Global Filters for Qualitative Behaviors[1]

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Abstract

Current methods in qualitative physic sometimes predict behaviors of physical systems that do not correspond to any real-valued solution. One reason is that the merging of distinct behaviors cannot be avoided by local criteria. It is necessary to determine the possible continuations of a qualitative behavior taking into account its **complete history**. Such **global** criteria for the partial elimination of spurious solutions are developed for 2nd order differential equations. The application of these filters is shown to reduce the set of behaviors for the mass-spring system predicted by other qualitative physics systems.

1 Introduction

Current methods in qualitative physics (QP) sometimes predict behaviors of physical systems that do not correspond to any real-valued solution.
The existence of spurious solutions and their origins have been analyzed for the treatment of differential equations [Kuipers, 86], [Schmid, 88] as well as of algebraic equations [Struss, 87], [Struss, 88a]. For the first case, the reason for the prediction of spurious behaviors is the **local** nature of the criteria for determining state changes.
This paper attempts a continuation and a refinement of this analysis. The problem is: how can we determine the possible continuations of a qualitative behavior taking into account its complete history? Such **global** criteria for the elimination of spurious behaviors are developed for 2nd order differential equations.

2 Outline of the Paper

The following section presents two simple questions for demonstrating limits of the current QP approaches that are mainly used for infering qualitative behaviors. In section 4, their common basis is formally described. A fundamental problem in behavior generation is discussed in section 5: the merging of different solutions. For this purpose, a brief introduction to the analysis of the so-called phase portrait of 2nd order differential equations is given. These techniques are then used in the next section to construct some necessary conditions for filtering out spurious behaviors. The application of these methods is demonstrated by partially answering the questions of section 3. Due to limitations of space, this presentation is restricted to the key ideas and sample examples. For more details, formal definitions, proofs and solutions, see [Struss, 88b].

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3 Some Questions

We consider examples from the problem class "mass on a spring", which is the "Tweety of qualitative physics" (Fig. 1).

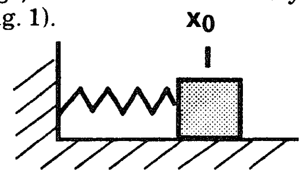


Figure 1 Mass on a spring

Question 1: If the mass is moved away from the equilibrium point, $x = 0$, given by the rest length of the spring, to $x = x_0 > 0$, will the mass, after one oscillation, return to exactly x_0, exceed it, or turn back before ?
[Kuipers, 86] showed that qualitative simulation in the style of QSIM cannot answer this question for the frictionless case based merely on the corresponding differential equation. It derives 3 branches of possible behavior after one oscillation, and worse :

Question 2: What can we tell about the sequence of the maxima during the oscillation ?
Since the above argument for the first oscillation applies to each oscillation, we get further branching in the course of the qualitative simulation (by more than a factor of 3, because new landmarks are introduced). The amplitude is allowed to change arbitrarily over time. This also applies to cases with friction.

4 Infering Qualitative Behavior

By (qualitative) behavior we mean the (qualitative) changes in characteristics of a physical system over time. We assume that a system is described by a finite number of characteristic **parameters** p_i. In the quantitative case, they are real-valued functions. Some of them may be the derivatives of others. In QP, these parameters take on **qualitative values**, i.e. essentially neighboring open intervals and the "**landmarks**" separating them, forming **quantity spaces** Q_i which are possibly parameter-specific.

A **qualitative state** is a tuple of qualitative values $(q_1, ..., q_n)$ for the parameters.

Let S denote the set of all states. **State transitions** are pairs of states forming a relation $T \subset S \times S$, where $(s,s) \notin T$.

Finally, a **behavior** is a sequence of states $b = (..., s_{-n}, ..., s_0, s_1, ..., s_n, ...) \in B$, either finite or infinite on one or both sides.

For a behavior $b = (..., s_{n-1}, s_n)$, finite on the right hand side, a (forward) **continuation** is a behavior $b_c = (..., s_{n-1}, s_n, s'_{n+1})$. Now we state our general problem:

Given a description of a system S, and an admissible behavior b, determine the admissible continuations of b.

