our belief that it is more in keeping with common sense use of temporal concepts to represent time by means of intervals, than to use the mathematical abstraction of points from the real number line (*op. cit.*). The Interval Calculus is particularly amenable to treatment by the methods of mathematical logic *[LadMad87.1, Lad87.2, Lad87.4]*, since it is complete, countably categorical (i.e. there is a unique countable model, up to isomorphism), decidable, and admits elimination of quantifiers (i.e. every first-order formula is equivalent to a quantifier-free formula), although it is NP-hard *[VilKau86]*. We shall show below that the Allen-Hayes reformulation is a strictly weaker theory than the Interval Calculus.

Overview of the Results

Allen and Hayes *[AllHay85]* introduced their axioms as a first-order logical formulation of the theory of intervals, guided by *[All83]*. We investigate their axioms in the slightly different form in which they are presented in *[AllHay87.1]*. Let $\mathcal{I}_{\mathcal{AH}}$ be the Allen-Hayes theory, i.e. the set of formulas that are consequences of the axioms. We present a complete categorisation of the models of $\mathcal{I}_{\mathcal{AH}}$. This enables us, via results in *[LadMad87.1]*, to directly compare the strengths of the various first-order theories of intervals in *[vBen83, AllHay85, LadMad87.1]*, and further to show that $\mathcal{I}_{\mathcal{AH}}$ is decidable *[Lad87.4]*. In this section, we survey the technical results described in this paper.

First we show that one axiom (Existential **M5**) is redundant. We then characterise the models of $\mathcal{I}_{\mathcal{AH}}$ and the important subtheory $\mathcal{I}_{SUB}$ by considering certain syntactic definitions and their properties. We introduce 'points' as a definable equivalence relation on pairs of intervals (the term '***intervals***' just refers to objects in the model). (Rather than develop a theory of pairs within the axioms, we use a syntactically definable relation with four interval arguments to define the equivalence relation on pairs of intervals). We call the equivalence classes ***pointclasses***.

We show that pointclasses are linearly ordered by a definable relation (which again has to be a relation on four intervals rather than on pairs of intervals), as a consequence of the axioms. We associate to each interval two pointclasses, representing the 'ends' of the interval, and show these pointclasses are unique, for a given interval. We show that one axiom (**M4**) guarantees also that there is a unique interval corresponding to a given pair of pointclasses. $\mathcal{I}_{SUB}$ does not contain **M4**. In fact, $\mathcal{I}_{SUB}$ with the addition of **M4** gives $\mathcal{I}_{\mathcal{AH}}$ (see below).

We can now show that the pairs of (ordered) distinct elements from an arbitrary unbounded linear order S, a structure which we call *INT(S)*, forms a model of $\mathcal{I}_{\mathcal{AH}}$, and conversely that any model of $\mathcal{I}_{\mathcal{AH}}$ is of the form *INT(S)*, for some unbounded linear order S.

When the axiom **M4** is dropped, there may be an arbitrary number of intervals with given *endpoint-classes*, and we show that the models of $\mathcal{I}_{SUB}$ are characterised by two parameters:

- the (unbounded) linear ordering of the pointclasses
- for each pair of pointclasses, the number of different intervals with that pair as the 'endpoints'.

Finally, we show how to complete the Allen-Hayes axioms by adding an axiom **N1**, so that they have *INT(Q)*, the rational intervals, as the only countable model up to isomorphism, if this is desired.

The results of this paper are essential for the proof of decidability of $\mathcal{I}_{\mathcal{AH}}$. However, the result and proof are beyond the scope of this paper. We refer the reader to *[Lad87.4]*.

What We Now Know

We indicate briefly here what is known concerning the various interval theories. We do not have the space to include a detailed comparison, but the interested reader may find one in the longer version of this paper, along with proofs of the results in the technical section *[Lad87.3]*.

Van Benthem considered first-order theories of intervals, first proved the countable categoricity of *Th(INT(Q))* (the full first-order theory of rational intervals) *[vBen83]* and indicated an axiomatisation in *[vBen84]*. Ladkin and Maddux *[LadMad87.1]* formulated the Interval Calculus as a *relation algebra* in the sense of Tarski *[JonTar52, Mad78]*, and associated with the algebra a first-order theory that they proved countably categorical, complete and decidable. It is a consequence of results in *[LadMad87.1]* on the interdefinability of the primitive relations that the formulations of van Benthem and Ladkin-Maddux define the same theory, even though they appear radically different - the theory of intervals over an unbounded, dense, linear order. Ladkin proved that the theory admits elimination of quantifiers, and exhibited an explicit decision procedure, making use of the Ladkin-Maddux extension of Allen's constraint satisfaction algorithm, and the quantifier elimination procedure, in *[Lad87.4]*.

