It's Not My Default: The Complexity of Membership Problems in Restricted Propositional Default Logics

Jonathan Stillman

Artificial Intelligence Program
General Electric Research and Development Center
P.O. Box 8
Schenectady, N.Y. 12301
e-mail: stillman@crd.ge.com

Abstract

We investigate the computational complexity of membership problems in a number of propositional default logics. We introduce a hierarchy of classes of propositional default rules that extends that described in [Kautz and Selman 1989], and characterize the complexity of membership problems in these classes under various simplifying assumptions about the underlying propositional theory. Our work significantly extends both that presented in [Kautz and Selman 1989] and in [Stillman 1990a].

Introduction

One of the central concerns of artificial intelligence research involves developing useful models of how one might emulate on computers the 'common-sense' reasoning in the presence of incomplete information that people do as a matter of course. Traditional predicate logics, developed for reasoning about mathematics, are inadequate as a formal framework for such research in that they are inherently *monotonic*: if one can derive a conclusion from a set of formulae then that same conclusion can also be derived from every superset of those formulae. It is argued that people simply don't reason this way: we are constantly making assumptions about the world and revising those assumptions as we obtain more information (see [McCarthy 1977] or [Minsky 1975], for instance). Many researchers have proposed modifications of traditional logic to model the ability to revise conclusions in the presence of additional information (see, for instance, [McCarthy 1986], [Moore 1983], [Poole 1986]). Such logics are called *nonmonotonic*. Informally, the common idea in all these approaches is that one may want to be able to "jump to conclusions" that might have to be retracted later. While a detailed discussion of nonmonotonic logics is outside the scope of this paper, a good introduction to the topic can be found in [Etherington 1988], and a number of the most important papers in the field have been collected in [Ginsberg 1987].

One of the most prominent of the formal approaches to nonmonotonic reasoning, developed by Reiter ([Reiter 1980]), is based on *default rules*, which are used to model decisions made in prototypical situations when specific or complete information is lacking. Reiter's *default logic* is an extension of first order logic that allows the specification of default rules, which we will summarize shortly. Unfortunately, the decision problem for Reiter's default logic is highly intractable in that it relies heavily on consistency checking for processing default rules, and is thus not even semi-decidable (this is not a weakness of Reiter's logic alone; it is common to most nonmonotonic logics). This precludes the practical use of Reiter's default logic in most situations.

The motivation for searching for computationally tractable inference mechanisms for subclasses of *propositional* default reasoning is based on the need to reason about relatively large propositional knowledge bases in which the default structures may be quite simple. Recent research involving inheritance networks with exceptions is particularly relevant, and is explored in depth in [Touretzky 1986] and in Chapter 4 of [Etherington 1988], where the close relationship between default logic and inheritance networks with exceptions is explored.

In order to gain computational tractability of reasoning in default logic, one must restrict the expressiveness considerably. Simply restricting the logic to reasoning about arbitrary propositions results in decision problems that are at least as hard as deciding standard propositional logic, regardless of restrictions on the types of default rules allowed. Since the satisfiability problem is intractable for propositional logic, one must consider further restrictions. Recently, Kautz and Selman [Kautz and Selman 1989] and Stillman [Stillman 1990a] have investigated default logics defined over subsets of propositional calculus with various restrictions on the syntactic form of default rules allowed. A partial order of such restrictions is described in [Kautz and Selman 1989], together with discussion of the complexity of several problems over this partial order when the propositional theory is restricted to consisting of a set of literals. Several of these restrictions were shown to result in polynomial-time tests for determining whether certain properties hold given such a restricted propositional theory. In particular, it was shown that one can decide in polynomial time whether there exists an *extension* that contains a given literal when the default rules are restricted to a class they called *Horn* default rules. They suggested that the ability to combine such default theories with non-

default propositional Horn theories would be particularly useful, but left open the question of whether the membership problem (i.e., determining whether there exists an extension of a given default theory containing a specified literal) for such a combination of theories is tractable. In [Stillman 1990a], we showed that a restriction of this problem is NP-complete, and presented several related results.

The remainder of this paper is organized as follows: we begin with a brief description of Reiter's default logic, followed by a short overview of NP-completeness, and a presentation of the restrictions considered by Kautz and Selman. Following this we introduce a hierarchy of classes of propositional default rules that significantly extends that presented in [Kautz and Selman 1989]. Next, we characterize the complexity of the membership problem for these classes. Finally, we summarize the results presented in this paper, and discuss related results and future work.

