Piecewise Linear Reasoning

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Abstract

This paper describes a new technique called piecewise linear reasoning (PLR) for analyzing dynamic systems describable by finite sets of ordinary differential equations. Current qualitative reasoning programs derive the abstract behavior of a system by simulating handcrafted "qualitative" versions of the differential equations that characterize it and summarizing the results. PLR infers more detailed information by constructing and examining piecewise linear approximations of the original equations. As evidence that PLR can provide useful information to engineers, its analyses of the Lienard and van der Pol equations are presented.

I. Introduction

This paper describes a new technique called piecewise linear reasoning (PLR) for analyzing dynamic engineering systems. Engineers treat many devices as dynamic systems and model them with sets of ordinary differential equations. They derive the behavior of the devices by analyzing the associated equations. Rather than treat individual devices directly, engineers aggregate them into classes that share common sets of parameterized differential equations. They analyze device classes abstractly and instantiate the results with appropriate numbers. This approach avoids redundancy, provides global insight, and facilitates design. For example, the parameterized equation $y'(t) = ay(t)$ describes the class of one-tank devices with instantaneous mixing. From the solution, $y(t) = y_0e^{at}$, and the physical constraint $y_0 > 0$, one sees that y increases toward infinity if a is positive, remains constant if a equals 0, and decreases asymptotically to 0 if a is negative. One can design a specific one-tank device by choosing an appropriate value of a.

PLR provides engineers with information they need about parameterized systems: local properties in interesting regions as well as global properties such as stability, periodicity, limit cycles, and asymptotic behavior. For systems of linear equations, this information can be derived through straightforward mathematical analysis. Nonlinear systems, however, generally require extremely sophisticated analysis and rarely yield to any known analytic technique. The central tenet of my research is to solve this problem by sacrificing generality for tractability: constructing and examining piecewise linear approximations of nonlinear systems instead of analyzing them directly.

The next section describes the PLR methodology and the following two sections demonstrate its capabilities. The final three sections contain a review of previous work, PLR's implementation status, and conclusions.

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II. The PLR Methodology

Engineers need to know the properties of parameterized systems of differential equations that model device classes. This section explains how PLR derives that information from piecewise linear approximations of the equations. PLR can produce straightforward approximations automatically, including both examples in this paper, but the ultimate responsibility for constructing adequate approximations rests with the user. Precedents for this division of labor certainly exist. Users of numerical packages must choose appropriate algorithms, error margins, initial guesses, and step sizes. Similarly, de Kleer and Brown [Bobrow, 1985, p. 26] note that qualitative reasoning requires users to derive the confluences for systems by themselves.

PLR determines the properties of a parameterized piecewise linear system in two analysis stages: local and global. Both stages employ a phase-space representation. Local analysis derives phase diagrams for each linear subregion of a piecewise linear system. It solves the differential equations symbolically with the familiar algorithm from the theory of linear systems—Laplace transform, partial fractions expansion, and inverse Laplace transform—and invokes the QMR mathematical reasoner [Sacks, 1985] to deduces the *qualitative properties* of the solutions: signs of the first and second derivatives, discontinuities, singularities, and asymptotes. It uses this information to construct a phase diagram consisting of one or more *significant regions* on which all solutions have identical qualitative properties.

Global analysis infers the joint phase diagram for a system from the local phase diagrams through a combination of algebraic and geometric reasoning. First, it concatenates the relevant portions of the individual diagrams and determines the significant regions. Next, it tests whether trajectories can cross the boundaries between pairs of adjoining regions and summarizes the results in a transition graph whose nodes and links represent regions and possible transitions. Whenever the out-degree of a region exceeds 1, PLR attempts to split it into subregions of lower degree by case analysis. Each walk through the graph denotes a trajectory in the joint phase diagram. Loops denote trajectories that remain in one region forever, whereas longer cycles denote trajectories that continually shift between a sequence of regions. PLR completes the phase diagram by sketching trajectories for all walks.

PLR has exponential time-complexity in the number of nonlinear components in a system. It must combine the solutions of 2^n sets of linear equations to perform global analysis on a system of n piecewise linear equations each composed of two lines. The original system would have to contain n nonlinear components for this situation to arise. Engineers rarely analyze systems with large numbers of nonlinearities. Doing so is a challenge for any intelligent agent.