We show in this paper that the Allen-Hayes axioms define precisely the theory of intervals over an unbounded linear order, not necessarily dense. Hence this theory is logically weaker than *Th(INT(Q))*, Since the addition of **N1** to the Allen-Hayes axioms assures density, this gives yet another axiomatisation of *Th(INT(Q))*.

Of course, *logically weaker* entails *more models*, which is what Allen and Hayes intended. They wanted the intervals over the *integers*, *INT(Z)*, as a possible model of their theory, as well as *INT(Q)*. The weaker theory is still decidable, but does not admit elimination of quantifiers *[Lad87.4]*.

So it all fits together very nicely and everyone should live happily ever after ....

Terminology

We assume that the reader has familiarity with the basic notions of first-order logic and model theory, as in *[ChaKei73, ManWal85]*. We include some reminders here. The only non-standard concept we use is that of an *atransitive* binary relation.

The language of time interval theories, in the Allen-Hayes version, has a single primitive binary relation symbol $\|$ for *meets*. Since all other relations may be defined from this in the Interval Calculus *[LadMad87.1]*, it suffices to use this simple language. All our definitions below will assume this language.

A *theory* T is a set of sentences that is closed under deduction. An *axiomatisation* of a theory T is a recursive set of sentences S such that T is the set of deductive consequences of S. T is *axiomatisable* if it has an axiomatisation.

A *structure* is a set of objects U, along with with a binary relation $\|_0$. We denote such a structure by $\langle U, \|_0\rangle$.

A *model* of a theory T is a structure such that all of the sentences in T are true in it. The class of all models of T is denoted *Mod(T)*.

The *theory of the model M* is the set of all sentences that are true in M, and is denoted by *Th(M)*. *Th(M)* is complete (by construction). Note that M is a model for *Th(M)*.

A function $\theta : M_1 \rightarrow M_2$ is a *homomorphism* of models $\langle M_1, \|_1\rangle$ and $\langle M_2, \|_2\rangle$ if and only if $(\forall x, y \in M_1)(x\|_1 y \leftrightarrow \theta(x)\|_2\theta(y))$. An *isomorphism* is a one-to-one, onto homomorphism. Two models are *isomorphic* iff there is an isomorphism between them.

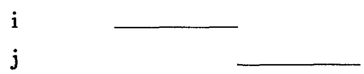
A theory T is *countably categorical* iff all countable models are isomorphic i.e. there is only one countable model, up to isomorphism.

A binary relation R (written infix) is **atransitive** iff $(\forall p,q,r)(pRq \;\&\; qRr \rightarrow (\neg pRr))$; an **ordering** iff it is irreflexive, asymmetric and transitive; an **unbounded** ordering iff it is an ordering, and also satisfies $(\forall p)(\exists q)(pRq) \;\&\; (\forall p)(\exists q)(qRp)$; a **linear ordering** iff it is an ordering and linear.

The following facts from model theory are relevant. A theory which is countably categorical is also complete. An axiomatisable, countably categorical theory is also decidable. The theory of unbounded dense linear orders is countably categorical. All countable models of the theory of unbounded dense linear orders are isomorphic to the rational numbers with the natural ordering, $\langle Q, <\rangle$. Finally, there are uncountably many non-isomorphic countable models of the theory of unbounded linear orderings.

2 The Allen-Hayes Theory $\mathcal{I}_{\mathcal{AH}}$

The Allen-Hayes axioms for $\mathcal{I}_{\mathcal{AH}}$ are motivated by considering intuitive properties of the relation *meets* over intervals from a linear order such as Q or Z. The intuitive definition of *meets* is given by the picture below:



We give the formal definition in terms of intervals as pairs-of-points over some arbitrary linearly-ordered domain S.

