On generalized interval calculi

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Abstract

The calculus of time intervals defined by Allen has been extended in various ways in order to accomodate the need for considering other time objects than convex intervals (eg.time points *and* intervals, non convex intervals). This paper introduces and investigates the calculus of generalized intervals, which subsumes these extensions, in an algebraic setting. The set of (p,q)-relations, which generalizes the set of relations in the sense of Allen, has both an order structure and an algebraic structure. We show that, as an order, it is a distributive lattice whose properties express the topological properties of the set of (p,q)-relations. We also determine in what sense the algebraic operations of transposition and composition act continuously on this set.

In Allen's algebra, the subset of relations which can be translated into conjunctive constraints on the endpoints using only <,>,=,≤,≥ has special computational significance (the constraint propagation algorithm is complete when restricted to such relations). We give a geometric characterization of a similar subset in the general case, and prove that it is stable under composition. As a consequence of this last fact, we get a very simple explicit formula for the composition of two elements in this subset.

1. Introduction

In [Vil82], Marc Vilain describes a logic for reasoning about time which is an extension of the logic defined by James Allen [All83]. Specifically, it is at its core composed of 13 relational primitives (Allen's relations) and of a body of inference rules (Allen's "transition table"). Moreover, it is extended to a logic which besides relations between two intervals, can also handle relations between two time points, or relations between a point and an interval. As concerns this extension, Vilain comments: "We should state that including [points] along with intervals in the domain of our system only minimally complicates the deduction algorithms. The polynomial complexity results and the consistency maintenance remain unaffected". In other words, allowing basic time objects to be either time intervals or time points does not alter in a significant way the framework of Allen's logic.

Ladkin [Lad86] introduced the notion of non convex temporal intervals and gave a taxonomy of important relations between them. He also argued for the convenience of a language based on non convex intervals in such applications as the specification of concurrent processes.

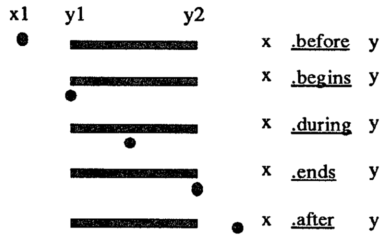

Fig. 1 : some of Vilain's point-to-interval relations

An interval in the sense of Allen is just a pair of ordered time points. A non convex interval (with a finite number of convex components) is entirely determined by the sequence of pairs associated to each convex component. This means that a non convex interval is an ordered sequence of an even number of time points.

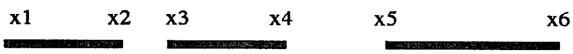

Fig. 2: a non convex interval

We extend somewhat further these remarks and define a generalized interval as an ordered, finite sequence of points in a linear order. We also call a generalized interval with n points a n-interval. More generally, for any subset S of the integers, a S-interval is a n-interval where n belongs to S. In this way, Allen's calculus is the calculus of 2-intervals; Vilain's universe is the set of {1,2}-intervals. The calculus of non convex intervals in the sense of Ladkin is the calculus of P-intervals, where P is the set of even integers.

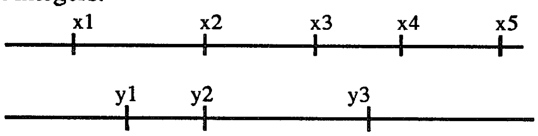

Fig. 3: generalized intervals

Remark Translating real-life intervals into n-intervals can raise problems of interpretation if both points and (ordinary) intervals are allowed in the interpretation of a single n-interval; e.g., a 3-interval could represent 3 points, or a point preceding an (ordinary) interval, or the other way round; a first solution is to consider extended and punctual entities separately; another would be to introduce types on the boundaries.

Allen's calculus is basically algebraic in nature. Ladkin and Maddux [LaMa87b, LaMa88] showed that it can be adequately described using the notion of a relation algebra as defined by Tarski and Jonsson [JoTa52]. Specifically, consider the set $\Pi(2,2)$ of 13 possible relations between two intervals. The set of subsets of $\Pi(2,2)$ is a Boolean algebra with additional structure, in particular, a transposition is defined on it, as well as an operation of composition (described by the transition table). This additional structure makes it an integral relation algebra A(2). The connexion between this algebra and the models of Allen's logic can be established using the algebraic notion of a weak representation, which generalizes the classical notion of a representation, as shown in [Lig90c].