Preliminaries

Reiter's Default Logic

For a detailed discussion of Reiter's default logic the interested reader is referred to [Reiter 1980]. In this section we will simply review some of the immediately pertinent ideas. A *default theory* is a pair (D, W), where W is a set of closed well-formed formulae (wffs) in a first order language and D is a set of *default rules*. A *default rule* consists of a triple $< \alpha, \beta, \gamma >$: α is a formula called the *prerequisite*, β is a set of formulae called the *justifications*, and γ is a formula called the *conclusion*. Informally, a default rule denotes the statement "if the *prerequisite* is true, and the *justifications* are consistent with what is believed, then one may infer the *conclusion*." Default rules are written

$$\frac{\alpha : \beta}{\gamma}$$

If the conclusion of a default rule occurs in the justifications, the default rule is said to be *semi-normal*; if the conclusion is identical to the justifications the rule is said to be *normal*. A default rule is *closed* if it does not have any free occurrences of variables, and a default theory is *closed* if all of its rules are closed.

The maximally consistent sets that can follow from a default theory are called *extensions*. An extension can be thought of informally as one way of "filling in the gaps about the world." Formally, an extension E of a closed set of wffs T is defined as the fixpoint of an operator Γ, where $\Gamma(T)$ is the smallest set satisfying:

- $W \subseteq \Gamma(T)$,
- $\Gamma(T)$ is deductively closed,
- for each default $d \in D$, if the *prerequisite* is in $\Gamma(T)$, and T does *not* contain the negations of any of the *justifications*, then the *conclusion* is in $\Gamma(T)$.

Since the operator Γ is not necessarily monotonic, a default theory may not have any extensions. Normal default theories do not suffer from this, however (see [Reiter 1980]), and always have at least one extension.

There are several important properties that may hold for a default theory. Given a default theory (D, W), perhaps together with a literal q, one might want to determine the following about its extensions:

Existence Does there exist *any* extension of (D, W)?

Membership Does there exist an extension of (D, W) that contains q? (This is called *goal-directed reasoning* by Kautz and Selman.)

Entailment Does every extension of (D, W) contain q? (This is closely related to *skeptical reasoning*, where a literal is believed if and only if it is included in all extensions.)

NP-complete Problems

NP is defined to be the class of languages accepted by a nondeterministic Turing machine in time polynomial in the size of the input string. The "hardest" languages[1] in NP are called NP-complete: all such languages share the property that all languages in NP can be transformed into them via some polynomial time transformation. To show that a problem in NP is NP-complete one must demonstrate a polynomial-time transformation of an instance of a known NP-complete problem to an instance of the problem under consideration in such a way that a solution to one indicates a solution to the other. For a thorough discussion of the topic the interested reader is referred to [Garey and Johnson 1979]. The fastest known deterministic algorithms for NP-complete problems take time exponential in the problem size. It is not known whether this is necessary: one of the central open problems in computer science is whether P = NP. Most researchers believe that P $\neq$ NP, and that NP-complete problems really do need exponential time to solve. Thus these problems are considered *intractable*, since if P $\neq$ NP, we cannot hope to solve arbitrary instances of them with inputs of nontrivial size.

Restricted Default Theories

If practical reasoning systems are to be developed, one cannot ignore computational complexity. Each of the questions mentioned above is at least as hard as deciding the underlying theory W. Thus, if W consists of arbitrary first-order formulae, none of these questions is even semi-decidable, and a practical system must consider stronger restrictions. If W is restricted to arbitrary propositional formulae, each of the questions require deterministic time proportional to that needed to determine propositional satisfiability (approximately 2^n where n is the number of atoms occurring in W, using the best algorithms currently known). It is unlikely that algorithms that perform significantly better will be developed in the future, under the assumption that P $\neq$ NP. Thus, a necessary condition that must be satisfied to guarantee efficient answers to the questions posed above is that we limit ourselves to even

---

[1]NP-completeness is often discussed in terms of *decision problems* rather than languages, although the two are interchangeable.

stronger restrictions on W. The propositional theories we will consider are described below.

Propositional literals: W consists of propositional atoms and their negations. In [Kautz and Selman 1989], this restriction is assumed throughout.

Horn clauses: W consists of a conjunction of propositional clauses, each of which contains at most one positive literal.

2-literal clauses: W consists of a conjunction of propositional clauses, each of which contains at most 2 literals. This restriction is assumed in *network default theories*, an important class of default theories described in detail in [Etherington 1988].

Each of these restricted propositional theories is known to be decidable in linear time. The first case is trivial. For the second and third, see [Dowling & Gallier 1984] and [Apsvall, Plass, & Tarjan 1979], respectively. These theories provide us with a good starting point for building simple default theories. Note that while the first restriction forms a subset of each of the others, the second and third are incomparable with respect to the formulae they contain. In subsequent sections we will examine the complexity of reasoning in a number of restricted default theories. We will consider default theories for which W falls into one of the three subclasses of propositional formulae presented above. For each of these, we will consider a number of restrictions on what classes of default rules are allowed. These restrictions are discussed below.