("admissible" intuitively means "corresponding to a real solution"). Hence, QP systems have to solve three subtasks (which are not necessarily separate, subsequent steps): If b is the empty tuple, this is

- **Filtering states:** Determine the possible qualitative states of a system, i.e. sets of qualitative values that satisfy the **equations** of its description.

For $b = (s_1)$ we have the task of

- **Filtering state transitions:** Determine the possible state transitions, i.e. changes from one state to another that are in accordance with the **derivative relations** and **continuity conditions**.

This reflects: admissible behaviors can only contain admissible states and state transitions:

$B_a \subset \{b = (s_i) \in B \mid \forall_i\ s_i \in S_a \ \wedge \ (s_{i-1}, s_i) \in T_a\}$

where the subscript "a" denotes "admissible".

In this paper we are mainly interested in the remaining case, which has not yet been tackled in a satisfactory way:

- **Filtering behaviors:** Determine the possible behaviors, i.e. "correct" sequences of states (or transitions).

The existing QP methods offer **no criteria** for checking the global correctness of these sequences, and, hence, they have to assume that each path through the state transition graph is an admissible behavior.

5 Filtering Behaviors

The generation of spurious behaviors has different sources [Struss, 88b]. One of them is the merging of different system instances or different behaviors of one system instance. In order to "see" this, we are

5.1 Making Differential Equations Visible: The Phase Portrait

For the subsequent analysis, we briefly introduce some basic ideas from a mathematical discipline called qualitative theory of dynamic systems (see e.g. [Andronov, 66]). In this theory, qualitative results about the solution space of differential equations are gained by applying topological methods. This is possible because of a correspondence between sets of differential equations and vector fields.

Consider again the mass on the spring. This system is described by some 2nd order differential equation

(5.1) $$d^2x/dt^2 = -M_0^+(x)$$

or the equivalent system of first order equations

(5.2) $$dx/dt = v \qquad dv/dt = -M_0^+(x)\ ,$$

where M_0^+ is a monotonic function with $M_0^+(0) = 0$.

(5.2) defines a vectorfield in the (x,v)-plain by mapping each point (x_0, v_0) of this plain to the vector of the derivatives in this point:

$$(dx/dt|_{(x_0,v_0)}, dv/dt|_{(x_0,v_0)}) = (v_0, -M_0^+(x_0)).$$

Solutions of (5.2) then correspond to those curves ("**trajectories**") in the plain that in each point have a tangent in the direction of the respective vector. Fig. 2 shows the construction of this vector field and a part of sample trajectory, t_0, which corresponds to a damped oscillation.

The collection of these trajectories, the so-called **phase-portrait** of the system, looks locally "in principle" like parallels (except for equilibrium points). This implies that trajectories do not intersect or branch etc. (see Fig. 3).

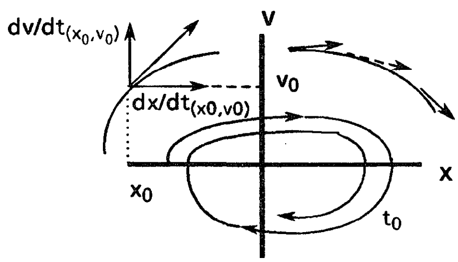


Figure 2 Construction of the phase portrait

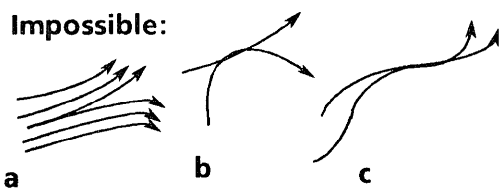
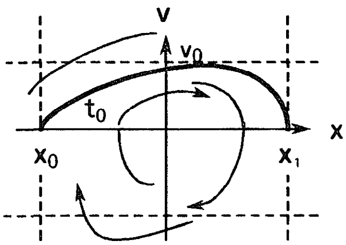


Figure 3 Impossible phase portraits

Some main charactereristics of the phase portrait of system (5.2) are indicated by Fig. 4. It expresses the oscillatory behavior, but does not decide upon the question whether the system really exhibits a **cyclic** behavior. Starting at an arbitrary point $(x_0, 0)$ on the negative x-axis, the respective trajectory, t_0, first stays in the quadrant $x<0$, $v>0$, then, intersecting the positive v-axis, continues in the quadrant $x>0$, $v>0$ and leaves it by reaching some point $(x_1, 0)$ on the x-axis.