I will illustrate local and global analysis with two examples that frequently arise in nonlinear oscillators: the Lienard equation and the van der Pol equation. Both examples are simple enough for mathematicians to analyze directly. The solutions, described in Brauer and Nohel [Brauer and Nohel, 1969], afford a standard against which to measure PLR.

III. The Lienard Equation

The Lienard equation takes on many forms. We will discuss the version

$$y'' + y' + y^2 + y = 0 \tag{1}$$

in this section. Approximating the nonlinear term $y^2 + y$ with two lines, as shown in Figure 1 yields the piecewise linear equations

$$y'' + y' - \frac{y}{2} - \frac{1}{2} = 0 \quad \text{for} \quad y \leq -\frac{1}{2} \tag{2}$$

$$y'' + y' + \frac{y}{2} = 0 \quad \text{for} \quad y \geq -\frac{1}{2} \tag{3}$$

PLR chooses this approximation by default because it is the simplest one that passes through the extrema and zeroes of $y^2 + y$. The results generalize to any bimodal linearization that contains these points. In this example, I have chosen numeric equations for expository ease. An example of global analysis of parameterized equations appears in the next section.

Figure 2 shows the phase diagrams that local analysis constructs for equations (2) and (3). Equation (2) has the solution

$$y_1(t) = a(y_0, y_0')e^{-\frac{\sqrt{3}+1}{2}t} + b(y_0, y_0')e^{\frac{\sqrt{3}-1}{2}t} - 1 \tag{4}$$

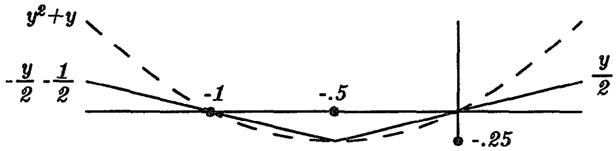

Figure 1: Piecewise linear approximation of $y^2 + y$

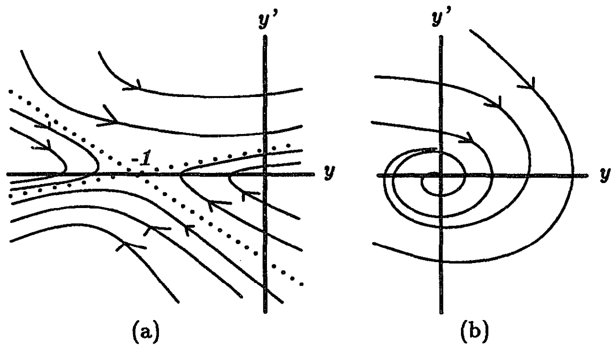

Figure 2: Phase diagrams for (a) equation (2) and (b) equation (3). The lines $a(y, y') = 0$ and $b(y, y') = 0$ are dotted.

with

$$a(y, y') = (3 - \sqrt{3})(y + 1) - 2y'/\sqrt{3} \tag{5}$$

$$b(y, y') = (3 + \sqrt{3})(y + 1) + 2y'/\sqrt{3} \tag{6}$$

where y_0 and y_0' denote the initial values of y and y'. Let us abbreviate $a(y_0, y_0')$ by a_0 and $b(y_0, y_0')$ by b_0. The function y_1 has four possible behaviors depending on the signs of a_0 and b_0.[2] It increases monotonically toward infinity for a_0 negative and b_0 positive, since the a_0 term increases toward 0 and the b_0 term increases toward infinity. Inspection of derivatives establishes that y_1 decreases to a minimum then increases toward infinity for a_0 and b_0 positive. By similar reasoning, it decreases toward negative infinity for a_0 positive and b_0 negative and increases to a maximum then decreases toward negative infinity for both negative. Two lines delimit the significant regions in which these four behaviors occur: a_0 is positive for points (y_0, y_0') below the line $a(y, y') = 0$ and negative for points above it, whereas b_0 is negative for points below the line $b(y, y') = 0$ and positive for points above it. The remaining analysis of equations (2) and (3) is analogous.