A well known theorem of Cantor states that the ordered set of the rational numbers Q is up to isomorphism, the only denumerable linear order which is both dense and unlimited on the left and on the right. This can be seen as a result about the calculus of points, or about the representations of the relation algebra A(1) associated to the set $\Pi(1,1)$ of 3 possible relations between two points: up to isomorphism, A(1) has only one denumerable representation. Ladkin [Lad87b] proved that a similar fact is true of A(2). In [Lig90c], we show that these two results are just the n=1 and n=2 cases of a quite general result: for any $n \geq 1$, A(n) has only one denumerable representation up to isomorphism, where A(n) is the representation algebra associated to the calculus of n-intervals.

But this is only part of the story. Consider again Allen's transition table, which gives the result of the composition of two primitive relations. Firstly, only 26 relations appear in it, among the 2^{13} possibilities in A(2). Then, as remarked by many people [Noe88, LiBe89a], there is a notion of neighbourhood between relations, which corresponds to the physical intuition of small moves of the intervals considered. For example, if an interval *i* meets another interval *j*, we can move *i* slightly to the right, and then *i* overlaps *j*; or to the left, and then *i* < *j*.

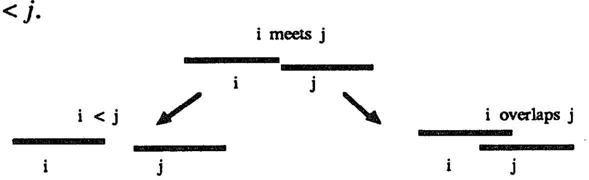

Fig. 4: neighbouring relations

Hence, relation m (meets) in some sense has o (overlaps) and < as immediate neighbours. In [LiBe89a, Lig90b] we describe how the topological structure of the set of relations in $\Pi(2,2)$ can be conveniently represented by a polygon; essentially the same picture (a lattice) was known to [Noe88]. Finally, the transition table makes apparent the fact that composition has some kind of continuity property: composing neighbouring relations, we tend to get neighbouring sets of results.

This is a first motivation for understanding more about the topology of relations. A further motivation is concerned with its computational relevance.

Allen's original publications described a polynomial algorithm for determining the consistency of a temporal network, ie. a graph of intervals with arcs labeled by elements of A(2). This algorithm was known not to be complete. Subsequently, Vilain and Kautz [ViKa86] showed that the problem in the general case is NP-complete.

However, restricting the labeling to a small subset of A(2) (83 elements) makes Allen's algorithm complete, as shown in [vBeek89, vBC90, ViKa86]. This subset can be defined as the set of relations which can be expressed as a conjunction of convex conditions on the endpoints of the intervals; or equivalently, it is the set of intervals in the lattice associated to $\Pi(2,2)$. This last characterization makes apparent the topological nature of this set of well behaved elements.

In this paper, we will be concerned both with the algebraic and the topological aspects of the calculus of generalized intervals. As the preceding discussion shows, giving this more general framework allows us to discuss in a unified way a whole body of results about representing and reasoning about time. The main purpose of the paper is to give a precise content to the remarks made above, by

- giving formal definitions of the interval algebras and their associated objects;
- proving the main facts about the algebraic and topological structures and the relationships between algebra and topology .

2. Generalizing Allen's relations: (p,q)- relations

2.1. (p,q)-relations

Definition Let T be a linear order. A n-interval in T is an increasing sequence $x = (x_1, x_2, ..., x_n)$ of elements of T : $x_1 < x_2 < ... < x_n$.

Let $x = (x_1, ..., x_p)$ be a p-interval, $y = (y_1, ..., y_q)$ a q-interval in a linear order T. The points $y_1, y_2, ..., y_q$ define a partition of T into 2q+1 zones in T, which we number from 0 to 2q (Fig.5):

zone 0 is $\{t \in T \mid t < y_1\}$;
zone 1 is y_1;
zone 2 is $\{t \in T \mid y_1 < t < y_2\}$;
...
zone 2q is $\{t \in T \mid t > y_q\}$.

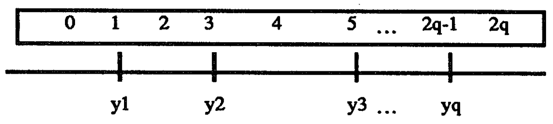

Fig. 5 : the partition of T determined by a q-interval.