- *(a,b)* is an *interval* if and only if $a < b$
- *(a,b) meets (c,d)* if and only if $b = c$
- *INT(S)* is the set of intervals on S, with the thirteen natural binary relations definable from the ordering on S

Note in particular that there is no question of intervals being *sets* of points, and therefore no issue as to whether they include endpoints or not. Intervals are just pairs of points, and an endpoint is just one of these points. It does turn out that the class of open, closed, and half-open (at either end) intervals on the rationals is also countably categorical, and we can provide an extension of the Allen-Hayes axioms that have this structure as the only countable model, up to isomorphism *[Lad87.5]*.

We give the Allen-Hayes axioms without much commentary, and refer the interested reader to *[AllHay85, AllHay87.1, AllHay87.2]* for further motivation. The theory $\mathcal{I}_{\mathcal{AH}}$ is axiomatised by **M1 - M5**; equivalently, as we shall show, by **M1 - M4**. The theory $\mathcal{I}_{SUB}$ is axiomatised by **M1 - M3** only, omitting **M4**. We use the symbol $\|$ for *meets*. The axioms are:

M1: $(\forall p,q,r,s)((p\|q) \,\&\, (p\|s) \,\&\, (r\|q) \rightarrow (r\|s))$
which is intended to make the 'meeting-places' unique

M2: $(\forall p,q,r,s)((p\|q) \,\&\, (r\|s) \rightarrow$
$(p\|s \otimes (\exists t)(p\|t\|s) \otimes (\exists t)(s\|t\|p)))$
where $\otimes$ is *exclusive or*, i.e. precisely one of the alternatives must hold.
This axiom is intended to linearly-order the meeting places

M3: $(\forall p)(\exists q,r)(q\|p\|r)$
which is intended to ensure that the intervals are unbounded at either end of the time line

M4: $(\forall p,q,r,s)(p\|q\|s \,\&\, p\|r\|s \rightarrow q = r)$
which is to ensure that there are unique intervals with particular given 'endpoints'

M5: Functional Form $(\forall p,q)(p\|q \rightarrow$
$(\exists r,s)(r\|p\|q\|s \,\&\, r\|(p+q)\|s))$
which is intended to guarantee the existence of a 'union' interval of two meeting intervals.

M5: Existential Form $(\forall p,q)(p\|q \rightarrow$
$(\exists r,s,t)(r\|p\|q\|s \,\&\, r\|t\|s))$
In fact, the axiom **existential M5** already follows from **M1 - M3** (below).

Existential M5 versus Functional M5

The operator $+$ in **Functional M5** may be introduced by *Skolemisation* in any given model of the axioms with **Existential M5**, i.e. such a model may be augmented with the addition of a function so that it becomes a model of **Functional M5** *[ChaKei73]*. We therefore prefer to use **Existential M5**, since **Functional M5** leads to technical difficulties which we prefer to avoid (*[Lad87.3]*). For example it dirties our tidy language .....

The Axiom M4

There are techniques for obtaining models of $\mathcal{I}_{\mathcal{AH}}$ from models of $\mathcal{I}_{SUB}$. The relation of *'having the same endpointclasses as'* is an equivalence relation that preserves the primitive relation *meets*, and therefore any model of $\mathcal{I}_{SUB}$ has a homomorphic image that is a model of $\mathcal{I}_{\mathcal{AH}}$, obtained by 'factoring through' the equivalence relation, i.e. by identifying objects iff they have the same equivalence class *[ChaKei73]*. However, the models of $\mathcal{I}_{SUB}$ are not the intended models of the interval theory, since in general they may have different intervals with identical endpoints. Hence even though **M4** is dispensable from the point of view of model theory, we need it to pick out precisely the intended models.

Technical Results

We present the definitions of 'points' in a model of the Allen-Hayes axioms, and analyse the models of the axioms.
The following lemma is due to Allen and Hayes:

Lemma 1 *The relation $\|$ is irreflexive, asymmetric and atransitive.*

The next lemma shows that axiom **M5** is dispensable:

Lemma 2 *The axiom* **Existential M5** *is a consequence of the axioms* **M1 - M3**.

The next lemma shows that the function introduced in **Functional M5** is dispensable. (This is just the theorem of *Function Introduction* in *[ManWal85]*, known as *Skolemisation* to model theorists *[ChaKei73]*).

Lemma 3 (Skolemisation) *: Every model of the axiom* **M5** *in the existential form may be extended (by adding a function) to a model of the axiom* **M5** *in the operator form.*

We define the four-argument predicate that generates the equivalence relation on pairs of meeting intervals.