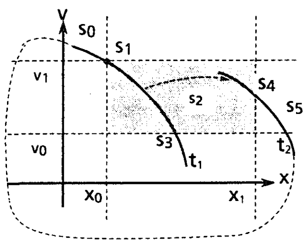
How does this representation relate to the description of the qualitative behaviors derived by QP methods? The quantity spaces for x and v impose a grid on the plain. Each of the rectangles correspond to 9 qualitative states: Its corners represent the states where both x and v take on landmark values (e.g. s_0 in Fig. 4), its interior is a state where x and v are between landmarks (e.g. s_1), and the edges exclusive of their endpoints (the corners) are states with only one variable crossing a landmark (e.g. s_2). Each trajectory defines a behavior in the sense of section 4, namely the sequence of states it crosses (Fig. 4).



Sequence of states for t_0

	x	v
s_0	x_0	0
s_1	$(x_0,0)$	$(0,v_0)$
s_2	0	$(0,v_0)$
s_3	$(0,x_1)$	$(0,v_0)$
s_4	$(0,x_1)$	v_0
s_5	$(0,x_1)$	$(0,v_0)$
s_6	x_1	0

Figure 4 Definition of a behavior by a trajectory



states

	x	v
s_0	$(0,x_0)$	(v_1,v_2)
s_1	x_0	v_1
s_2	(x_0,x_1)	(v_0,v_1)
s_3	(x_0,x_1)	v_0
s_4	x_1	(v_0,v_1)
s_5	(x_1,x_2)	(v_0,v_1)

Figure 5 Merging different behaviors

5.2 Merging Different Solutions

Consider the pieces of the trajectories, t_1 and t_2, in Fig. 5. They introduce the admissible state transitions (s_0, s_1), (s_1, s_2), (s_2, s_3), (s_2, s_4), (s_4, s_5). Having constructed a behavior $b_1 = (s_0, s_1, s_2)$ (which is admissible, since it represents a piece of t_1), we have (at least) two possible continuations for b_1, $b_2 = (s_0, s_1, s_2, s_3)$ and $b_3 = (s_0, s_1, s_2, s_4)$. Neither of them can be ruled out by the step of state transition filtering, although only b_2 corresponds to a trajectory of the **specific system under consideration**, whereas b_3 merges two solutions of the system with different initial conditions. This is the reason why question 1 in section 3 is answered by **QP** with a branching of the behavior. It was correctly identified in [Kuipers, 86] as the merely local nature of the state transition filtering. We can not unambiguously infer a damped behavior for the occurance of friction: Fig. 5 indicates that if the damped trajectory t_2 returns to state s_2 as the arc t_1, the existing filters do not forbid a "jump" back to t_2 and, hence, a spurious cyclic behavior.

5.3 Merging Distinct Systems

Of course, we can argue that the behavior $b_3 = (s_0, s_1, s_2, s_4)$ may be not admissible for the system sketched in Fig 5. However, the qualitative description covers a whole family of systems. Could there not be an appropriate choice of a parameter such that b_3 is admissible for the corresponding system (and b_2 is not) ? Yes, this might happen, although it is hard to prove it for a specific case. But, subsequent continuations of b_3 may require choices between state transitions that imply a different choice for the range of the parameter. Hence, in combining admissible state transitions we are not prevented from jumping between **different instances of a class of systems and merge their behaviors** thus potentially generating behaviors which are not admissible for any single system.

In the following section, we construct some filters for sequences of state transitions.