Prior Work on Restricted Default Theories

In [Kautz and Selman 1989], Kautz and Selman presented a taxonomy of propositional default theories. They restricted W to contain only propositional literals, and restricted default rules to be semi-normal, with the precondition, justifications, and conclusions of each default rule consisted of conjunctions of literals (this restriction makes consistency checking a simple task). They also considered the following further restrictions on the default rules allowed.

Unary The prerequisite of each default must be a positive literal, and the conclusion must be a literal. If the consequence is positive, the justification must be the conjunction of the consequence and a single negative literal; otherwise, the justification must be the consequence.

Disjunction-Free Ordered The interested reader is referred to [Etherington 1987] for a formal definition of ordered default theories, which we omit here. Intuitively, in an *ordered* semi-normal default theory the literals can be ordered in such a way that potentially unresolvable circular dependencies cannot occur. The interested reader is referred to [Etherington 1987] for a formal definition of ordered default theories, which we omit here. Intuitively, in an *ordered* semi-normal default theory the literals can be ordered in such a way

that potentially unresolvable circular dependencies cannot occur.

Ordered Unary These combine the restrictions of the first two theories described above. Kautz and Selman remark that these theories appear to be the simplest necessary to represent inheritance hierarchies with exceptions (see [Touretzky 1986; Etherington 1988]).

Disjunction-Free Normal These are disjunction-free ordered theories in which the consequence of each default rule is identical to the justification.

Horn The prerequisite literals in these default rules must each be positive, and the justification and consequence are each a single literal.

Normal Unary The prerequisite in each of these default rules consists of a single positive literal, the conclusion must be a literal, and the justification must be identical to the consequence. These form the most simple class of default rule that is considered in [Kautz and Selman 1989].

These restricted theories are related in a partial order as shown in Figure 1 below. Kautz and Selman examined the extension existence, membership, and entailment questions for these theories in [Kautz and Selman 1989].
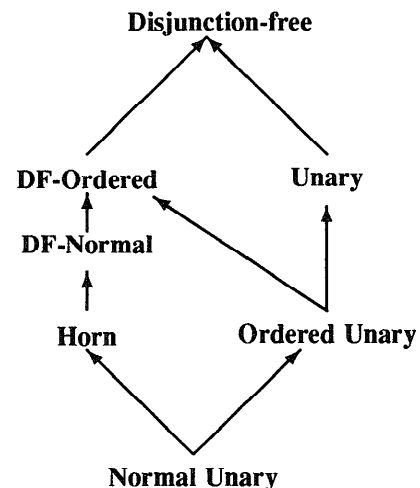

Figure 1: Kautz and Selman's hierarchy of restricted default theories.

Prompted by a gap in the characterization of restricted default theories, we showed recently in [Stillman 1990a] that the following problem is NP-complete.

Horn Clauses with Normal Unary Default Rules (HC-NU)
Instance: A finite set H of propositional Horn clauses, together with a finite set D of normal, unary, propositional default rules, and a distinguished literal q.

Question: Does there exist an extension of (D, H) that contains q ?

This result subsumed an open question cited in [Kautz and Selman 1989]: Kautz and Selman were interested in whether one could add Horn default rules to Horn propositional theories without introducing intractability. Unfortunately, our result answers this question negatively. Among other related results, we showed that the entailment problem is co-NP-complete for these default theories.

We subsequently examined even stronger restrictions on the classes of default rules allowed, hoping to find a class of rules that could be combined with Horn clauses while retaining the tractability of propositional Horn clause reasoning. We also examined the complexity of restricted default reasoning under other restrictions on the propositional theories allowed, as described above. In the following sections, we report on the results of this work.

Expanding the Horizons

Our investigation suggested a richer hierarchy of default rules, most of which result from disallowing any prerequisites in rules. This corresponds to introducing a "context-free" element to the reasoning, and seems to constitute the most simple type of default rule that is not completely trivial. In this section, we explore the complexity of membership problems in default theories in which W belongs to one of the classes of formulae listed above, and in which D belongs either to one of the classes of default rules discussed above or to one of the following:

Prerequisite-Free Disjunction-free default rules with no prerequisites.

Prerequisite-Free Unary The prerequisite of each rule is empty, and the conclusion must be a literal. If the consequence is positive, the justification must be the conjunction of the consequence and a single negative literal; otherwise, the justification must be the consequence.