Figure 3 contains the transition graph and phase diagram produced by global analysis. The significant regions are labeled A–E with region E subdivided into E_1, E_2, and

[2] This discussion excludes the following degenerate cases: if a_0 and b_0 both equal 0, y_1 equals -1 identically; if a_0 equals 0, y_1 moves away from -1 along the line $b(y, y') = 0$; and if b_0 equals 0, y_1 approaches -1 along the line $a(y, y') = 0$.

E_3. Trajectories in region A cannot cross into any other region. Trajectories in region B cross into region E because they increase toward infinity in the y direction. If the height, h, at which a trajectory crosses into E is less than h_1, it enters E_1 and remains there forever, spiraling around the origin. For h between h_1 and h_2, the trajectory crosses into region E_2 then into region C then back into E_1 for good. It cannot cross from region C to E_2 or E_3 because the upper boundary of C, the line $a(y, y') = 0$, intersects the boundary of E below h_1. For h greater than h_2, the trajectory crosses from region B to E_3 then enters region D and remains there. After performing this analysis, PLR sketches the phase diagram. We can verify the results by comparing Figure 3 with Figure 4, the phase diagram for the original Lienard equation (1), as given by Brauer and Nohel [Brauer and Nohel, 1969, p. 220].

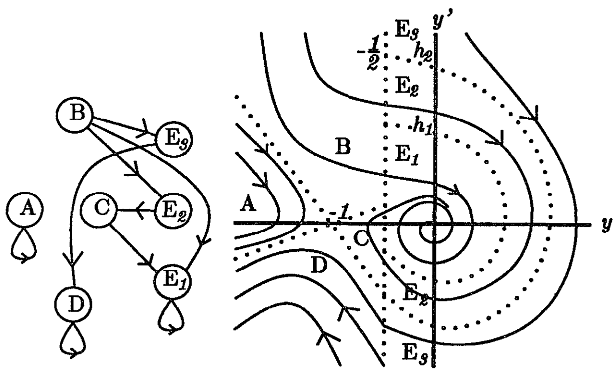

Figure 3: Transition graph and phase diagram for the piecewise Lienard equations

IV. The Van der Pol Equation

Van der Pol equations often arise in oscillatory dynamic systems. Figure 5 depicts a simple example from network theory: a capacitor, an inductor, and a nonlinear resistor connected in series. By Kirchoff's laws, the current through the circuit, I, obeys the equation

$$I'' + \frac{k}{L}(3I^2 - 1)I' + \frac{1}{LC}I = 0 \tag{7}$$

with C the capacitance, L the inductance, and k a positive scaling factor. Intuitively, the system oscillates because the nonlinear resistor adds energy to the circuit at low currents and drains energy at high currents. One obtains a piecewise linear approximation of equation (7) by replacing the nonlinear resistor model with a piecewise linear one, as illustrated in Figure 6. The analysis of the resulting equations

$$I'' - \frac{2k}{3L}I' + \frac{1}{LC}I = 0 \quad \text{for} \quad |I| \leq \frac{1}{\sqrt{3}} \approx .58 \tag{8}$$

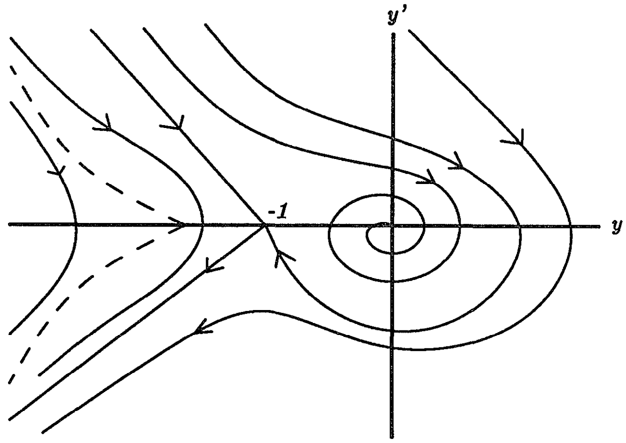

Figure 4: Phase diagram for the actual Lienard equation.

$$I'' + \frac{2k}{3L}I' + \frac{1}{LC}I = 0 \quad \text{for} \quad |I| > \frac{1}{\sqrt{3}} \tag{9}$$

follows the general pattern described in the previous section, although the symbolic parameters, k, L, and C, complicate the process somewhat. PLR must consider two cases, depending on whether the characteristic equations have real or complex roots. I will discuss only real roots; the complex case is similar.