Define *Equiv(p, q, r, s)* if and only if $p \| q$ & $r \| s$ & $p \| s$.

We use the notation *[p, q]* for the pair of intervals p and q, whenever $p \| q$. The notation thus includes an implicit assertion of $\|$. We shall write *Equiv(p, q, r, s)* as *[p, q]* $\sim$ *[r,s]*. Using our notation, we could define *Equiv(p, q, r, s)* by the biconditional: *[p, q]* $\sim$ *[r,s]* if and only if $p \| s$. Technically, the notation [p, q] is only a convenience, and assertions involving terms of this form and $\sim$ are just shorthand for assertions involving the 4-ary relation *Equiv*. The next lemma uses this shorthand.

Lemma 4 ($\sim$ is an Equivalence Relation) *:*
(a) [p, q] $\sim$ *[p, q]*
(b) [p, q] $\sim$ *[r, s]* $\Leftrightarrow$ *[r, s]* $\sim$ *[p, q]*
(c) [p, q] $\sim$ *[r, s]* $\sim$ *[u,v]* $\Rightarrow$ *[p, q]* $\sim$ *[u, v]*

We call the equivalence classes *pointclasses*, and we denote the equivalence class of [p, q] by *[[p, q]]*. They will represent the 'points' in any model of the axioms $\mathcal{I}_{\mathcal{AH}}$.

Define the 4-ary relation *PointLess(p, q, r, s)* as follows: *PointLess(p, q, r, s)* if and only if

$$(\exists u, v, w)([p, q] \sim [u, v] \;\&\; [r, s] \sim [v, w])$$

PointLess is heterological; that is, it's not a pointless relation. We denote *PointLess(p, q, r, s)* by the rather more perspicuous notation *[[p, q]]* $\prec$ *[[r, s]]*. This notation is also just a convenience.

Lemma 5 ($\prec$ is linear) *$\prec$ linearly orders the equivalence classes of $\sim$*

Theorem 1 (Models I) *Given an arbitrary unbounded linear order $<$ on a set S, the intervals of S, INT(S), form a model of $\mathcal{I}_{\mathcal{AH}}$ under the definition of $\|$ given earlier. Furthermore, the ordering $\prec$ on equivalence classes of meeting intervals is isomorphic to the ordering $<$ on S.*

Sketch of Proof: If two intervals meet, they have a member of S in common. It's easy to check that the equivalence classes have the same member of S associated with each pair in the class, and that each member of S is associated with an equivalence class. To construct the required isomorphism, map [(a, b), (b, c)] to *b*. It is easy to see that $\prec$ on the classes is preserved as $<$ on S.
End of Sketch.

Corollary 1 *There are uncountably many countable models of the axioms $\mathcal{I}_{\mathcal{AH}}$.*

We shall show that the models of the theorem are the only models of $\mathcal{I}_{\mathcal{AH}}$. We accomplish this by characterising the models of $\mathcal{I}_{SUB}$, in such a way that the models of M4 are homomorphic images of these.

Lemma 6 (Endpointclasses) *For any p, there are unique equivalence classes P_1 and P_1 such that* $(\exists q)([p, q] \in P_1)$ & $(\exists r)([r, p] \in P_2)$

Summarising what we have so far: associated with any object p in a model for $\mathcal{I}_{SUB}$ is a unique pair of equivalence classes. All intervals which meet p are included in some pair in one equivalence class, as are all intervals which are-met-by one of those. In the other are included in some pair all intervals which are-met-by p, and all intervals which meet one of those. The equivalence classes are linearly ordered.

Given any model M of $\mathcal{I}_{SUB}$, form the set *M'* of pairs of equivalence classes of meeting intervals under $\sim$, and using the linear order $\prec$, form the *intervals*, and the *meets* relation on these by using the standard definition for pairs from a linearly ordered set. Call the resulting model *INT(M)*, the *interval structure* of *M*.

We can now state and prove our main result categorising the models of $\mathcal{I}_{\mathcal{AH}}$. All of them are isomorphic to their interval structures.