Prerequisite-Free Ordered Again, the reader is referred to [Etherington 1988] for a formal definition of ordered theories; A prerequisite-free ordered theories is a disjunction-free ordered theory in which the prerequisite is empty.

Prerequisite-Free Ordered Unary These combine the restrictions of the first two theories described above.

Prerequisite-Free Normal These are prerequisite-free ordered theories in which the consequence of each default rule is identical to the justification.

Prerequisite-Free Normal Unary The prerequisite in each of these default rules is empty, the conclusion must be a literal, and the justification must be identical to the consequence.

Prerequisite-Free Positive Normal Unary The prerequisite in each of these default rules is empty, the conclusion must be a positive literal, and the justification must be identical to the consequence.

These restricted theories are related in a partial order. The hierarchy is shown in Figure 2.

Horn Clause Theories

After showing that the problem HC-NU was NP-complete, we looked for even tighter restrictions on the default rules allowed that would provide us with tractable default reasoning where the propositional theory consisted of Horn clauses. The results reported here were somewhat surprising. Unfortunately, they are also largely negative. The membership problem remains intractable under very tight restrictions. In particular, for the following problem

Horn Clauses with Prerequisite-Free Positive Normal Unary Default Rules (HC-2)
Instance: A finite set H of propositional Horn clauses, together with a finite set D of prerequisite-free positive normal, unary, propositional default rules, and a distinguished literal q.
Question: Does there exist an extension of (D, H) that contains q ?
we prove:

Theorem 1 *HC-2 is NP-complete.*

Proof: It is not difficult to demonstrate membership in NP: although the extension may be too large to describe explicitly, it suffices to provide the original set of Horn clauses, together with those default rules that were applied, and verify that the default rules form a maximal set and can actually be applied consistently. Since these are disjunction-free, this can be done efficiently.

To demonstrate NP-hardness we transform an instance of NOT-ALL-EQUAL SATISFIABILITY to one of HC-2. NOT-ALL-EQUAL SATISFIABILITY can be stated as follows.

> Given sets $S_1, S_2, \ldots, S_m$, each having 3 members, can the members be colored with two colors so that no set is all one color?

In [Shaefer 1978] it is shown that NOT-ALL-EQUAL SATISFIABILITY is NP-complete. Given an instance I of NOT-ALL-EQUAL SATISFIABILITY, let Σ be the set of all elements appearing in any S_i. For each such element σ, introduce the a new propositional atom σ, and add the following default rule to D:

$$\frac{: \sigma}{\sigma}$$

Next, for each set $S_i = \{\sigma_{i_1}, \sigma_{i_2}, \sigma_{i_3}\}$ in I introduce a new propositional atom S_i, and add the following clauses to W:

$$(\neg\sigma_{i_1} \vee \neg\sigma_{i_2} \vee \neg\sigma_{i_3})$$
$$(\neg\sigma_{i_1} \vee S_i)$$
$$(\neg\sigma_{i_2} \vee S_i)$$
$$(\neg\sigma_{i_3} \vee S_i)$$

Finally, introduce a new propositional atom q and add the following clause to W:

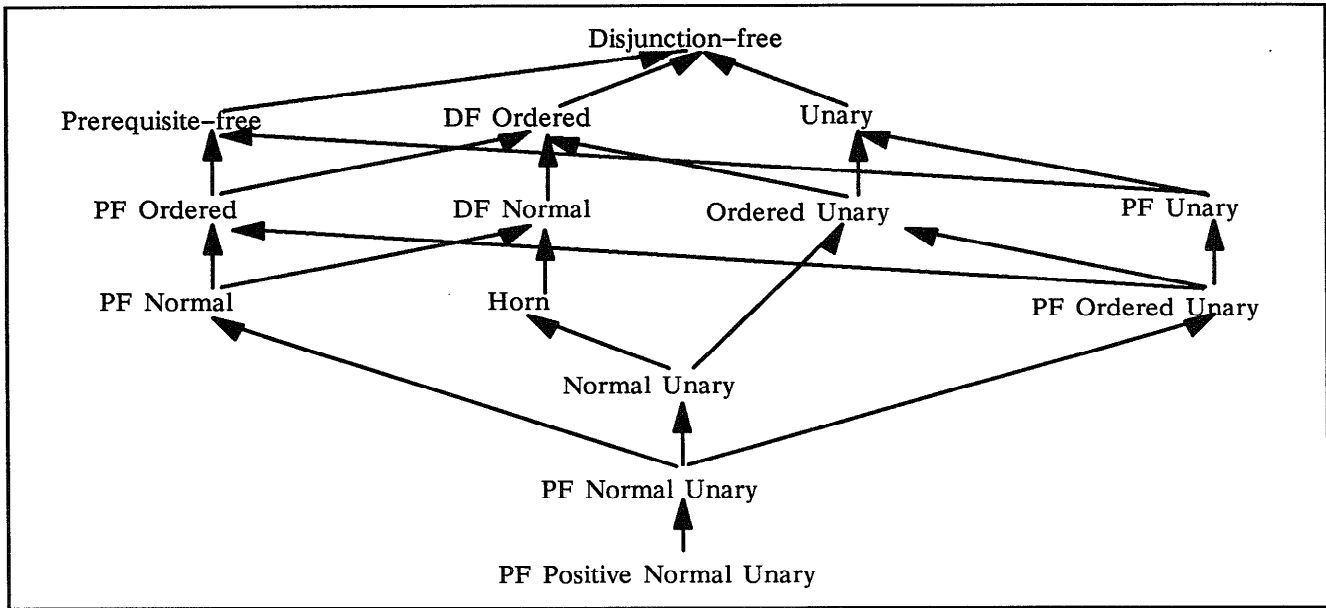$$(\neg S_1 \vee \neg S_2 \vee \ldots \vee \neg S_m \vee q).$$