Now the relation of x relative to y is entirely determined by specifying for each x_i which zone it belongs to; a constraint is that each oddly numbered zone contains one of the x_i's at most. This motivates the following:

Definition The set $\Pi(p,q)$ of (p,q)-relations is the set of non-decreasing sequences π of p integers between 0 and 2q, where each odd integer occurs at most once.

Remark In [Lig90c] we give another equivalent definition of the set of (p,q)-relations; it has the advantage of being more symmetrical; however, the present definition makes the order structure of this set more obvious.

Examples (i) $\Pi(1,1)$ has 3 elements 0, 1, 2, corresponding respectively to x <y, x=y, y<x.

(ii) $\Pi(1,2)$ has 5 elements 0, 1, 2, 3, 4; using the terminology of Vilain in [Vil82], we can call them respectively .before, .begins, .during, .ends, .after . On the other hand, $\Pi(2,1)$ also has 5 elements, namely (0,0), (0,1), (0,2), (1,2), (2,2); in Vilain's notation, these are called before., ended-by., contains., begun-by., after., respectively. (More generally, exchanging the roles of x and y, we see that $\Pi(p,q)$ and $\Pi(q,p)$ contain the same number of elements; more about this later).

(iii) $\Pi(2,2)$ has 13 elements, namely Allen's primitive relations; in particular, (1,3) is equality; six elements are the following (the remaining ones are obtained by switching roles between x and y):

(0,0)=< (before); (0,1)=m (meets); (0,2)=o (overlaps);
(2,2)=d (during); (2,3)=e (ends); (1,2)=s (starts).

2.2. Associated inequations

Each element π in $\Pi(p,q)$ corresponds to a set of equations and inequations $E(\pi)$:

$$(1)\ E(\pi)\ \left|\ \begin{array}{ll} x_i = y_{(\pi(i)+1)/2} & \text{if } \pi(i) \text{ is odd;} \\ x_i > y_{\pi(i)/2} & \text{if } \pi(i) \text{ is even, } \pi(i) < 2q; \\ x_i < y_{(\pi(i)+2)/2} & \text{if } \pi(i) \text{ is even, } \pi(i) > 0 \end{array}\right.$$

for i =1,...,p.

As a consequence, any subset of $\Pi(p,q)$ corresponds to a formula in the language with equality with variables $x_1,\ldots, x_p$, $y_1,\ldots, x_q$ and predicate "<". We extend this language with >, ≤, ≥, considered as abbreviations.

3. The order structure of (p,q)- relations

3.1. The lattice of (p,q)-relations

We have defined the set of (p,q)-relations as a subset of $\mathbf{N}\times \ldots \times \mathbf{N}$ (p times); hence the product order on $\mathbf{N}\times \ldots \times \mathbf{N}$ defines an order on $\Pi(p,q)$, namely:

Definition Let π, π' be two elements of $\Pi(p,q)$; then $\pi \le \pi'$ if and only if $\pi(i) \le \pi'(i)$ for i=1, ...,p.

Proposition $(\Pi(p,q), \le)$ is a distributive lattice.

Proof A product of linear orders is a distributive lattice. The sup and inf of two elements can be computed componentwise. $\Pi(p,q)$ is a subset of such a product which contains sup's and inf's. Q.E.D.

Again because of its very construction, $\Pi(p,q)$ is a subset of $\mathbf{R}^p$; we can consider its Hasse diagram, which is by definition the graph with $\Pi(p,q)$ as its set of vertices, where an arc joins π to π' if π' is an immediate successor of π. In this way, the Hasse diagram of H(p,q) is naturally embedded into $\mathbf{R}^p$. Figure 6 represents H(1,1), H(1,2), H(2,1), H(2,2). Intuitively, two relations are neighbours if passing from one to the other only involves changing the relation of one pair (x_i,y_j).

We claim that H(p,q) (with its canonical embedding in $\mathbf{R}^p$) gives an adequate representation of the topology of (p,q)-relations.