6 Exploiting the Phase Portrait

We do so mainly by taking advantage of the property that trajectories cannot intersect, because otherwise we would get different solutions for the same initial conditions. The filters apply to 2nd order differential equations. We have to emphasize that this choice has not only been made for the sake of simplicity of the examples. It mainly reflects the fact that only for this case (i.e. vectorfields in the plain), we expect strong results (with the 3rd dimension, chaos starts).

6.1 Avoidance of System Merging

Filters aiming at this goal need to identify behaviors that do not correspond to solutions of the same system. We introduce a symmetric binary relation **exclusive**$\subset B \times B$. exclusive(b_1, b_2) means that there exists no system instance for which both behaviors, b_1 and b_2, are admissible.

We know that different trajectories passing through one point of the plain cannot belong to the same system (Fig. 3). The problem is how to recognize the respective behaviors, which run through qualitative states instead of points ?

One case is easily solved, namely if states are involved that correspond to points. Hence we call a state $s = (q_1, q_2)$ a **landmark state** if all the q_i are landmarks. If exactly one of the q_i is a landmark, we call it **edge state**.

We define the relation **convergent** $\subset B \times B \times S$:

convergent (b, b', s_0)

$:\Leftrightarrow b = (\ldots, s_{-1}, s_0) \wedge b' = (\ldots, s'_{-1}, s_0) \wedge s_{-1} \neq s'_{-1}$

and analogously a relation **divergent** for $b = (s_0, s_1, \ldots), b' = (s_0, s'_1, \ldots)$. Then we have the

Proposition 6.1

If b and b' are convergent to or divergent from a landmark state, or an edge state, then exclusive (b, b') holds.

If two behaviors converge to a state that is not an edge or landmark state, it appears to be difficult to detect a case like the one shown in Fig. 3 b. There is some hope, however, to catch situations like Fig. 3 c on the qualitative level. If a behavior b approaches another one, b', "from the left" and leaves it "to the right", we have the relation **crossing**(b,b'). Assume we have ways to check whether this relations holds, then we can make use of it by the

Proposition 6.2

Crossing behaviors are exclusive :

crossing (b,b') $\Rightarrow$ exclusive (b,b').

We may detect spurious behaviors with the obvious

Proposition 6.3

exclusive (b, b) $\Rightarrow$ spurious(b).

For example, a behavior crossing itself is spurious.

Exclusiveness (and crossing) is monotonic w.r.t. **behavior inclusion** : $b \subset b'$ is true, when there is a sequence of continuations of b establishing b'.

Proposition 6.4

$b_1 \subset b'_1 \wedge$ exclusive$(b_1, b_2) \Rightarrow$ exclusive(b'_1, b_2),

$b_1 \subset b'_1 \wedge$ crossing$(b_1, b_2) \Rightarrow$ crossing(b'_1, b_2)

and, hence,

$b_1 \subset b \wedge b_2 \subset b \wedge$ exclusive(b_1, b_2)

$\Rightarrow$ spurious(b)

Sometimes we can infer the negation of exclusiveness, i.e. for some b, b' there exists a system allowing both b and b' (see section 6.3). We are allowed to combine behaviors only if they belong to the same solution. For $b = (\ldots, s_{-2}, s_{-1}, s_0)$ and $b' = (s_0, s_1, s_2, \ldots)$, we define the **behavior union**

$b \cup b' = (\ldots, s_{-2}, s_{-1}, s_0, s_1, s_2, \ldots)$.

Proposition 6.5

Let s_0 be a landmark state and $b = (\ldots, s_{-2}, s_{-1}, s_0)$ and $b' = (s_0, s_1, s_2, \ldots)$, then

$\neg$ exclusive (b, b') $\Rightarrow$ $\neg$ exclusive $(b \cup b', b)$ $\wedge$ $\neg$ exclusive $(b \cup b', b')$

Before demonstrating the use of these criteria, we have to provide a way to check the crossing relation.

6.2 Detecting Crossing Behaviors

As stated above, we hope to check qualitatively whether a behavior approaches another one from one side and leaves it towards the other side. In between, they may share a sequence of states.