Figure 2: An expanded hierarchy of default rules.

This completes the transformation, which results in only a linear increase in the size of the problem. It is a simple matter to verify that the transformed instance satisfies the restrictions on W and D, i.e., the clauses are Horn and the default rules are prerequisite-free positive normal unary. We now show that there exists an extension of (D, W) that contains q if and only if the original instance I of NOT-ALL-EQUAL SATISFIABILITY is satisfiable.

($\Rightarrow$). Suppose I is satisfiable. Then the elements of Σ must be two-colorable in such a way that none of the sets S_i has all its elements the same color. Let us assume that the two colors correspond to the truth values *true* and *false*. There must exist a satisfying assignment to the elements of Σ in which a maximal number of the elements of Σ are colored *true*. We must show that we can, given such a maximal satisfying assignment α for I, construct an extension of (D, W) that contains q.

We proceed as follows. Each of the sets in S must have had at least one of its elements assigned the value *true*. For each such element, assign the corresponding atom in the instance of HC-2 the value *true*. This can be done using the default rules that were added for each of the set elements. It is not hard to see that this can always be done consistently: the three element clauses introduced into W will not be contradicted since they correspond to at least one of the elements of each set being assigned the value *false*. We know that this can be done because we are given a solution to I. Since the assignment in I is maximal as described above, no other set elements can be made true without forcing at least one of the sets to have all its elements take the same value. Thus, none of the remaining default rules can be applied. Since each set has at least one of its members assigned the value *true*, each

of the propositional atoms S_i are true in the extension we are constructing. Thus, due to the clause

$$(\neg S_1 \lor \neg S_2 \lor \ldots \lor \neg S_m \lor q)$$

in W, the extension must contain the literal q. At this point it is easy to see that an extension containing q exists.

($\Leftarrow$). Suppose there exists an extension of (D, W) that contains q. We note that since it only contains non-unit Horn clauses, W is easily seen to be consistent. Thus (D, W) has only *coherent* extensions. It follows that each of the literals of the form $S_i : 1 \le i \le m$ must be *true* (this is the only way to force q to be *true*). Furthermore, it follows that for each such literal, S_i, at least one of the literals in the set $\{\sigma_{i_1}, \sigma_{i_2}, \sigma_{i_3}\}$ must be *true*. The clause in W of the form

$$(\neg \sigma_{i_1} \lor \neg \sigma_{i_2} \lor \neg \sigma_{i_3})$$

forces at least one of these to be *false* as well. This provides us with at least one element of each set $S_i : 1 \le i \le m$ that is *true*, and at least one that is *false*. Given this, it is easy to construct a satisfying assignment for for the instance I of NOT-ALL-EQUAL-SATISFIABILITY. $\square$

The implications of this result on the hierarchy above are summarized in Figure 3 below.