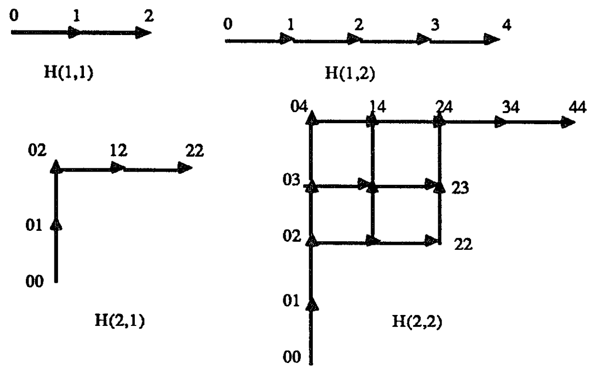

Fig. 6 : Hasse diagrams

3.2. Characterizing the intervals in $\Pi(p,q)$

Recall that in any lattice L, an interval is any subset of the form $I(l_1,l_2) = \{l \in L \mid l_1 \le l \le l_2\}$ where $l_1 \le l_2$ are two elements of L. The central theme in this section will be the set of intervals in $\Pi(p,q)$. We first give a characterization of the set of intervals in terms of inequations.

Proposition The set of intervals in $\Pi(p,q)$ coincides with the set of subsets associated to conjunctions of formulas $x_k \psi\, y_s$, where $\psi \in \{=, <, >, \le, \ge\}$.

Proof By definition of $\Pi(p,q)$, we have

$\pi(k) = 2s - 1$ if and only if ($x_k = y_s$);

(2) $\pi(k) = 2s$ if and only if ($y_s < x_k < y_{s+1}$) ($s \neq 0, 2q$);

$\pi(k) = 0$ if and only if $x_k < y_1$;

$\pi(k) = 2q$ if and only if $y_q < x_k$.

Hence each one of formulas (a,b,c,d,e) defines the subset of π's verifying a corresponding condition:

(a) $x_k = y_s$ defines $\{ \pi \mid \pi(k) = 2s - 1\}$;
(b) $x_k < y_s$ defines $\{ \pi \mid \pi(k) \leq 2s - 2\}$;
(c) $x_k > y_s$ defines $\{ \pi \mid \pi(k) \geq 2s \}$;
(d) $x_k \leq y_s$ defines $\{ \pi \mid \pi(k) \leq 2s - 1\}$;
(e) $x_k \geq y_s$ defines $\{ \pi \mid \pi(k) \geq 2s - 1\}$.

Obviously, the subsets defined in $\Pi(p,q)$ by (a,b,c,d,e) are intervals in $\Pi(p,q)$; hence any subset defined by a conjunction of such formulas is an interval.

Conversely, it suffices to show that any interval of the form

$I_{p,q}(k,[m,n]) = \{ \pi \mid \pi(k) \in [m,n]\}$ is defined by such a formula.

By (2), $\pi(k) \in [m,n]$ is equivalent to

$y_{m/2} < x_k < y_{(n+2)/2}$ if m,n are even;
$y_{(m+1)/2} \leq x_k < y_{(n+2)/2}$ if m is odd, n is even;
$y_{m/2} < x_k \leq y_{(n+1)/2}$ if m is even, n is odd;
$y_{(m+1)/2} \leq x_k \leq y_{(n+1)/2}$ if m,n are odd;

(we use the convention to leave out the inequations involving y_0 or y_{q+1}; e.g. $y_0 < x_k < y_{(n+2)/2}$ is replaced by $x_k < y_{(n+2)/2}$).

The general result follows from this fact. Q.E.D.

This result generalizes what was essentially known for $\Pi(2,2)$; intervals are called "convex relations" in Nökel [Noe88]. The algorithmic properties of the set of intervals in $\Pi(2,2)$ are examined in van Beek [vBeek89, vBeek90] and van Beek and Cohen [vBC90].

4. Operations and intervals

4.1.Operations on $\Pi(p,q)$

Transposition As already remarked, switching roles between x and y sends a (p,q)-relation π to a (q,p)-relation π^t. We give a precise description of this operation, which is called transposition:

Definition Let π be an element of $\Pi(p,q)$. Then $\pi^t \in \Pi(q,p)$ is defined as follows:

Consider the first q odd integers 1, 3,..., 2q-1; call odd(i) = 2i - 1 the i-th odd number; consider each odd(i) in sequence, and position it in the sequence $\pi(1),\ldots,\pi(p)$:

if odd(i) $< \pi(1)$, then $\pi^t(i) = 0$;
if $\pi(p) <$ odd(i), then $\pi^t(i) = 2p$;
if odd(i) $= \pi(j)$, then $\pi^t(i) = 2j-1$;
if $\pi(j) <$ odd(i) $< \pi(j+1)$, then $\pi^t(i) = 2j$.