Proposition 6.6

Let $b = (\ldots, s_{c-1}, s_c, \ldots, s_d, s_{d+1}, \ldots)$ and $b' = (\ldots, s'_{c-1}, s_c, \ldots, s_d, s'_{d+1}, \ldots)$.

convergent-left(b, b', s_c)

$\wedge$ divergent-right$(b, b', s_d) \Rightarrow$ crossing(b, b').

We will demonstrate the algorithm for checking the relation convergent-left denoting that the first behavior joins the second from the left. The other check is then obvious. Because of Proposition 6.1 we need only consider the case, where s_c is the interior of a rectangle. Such a state can be entered from at most 8 neighboring states An example is shown in Fig. 6. Fig. 6a shows the trajectories in the landmark grid. In Fig. 6b, the same situation is transfered to the state space under preservation of the topology of the plain. b' is assumed to proceed from s_c to s_3.

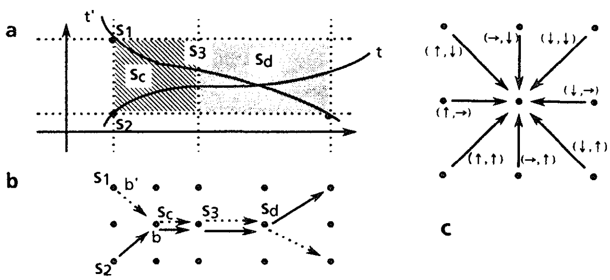


Figure 6 a Crossing trajectories
b crossing behaviors
c Transitions and their Δ

We characterize each of the eight transitions, (s, s_c) by $\underline{\Delta}(s, s_c)$ which is the pair of the changes, Δ, of the respective qualitative values (x and v, in our case):

$\underline{\Delta}(s, s') = \underline{\Delta}((x,v), (x', v'))$

$= (\Delta(x, x'), \Delta(x, v'))$, where

$\Delta(x, x') = \downarrow \quad :\Leftrightarrow \quad x' < x$

$\Delta(x, x') = \rightarrow \quad :\Leftrightarrow \quad x' = x$

$\Delta(x, x') = \uparrow \quad :\Leftrightarrow \quad x' > x$

In our example, we have $\underline{\Delta}(s_1, s_c) = (\uparrow, \downarrow)$ and $\underline{\Delta}(s_c, s_3) = (\uparrow, \rightarrow)$. Running counterclockwise through the state transitions to s_c (see Fig. 6 c) corresponds to stepping through the circular list

$L_T = ((\uparrow, \uparrow), (\rightarrow, \uparrow), (\downarrow, \uparrow), (\downarrow, \rightarrow), (\downarrow, \downarrow), (\rightarrow, \downarrow), (\uparrow, \downarrow), (\uparrow, \rightarrow), (\uparrow, \uparrow), \ldots)$

Now, we formulate our **criterion** :

Let $b = (s_1, s_c, \ldots)$ and $b' = (s_2, s_c, s_3)$ with convergent(b, b', s_c). If $\underline{\Delta}(s_1, s_c)$ is in the sublist of L_T that is started by $\underline{\Delta}(s_3, s_c)$ and ended by $\underline{\Delta}(s_2, s_c)$, then convergent-left (b, b', s_c).

In the example, we start at $\underline{\Delta}(s_3, s_c) = (\downarrow, \rightarrow)$ and end in $\underline{\Delta}(s_2, s_c) = (\uparrow, \uparrow)$. $\underline{\Delta}(s_1, sc) = (\uparrow, \downarrow)$ is between these elements, hence, the criterion is satisfied.

Since divergent-right can be checked in a similar way, we are now able to detect crossing behaviors. We demonstrate that some progress is achieved by answering questions of section 3.