2-Literal Clauses

A second interesting subclass of propositional formulae is 2-literal clauses. The classes formed by combining theories consisting of 2-literal clauses with restricted default theories is assumed in *network default theories*, described in [Etherington 1988]. We have investigated the complexity of membership problems for this class given the above
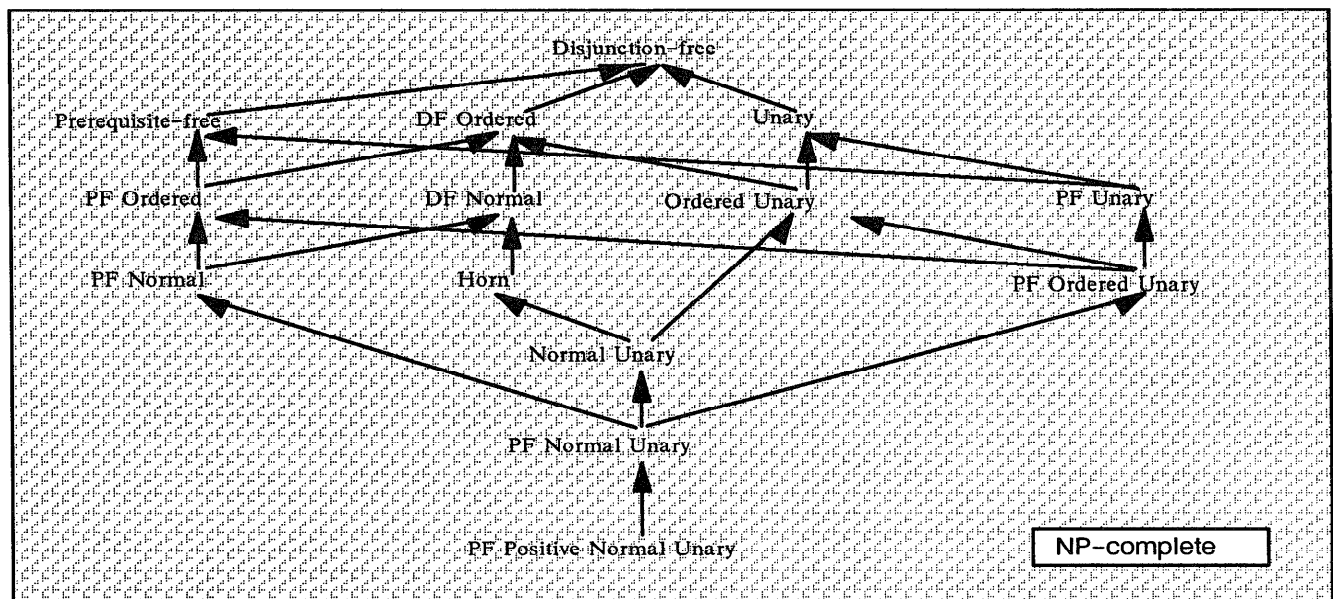
Figure 3: The complexity of membership problems with Horn theories.

hierarchy of restrictions on D shown above. For the problem

2-Literal Prerequisite-Free Normal
Instance: A finite set W of propositional 2-literal clauses, together with a finite set D of prerequisite-free normal propositional default rules, and a distinguished literal q.
Question: Does there exist an extension of (D, W) that contains q ?

we have the following theorem:

Theorem 2 *2-Literal Prerequisite-Free Normal can be solved in polynomial time.*

We present an $O(n^3)$ algorithm deciding the membership problem for this class in [Stillman 1990b]. The basic idea is to exploit the structural property of 2-literal clauses that they resemble binary relations. As a result, we can effectively compute an implicational "closure" of the underlying propositional theory. Once this is done, it is relatively easy to determine whether there is a default rule that can be used to force q to be included in the extension. For the problem

2-Literal Normal Unary
Instance: A finite set W of propositional 2-literal clauses, together with a finite set D of normal unary propositional default rules, and a distinguished literal q.
Question: Does there exist an extension of (D, W) that contains q ?

we prove the following:

Theorem 3 *2-Literal Normal Unary is NP-complete.*

The proof is complex, and space restrictions do not allow its inclusion herein. A complete proof is available in [Stillman 1990b]. For the problem

2-Literal Prerequisite-Free Ordered Unary
Instance: A finite set W of propositional 2-literal clauses, together with a finite set D of prerequisite-free ordered unary propositional default rules, and a distinguished literal q.
Question: Does there exist an extension of (D, W) that contains q ?

we have

Theorem 4 *2-Literal Prerequisite-Free Ordered Unary is NP-complete.*

This follows from the proof of Theorem 5 below. The proof is complex and thus omitted. These results are summarized in Figure 4 below.