Proposition Transposition is an order reversing bijection from $\Pi(p,q)$ onto $\Pi(q,p)$.

Proof Using the definition, it is easily shown that if π' is an immediate successor of π, then π'^t is an immediate predecessor of π^t. Q.E.D.

Examples Fig.7 resp. Fig 8 illustrates the correspondance between H(1,2) and H(2,1), resp. H(2,3) and H(3,2).

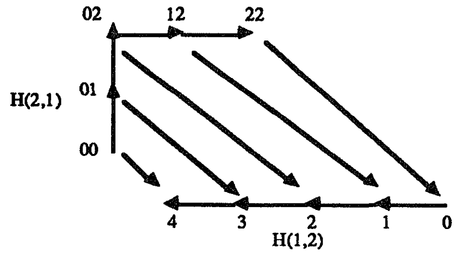

Fig. 7

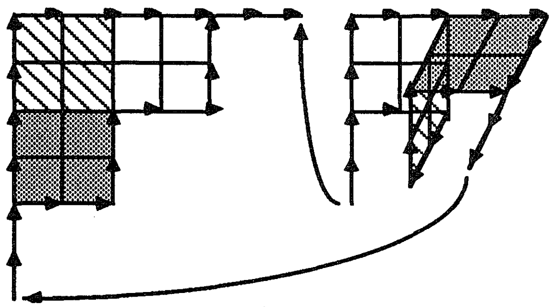

Fig. 8

Composition If a p-interval x is in relation π relative to a q-interval y, and y itself is in relation π' relative to a r-interval z, there is a finite number of elements in $\Pi(p,r)$ representing the possible relations of x relative to z. Translating this fact into our notations, we arrive at the definition of composition. In order to express it conveniently, we need the following:

Notation If $m \leq n$ are two integers, [[m,n]] denotes [m,n] if m and n are even, [m+1,n] if m is odd, n is even, [m,n-1] if m is even, n is odd, [m+1, n-1] if both m and n are odd (in other words, we leave out the odd endpoints).

Convention For any $\pi \in \Pi(s,t)$, $\pi(0) = 0$; $\pi(i) = 2t$ if $i > s$.

Definition Let $\pi \in \Pi(p,q)$, $\pi' \in \Pi(q,r)$. Then $(\pi;\pi')$ is the set of elements $\pi'' \in \Pi(p,r)$ such that, for every j, $1 \leq j \leq p$:

$\pi''(j) = \pi'((\pi(j)+1)/2)$ if $\pi(j)$ is odd;
$\pi''(j) \in [[\pi'(\pi(j)/2), \pi'((\pi(j)+2)/2)]]$ if $\pi(j)$ is even.

Examples (i) Take $\pi = \pi' = (0,2) \in \Pi(2,2)$ (the o = "overlaps" relation); since $\pi(1)$ and $\pi(2)$ are even, we get the set of π'' such that:

$\pi''(1) \in [[\pi'(0),\pi'(1)]] = [[0,0]] = 0$;

$\pi''(2) \in [[\pi'(1),\pi'(2)]] = [[0,2]] = [0,2]$;
hence $(\pi;\pi')$ is the set $\{(0,0),(0,1),(0,2)\} = \{<,m,o\}$.

(ii) Take $\pi = (2,3)$, $\pi' = (3,4) \in \Pi(2,2)$; then
$\pi''(1)$ is $\in [[\pi'(1),\pi'(2)]] = [[3,4]] = 4$;
$\pi''(2) = \pi'(2) = 4$;
hence $(\pi;\pi')$ is the set reduced to (4,4) (the ">" relation).

Remark The above definition clearly shows that, for $\pi \in \Pi(p,q)$ and $\pi' \in \Pi(q,r)$, $(\pi;\pi')$ is an interval in $\Pi(p,r)$. In other words, the entries in the corresponding transition table are intervals. Moreover, they are intervals of a special kind: their projection on each component is either an integer or an interval with even boundaries.

In particular, the entries in Allen's transition table should be of this type. Checking all possibilities, we get a set of 28 intervals among which 26 do appear in the transition table.

We show in the next section that this fact is only a particular aspect of the stability of intervals with respect to composition.