Theorem 2 (Models II) *INT(M) is a homomorphic image of M, and is a model of $\mathcal{I}_{\mathcal{AH}}$. Furthermore, if M is a model of $\mathcal{I}_{\mathcal{AH}}$, they are isomorphic.*

Sketch of Proof: The mapping is $p \mapsto ([[q, p]], [[p, r]])$ for any q, r that meet, respectively, are-met-by p. It's easy to check that the relation $\|$ is preserved by this mapping, and that the mapping is onto. Since this is the only primitive in the theory, this suffices for the homomorphism. To show isomorphism if **M4** is true in M, note that if $p, p' \mapsto ([[q, p]], [[p, r]])$, then $q \| p'$ and $p' \| r$ and hence $p = p'$, so the map is one-to-one.
End of Sketch.

Since the interval structures *INT(M)* are homomorphic images of each model M of $\mathcal{I}_{SUB}$, it follows that to discover the structure of models of $\mathcal{I}_{SUB}$, it suffices to look at the kernel of the homomorphism, which in each case is the equivalence relation

$$p \simeq q \text{ if and only if}$$

$$(\exists r, r', s, s')(([[r, p]], [[p, r']]) = ([[s, q]], [[q, s']]))$$

This is the equivalence relation of *'having-the-same-endpoints-as'*, and it's easy to check that the same intervals meet p as meet q, and the same intervals are-met-by p as are-met-by q, when p $\simeq$ q. Hence the number of intervals in each $\simeq$ equivalence class may be chosen independently for each equivalence class. This may be stated more precisely in the following way:
Let *endpoints(p)* be the pair ([[r, p]], [[p, r']]).
Let *MULTI-INT(M)* consist of the pairs *(endpoints(p), p)*, with the relation of $\|$ defined as
(endpoints(p), p) $\|$ *(endpoints(q), q)* if and only if $p \| q$.

It's easy to check that $p \| q$ if and only if *endpoints(p)* $\|$ *endpoints(q)*.

Lemma 7 *MULTI-INT(M) is isomorphic to M.*

The isomorphism is defined by $p \mapsto$ *(endpoints(p), p)*. Another way of constructing *MULTI-INT(M)* is simply by taking *INT(M)* and, for each $(a, b) \in$ *INT(M)*, adding an element *((a, b), p)* for each p such that *(a, b) = endpoints(p)*. This is summarised in the following theorem.

Theorem 3 (Models III) *The models of* $\mathcal{I}_{SUB}$ *are completely characterised by*
(a) the linear ordering $\prec$ *on the equivalence classes of* $\sim$;
(b) the number of elements in each equivalence class of $\simeq$.

Sketch of Proof: Given a model of the form *MULTI-INT(M)*, we define a model *M'* with the elements $((a, b), \beta)$ for each $\beta < \alpha$, where α is the cardinality (number) of the p such that *endpoints(p) = (a, b)*. Define $\|$ on this model the same way as in *MULTI-INT(M)*. We construct an isomorphism between the two models.
End of Sketch.

We have completely characterised the models of $\mathcal{I}_{SUB}$, and the models of $\mathcal{I}_{AH}$.

Extending the Theory

We now give an axiom **N1** that, added to $\mathcal{I}_{AH}$, gives *Th(INT(Q))*. Thus this axiom completes the theory $\mathcal{I}_{AH}$.

- **N1:** $(\forall p, q, r, s)$
  (*PointLess(p, q, r, s)* $\rightarrow$
  $(\exists x, y)$
  (*PointLess(p, q, x, y)* & *PointLess(x, y, r, s)*))

  N1 expresses the density of the ordering $\prec$ on point-classes. Translating it into the $\prec$ notation should make this clear.

Theorem 4 (Completion) *The theory axiomatised by* **M1 - M4, N1** *is countably categorical, with all countable models isomorphic to INT(Q), and hence is Th(INT(Q)).*

3 Summary

We have characterised the models of the Allen-Hayes axioms for time intervals, as structures of intervals over an arbitrary unbounded linear order. The characterisation shows that the Allen-Hayes axioms serve the purposes for which they were introduced. The characterisation has enabled a direct comparison of the different first-order theories of intervals. The Allen-Hayes theory is incomplete, which was intended, and is weaker than the Ladkin-Maddux-van Benthem theory. We indicated how to complete the Allen-Hayes theory. We have noted that both the Allen-Hayes theory, and the stronger complete theory, are decidable.

Acknowledgements

We thank Roger Maddux and Pat Hayes for much lively discussion, and Cordell Green, Director of Kestrel Institute, for giving me *time* to think about all this.

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