Single Literal Theories

As mentioned above, this is the class that was investigated in [Kautz and Selman 1989]. The complexity of reasoning in the theories they considered is described in [Kautz and Selman 1989]; their results, together with ours, are illustrated in Figure 5. Since these theories are contained in both of those considered above, problems easy for them are also easy for these. The new result we present for these theories is given below:
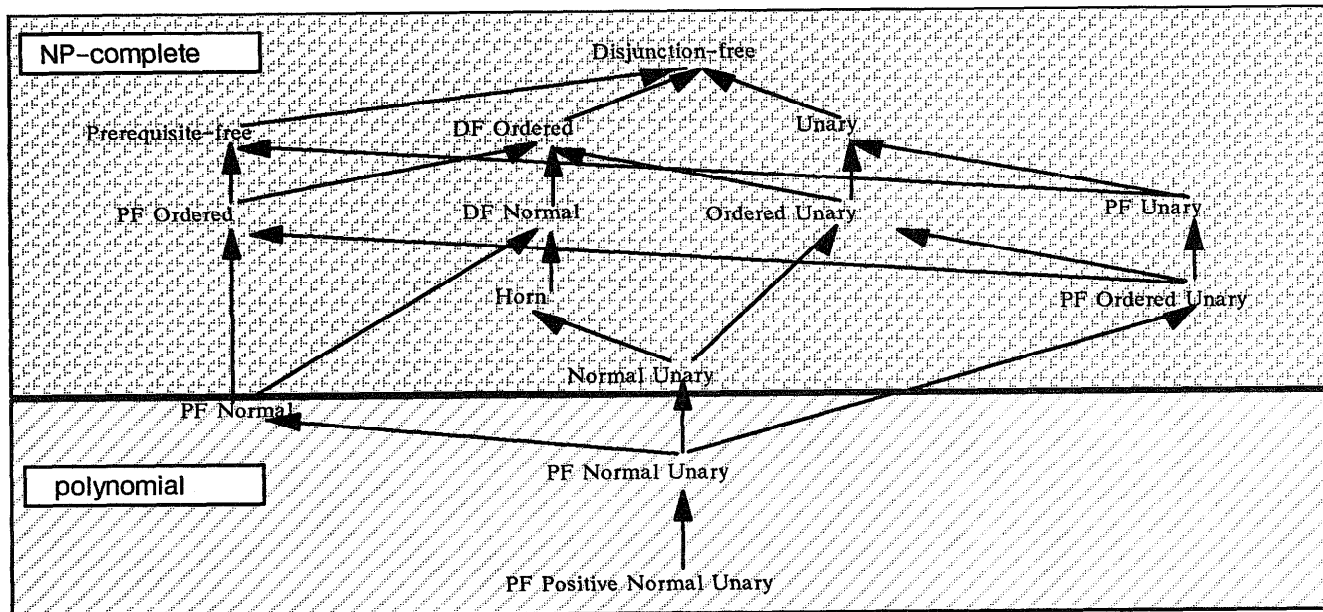
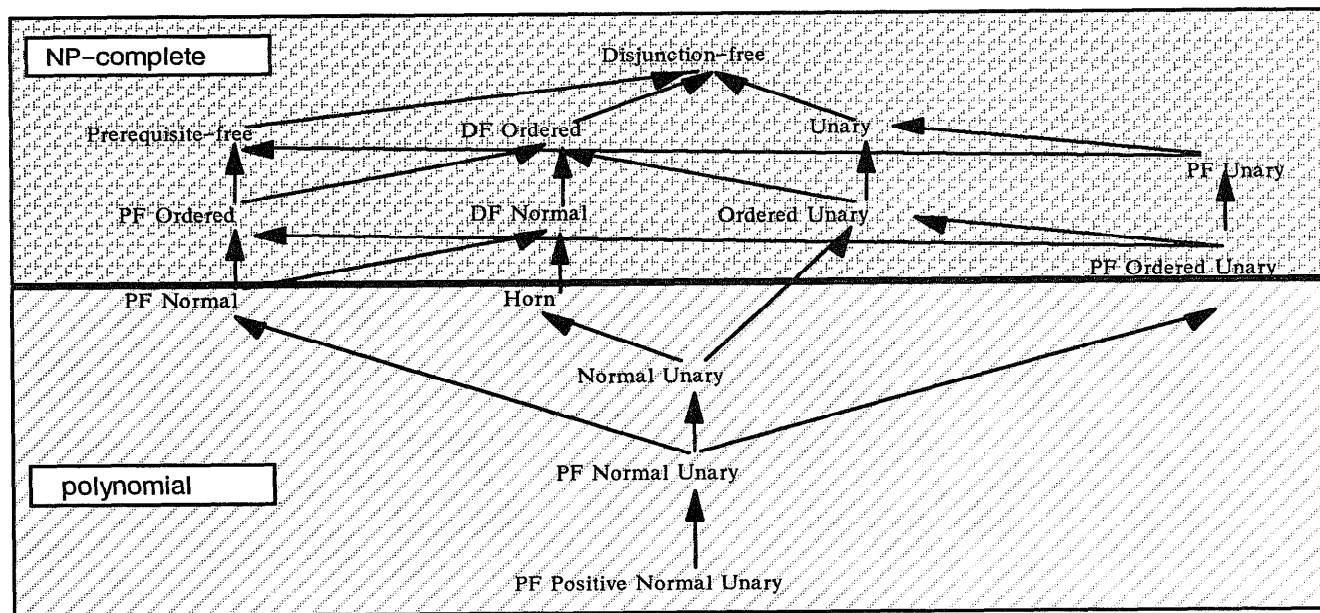Figure 4: The complexity of membership problems with 2-literal theories.