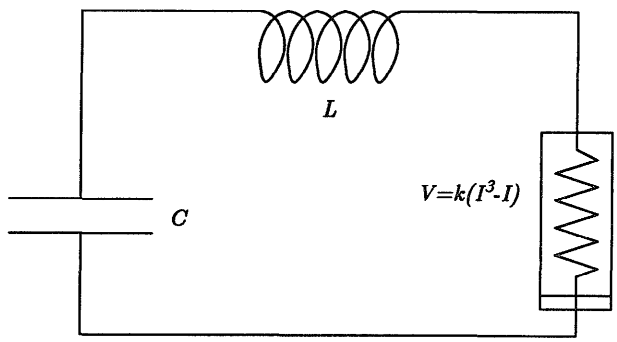

Figure 5: A circuit governed by van der Pol's equation

In the case of real characteristic roots, equation (8) has positive roots, while equation (9) has negative ones. Figure 7 depicts the complete phase diagrams for both equations along with their regions of applicability. Equation (8) holds in region G, while equation (9) holds in F and H. As with the Lienard equation, PLR infers the joint phase diagram (Figure 8) from the individual ones by constructing a transition graph. The significant regions are F, G, and H with region G subdivided into G_1 above the I

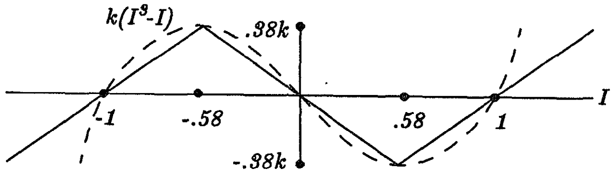

Figure 6: Piecewise linear approximation of $k(I^3 - I)$

axis and G_2 below. Figure 7b shows that trajectories in region F of the joint phase diagram eventually enter G_1. From there, they go up and right until they enter H. Similarly, Figure 7a shows that trajectories in region H eventually enter G_2 and continue down and left into F. Hence, the transition graph consists of a single cycle. The phase diagram contains a unique limit cycle toward which all non-periodic trajectories spiral, but PLR currently lacks the tools to derive this fact.

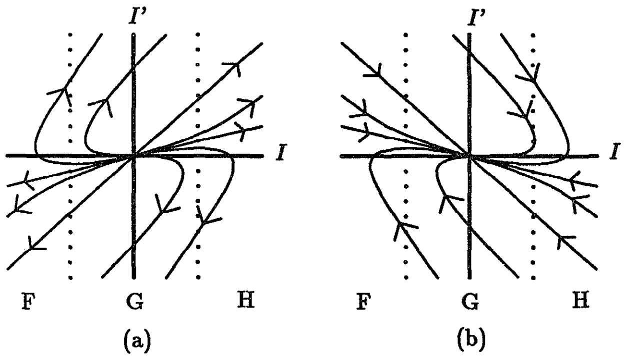

Figure 7: Phase diagrams for (a) equation (8) and (b) equation (9)

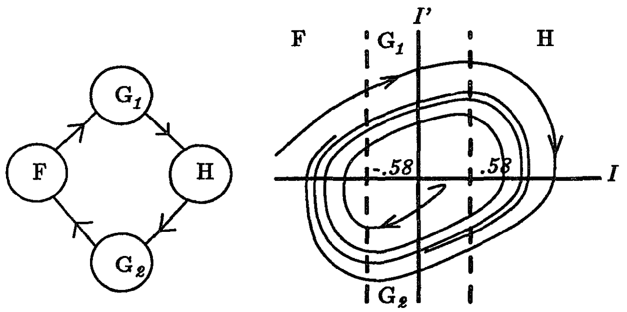

Figure 8: Transition graph and phase diagram for the piecewise van der Pol equations

V. Previous Work

Commonly used tools for analyzing nonlinear device models fall into the following categories: theoretical methods, experimentation, numeric simulation, piecewise linear approximation, and qualitative reasoning. Although theoretical methods can be extremely powerful, engineers try to avoid them because of their complexity and limited applicability. Experiments and simulations yield low-level, numeric data about individual devices. Engineers must interpret the data and generalize the results to device classes. This process becomes difficult for systems containing many parameters. In interpretation, engineers can miss important properties of the model for lack of raw data. For example, discontinuities and extrema might occur between the observed or simulated points, while asymptotes may arise beyond their range. Engineers can also overlook important properties due to the sheer volume of raw data. Generalization can fail too, since a model need not behave in a certain manner for *all* parameter values just because it does so for certain ones.