6.3 Symmetry - Infering Cyclic Behavior

Consider again question 1 of section 3. We are now able to infer the cyclic behavior of all solutions to (5.2). The idea is the following: Looking at (5.2), we realize that the application of the transformations $t'=-t$ and $v'=-v$ leads to

$$(6.1) \quad dx/dt' = v' \qquad dv'/dt' = -M_0^+(x) ,$$

which is of the same form as (5.2). This means we are able to derive the phase portrait in the half plain $v < 0$ by merely mirroring the $v > 0$ half plain at the x-axis (and reversing the orientation of the trajectories). Hence the trajectory continuing beyond $(x_1, 0)$ is the mirror image of the curve we started with and therefore hits the x-axis again at $(x_0, 0)$ thus establishing a closed curve (Fig. 7) that corresponds to oscillation with a constant amplitude.

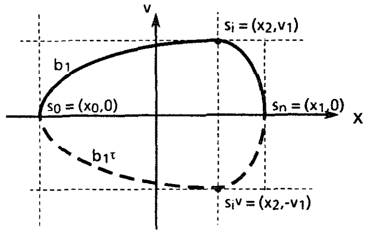


Figure 7 Symmetry of behavior

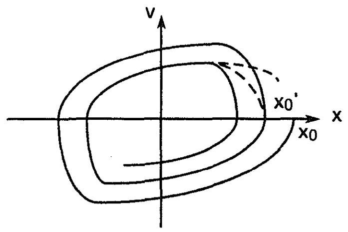


Figure 8 Damped oscillation

In the framework introduced in this paper, we can express the following general principle:

Consider two systems sys_1 and sys_2 and the transformations $\tau_t : t \rightarrow -t,$, $\tau_v : v \rightarrow -v$.

Proposition 6.7

Let $sys_2 = \tau_v(\tau_t(sys_1))$ and for $s = (x,v)$, $s^v := (x,-v)$.
$b = (\ldots, s_1, s_2, \ldots) \wedge b^\tau = (\ldots, s_2^v, s_1^v, \ldots)$
$\Rightarrow \neg$ exclusive(b, b^τ).

If we have generated a behavior $b_1 = (s_0, s_1, \ldots, s_n)$ (see Fig. 7), we know for $b_1^\tau = (s_n, s_{n-1}^v, \ldots, s_1^v, s_0)$ that $\neg$ exclusive(b_1, b_1^τ) holds. (Note that $s_0 = s_0^v$, $s_n = s_n^v$, because $v=0$). Since s_n is a landmark state, Proposition 6.5 then yields

Corollary 6.8

For the system (5.2), $\neg$ exclusive$(b_1 \cup b_1^\tau, b_1)$, and $b_1 \cup b_1^\tau$ is cyclic!

6.4 Infering Steady Damping

Question 2 is concerned with the identification of a global tendency of behavior, namely with the problem of arbitrary changes in the subsequent maxima of the oscillation. This problem is solved for the frictionless case by the result of the previous section. However, it also occurs for the case with friction. Using our filter criteria, we are now able to deduce immediately that if the the oscillation is damped in the first period, it will always be damped: Let $s_0 = (x_0, 0)$ be one maximum, and $s_0' = (x_0', 0)$ with $x_0' < x_0$ the next one (Fig. 8). Since the maximum could only be increased again if the solution crosses itself, Propositions 6.3 and 6.4 detect it to be spurious. The return to s_0' is also excluded, because in this case

convergent$((s_0, \ldots, s_0'), (s_0', \ldots, s_0'))$
$\Rightarrow$ exclusive$((s_0, \ldots, s_0'), (s_0', \ldots, s_0'))$ by Prop. 6.1
$\Rightarrow$ exclusive$((s_0, \ldots, s_0', \ldots, s_0'), (s_0, \ldots, s_0', \ldots, s_0'))$ by Prop. 6.4
$\Rightarrow$ spurious$((s_0, \ldots, s_0', \ldots, s_0'))$ by Prop. 6.3.

7 Summary

Our approach to expressing restrictions imposed by 2nd order differential equations is essentially based on the uniqueness of solutions for fixed initial conditions. The criteria can be used to discriminate behaviors that belong to different system instances and to discover spurious behaviors. They enable us to derive cyclic behavior for the frictionless mass-spring system and the principle "once-damped-always-damped" for the case with friction.
Similar methods can be used, for example, to infer damping for the mass with friction [Struss, 88b].

Acknowledgements:

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