Figure 5: The complexity of membership problems with single literal theories.

Single Literal Prerequisite-Free Ordered Unary
Instance: A finite set W of propositional single literal clauses, together with a finite set D of prerequisite-free ordered unary propositional default rules, and a distinguished literal q.
Question: Does there exist an extension of (D, W) that contains q ?

Theorem 5 *Single Literal Prerequisite-Free Ordered Unary is NP-complete.*

The complete proof appears in the full version of this paper. These results are summarized in Figure 5 below.

Conclusions and Future Research

We have presented a number of results that characterize the complexity of the membership problem for restricted default theories. This work significantly extends that presented in [Kautz and Selman 1989] and [Stillman 1990a]. Our work considers very tight restrictions on the expressiveness of default rules as well as the underlying propositional theory. Unfortunately, our results show that even under these restrictions, membership problems almost invariably remain intractable. This suggests that if practical default reasoning systems are desired, one must either consider extremely restricted expressiveness or work to identify subcases of otherwise intractable classes that yield feasible complexity.

A number of related results pertaining to the complexity of extension existence and entailment over the classes we have considered can be answered easily given minor modifications of the proofs of the complexities membership problems. These results are presented in the full version of this paper ([Stillman 1990b]).

Acknowledgements

References

Aspvall, V., Plass, M.F., and Tarjan, R.E. 1979. A linear-time algorithm for testing the truth of certain quantified Boolean formulas. *Information Processing Letters* 8 (3).

Dowling, W.F., and Gallier, J.H. 1984. Linear time algorithms for testing the satisfiability of propositional horn formulas. *Journal of Logic Programming*, 3:267-284.

Etherington, D.W. 1987. Formalizing Non-Monotonic Reasoning Systems. *Artificial Intelligence* 31:41-85.

Etherington, D.W. 1988. *Reasoning with Incomplete Information.* Pitman, London.

Garey, M.R., and Johnson, D.S. 1979. *Computers and Intractability.* W.H. Freeman, New York.

Ginsberg, M.L., editor. 1987. *Readings in Nonmonotonic Reasoning.* Morgan Kaufman, Los Altos, CA, 1987.

Kautz, H.A., and Selman, B. 1989. Hard problems for simple default logics. In Proceedings of the First International Conference on Principles of Knowledge Representation and Reasoning, 189-197, Toronto, Ontario, Canada.

McCarthy, J. 1977. Epistemological problems of artificial intelligence. In Proceedings of the Fifth International Joint Conference on Artificial Intelligence, 1038-1044. International Joint Conferences on Artificial Intelligence, Inc.

McCarthy, J. 1986. Applications of circumscription to formalizing commonsense knowledge. *Artificial Intelligence*, 28:89–166.

Minsky, M. 1975. A framework for representing knowledge. In Patrick Winston, editor, *The Psychology of Computer Vision*, pages 211–277. McGraw-Hill, New York.

Moore, R.C. 1983. Semantical considerations on non-monotonic logic. In Proceedings of the Eighth International Joint Conference on Artificial Intelligence, 272–279, Karlsruhe, West Germany, International Joint Conferences on Artificial Intelligence, Inc.

Poole, D.L. 1986. Default reasoning and diagnosis as theory formation. Technical Report CS-86-08, Dept. of Computer Science, University of Waterloo.

Reiter, R. 1980. A logic for default reasoning. *Artificial Intelligence*, 13:81–132.

Shaefer, T.J. 1978. The Complexity of Satisfiability Problems. In Proceedings of the 10th Annual ACM Symp. on Theory of Computing, Association for Computing Machinery, New York.

Stillman, J.P. 1990a. The Complexity of Horn Theories with Normal Unary Defaults. In Proceedings of the 8th Canadian Artificial Intelligence Conference, (CSCSI/SCEIO-90), Ottawa, Canada.

Stillman, J.P. 1990b. The Complexity of Reasoning in Restricted Propositional Default Logics. In preparation.

Touretzky, D.S. 1986. *The Mathematics of Inheritance Systems.* Pitman, London.