The third method of analyzing a nonlinear system consists of constructing a piecewise linear approximation, simulating it for various parameter values, and scrutinizing the results. Piecewise linear approximation offers a convenient representation for nonlinear engineered systems. However, analysis by simulation and scrutiny suffers from the same limitation as experimentation and simulation: it provides raw data about individual devices rather than abstract properties of device classes. PLR exploits the piecewise linear representation, but replaces the simulation algorithm with one that derives higher-level information.

Qualitative reasoning [Bobrow, 1985] (QR) derives the abstract behavior of dynamic systems by simulating handcrafted "qualitative" versions of their differential equations and summarizing the results. In its current form, QR falls far short of telling an engineer what he needs to know about a nonlinear system. It can only provide extremely abstract descriptions, such as "the quantity f increases for a while, reaches a maximum, and decreases thereafter." More information is required to design, analyze, and debug actual devices: local properties in interesting regions such as estimates of maxima, minima, and rates of change as well as global properties such as stability, periodicity, limit cycles, and asymptotic behavior. QR abstracts away the details required to derive this information by representing dynamic systems with confluences instead of differential equations. It cannot even express many functional properties that engineers find useful, such as linearity, exponential decay, asymptotic approach, oscillation, damped oscillation, stability and limit cycles.

QR also generates spurious behaviors. One cause, described by Kuipers [Kuipers, 1985b], is the local character of its analysis. In addition, the abstract nature of confluences introduces ambiguities that differential equations preclude. For example, the equation $y' = y - y^2$ implies that y' is negative whenever y exceeds 1, whereas the corresponding confluence leaves the sign completely ambiguous.[3] Consequently, QR concludes that y can increase toward infinity, even though it is bounded from above. Kuipers [Kuipers, 1985a] notes that this type of ambiguity crops up in almost every clinical system of second-order or higher. The same result holds for other domains. The problem is that QR focuses on the abstract behavior of extremely general systems, whereas engineers require detailed information about more-specific ones. It might be possible to attain this level of detail with an extended version of QR that included a richer set of confluences and stronger analysis algorithms. I have found it more promising to extend the piecewise linear approach, although PLR takes ideas from QR as well.

VI. Implementation Status

The local analysis component of PLR is fully implemented, but the global one is under development. The primary component of global analysis is a program that derives transition graphs from the results of local analysis. A second program will help users construct piecewise linear approximation of nonlinear systems. It will construct an initial approximation to a system $y' = f(y)$ by connecting the zeroes and extrema of each f_i with lines. This model will be refined when additional significant points arise in analysis or are specified externally. I will also extend the QS qualitative sketcher [Sacks, 1987b] to draw phase diagrams of transition graphs. The local analysis program consists of an algebraic simplifier, the QMR mathematical reasoner, and the BOUNDER inequality reasoner [Sacks, 1987a]. PLR invokes QMR to derive the qualitative properties of the solutions to systems of linear differential equations and answer questions about the results. Every component of PLR uses BOUNDER.

VII. Conclusions

This paper has described piecewise linear reasoning, a technique for deriving the properties of nonlinear dynamic systems from their piecewise linear approximations. The purpose of PLR is to enable computers to analyze man-made devices at an appropriate level of detail for engineering applications. The fact that engineers normally find piecewise linear equations adequate for modeling devices provides empirical support for the PLR paradigm. As more concrete evidence, I have demonstrated its analyses of the Lienard and van der Pol equations. Although mathematicians can handle both equations analytically, the prospect of encoding the prerequisite creativity, sophistication, and knowledge is daunting. PLR provides an algorithmic alternative. It can also help understand nonlinear systems that defy known analytic techniques.

[3]Raiman [Raiman, 1986] addresses a special case of this problem by incorporating assertions of the form "quantity a is negligible in relation to quantity b" into QR. His extension does not solve our example because neither y^2 nor y is negligible with respect to the other.

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