4.2. Stability of the set of intervals

We first extend composition to sets in a natural way:

Definition Let E, F be subsets of $\Pi(p,q)$, $\Pi(q, r)$ resp. Then we denote by (E;F) the union of all $(\pi;\pi')$, where $\pi \in \Pi(p,q)$ and $\pi' \in \Pi(q,r)$. In the same manner, E^t is the set of π^t, where $\pi \in E$.

The interaction of composition with transposition is described by:

Proposition For all $\pi \in \Pi(p,q)$, $\pi' \in \Pi(q,r)$,
$(\pi; \pi')^t = (\pi'^t; \pi^t)$.

The next proposition expresses the fact that composition is non decreasing relative to its arguments:

Proposition If $\pi \le \pi_1$, $\pi' \le \pi'_1$ then :
(a) $\inf(\pi;\pi') \le \inf(\pi_1;\pi')$ and $\sup(\pi;\pi') \le \sup(\pi_1;\pi')$; hence
(b) $\inf(\pi;\pi') \le \sup(\pi_1;\pi'_1)$.

Proof Suppose π_1 is an immediate successor of π; then all components of π and π_1 coincide, except for one; let i_0 such that $\pi_1(i_0) = \pi(i_0) + 1$. Then the definitions of $(\pi;\pi')$ and $(\pi_1;\pi')$ only differ on their i_0-th components: if $\pi(i_0) = m$ is odd, we get $\pi'((m+1)/2)$ for $(\pi;\pi')$ and $[[\pi'((m+1)/2), \pi'((m+3)/2)]]$ for $(\pi_1;\pi')$; if $\pi(i_0) = m$ is even, we get $[[\pi'(m/2), \pi'((m+2)/2)]]$ for $(\pi;\pi')$ and $\pi'((m+2)/2)$ for $(\pi_1;\pi')$; in all cases, $\inf(\pi;\pi') \le \inf(\pi_1;\pi')$, and $\sup(\pi;\pi') \le \sup(\pi_1;\pi')$; hence
$\inf(\pi;\pi') \le \sup(\pi_1;\pi')$.
A similar, easier reasoning works for the right argument; (alternatively, transposition can be used to deduce the result from the preceding one). By induction, we get the general result. Q.E.D.

We now state the main result about stability:

Proposition The set of intervals is stable by intersection, transposition, and composition.

The last statement means that, if I in $\Pi(p,q)$ and J in $\Pi(q,r)$ are two intervals, then (I;J) is an interval in $\Pi(p,r)$.

Proof The part about intersection and transposition is obvious. For composition, we use the more precise
Lemma Let $I_{p,q}(k,[m,n]) = \{ \pi \mid \pi(k) \in [m,n]\}$; then
$(I_{p,q}(k,m) ; \pi') = I_{p,r}(k, \pi'((m+1)/2))$ if m is odd;
$(I_{p,q}(k,m) ; \pi') = I_{p,r}(k, [[\pi'(m/2), \pi'((m+2)/2)]])$, if m is even.

Putting together the preceding results, we get an explicit formula for computing the composition of two intervals:

Theorem Let $[\alpha,\beta]$ be an interval in $\Pi(p,q)$, $[\gamma,\delta]$ an interval in $\Pi(q,r)$; then
$([\alpha,\beta] ; [\gamma,\delta])$ is the interval $[\inf(\alpha;\gamma), \sup(\beta,\delta)]$ in $\Pi(p,r)$.

Proof Since $([\alpha,\beta] ; [\gamma,\delta]) \supseteq (\alpha;\gamma)$, $\inf(\alpha;\gamma) \in [\alpha,\beta] ; [\gamma,\delta]$. The same holds for $\sup(\beta,\delta)$. Since $([\alpha,\beta] ; [\gamma,\delta])$ is an interval, $([\alpha,\beta] ; [\gamma,\delta]) \supseteq [\inf(\alpha;\gamma), \sup(\beta,\delta)]$. Conversely, let $t \in (\pi;\pi')$, for $\pi \in [\alpha,\beta]$, $\pi' \in [\gamma,\delta]$. Since $\alpha \le \pi$ and $\gamma \le \pi'$, we have $\inf(\alpha;\gamma) \le \inf(\pi;\pi') \le t$. In the same way, we show that $t \le \sup(\beta,\delta)$. Q.E.D.

5. Interval calculus as algebra

5.1. Constructing relation algebras

We now look in more detail at the algebraic structure of $\Pi(p,q)$.
Notations For $p \ge 1$, $\mathbf{1'}_{p,p}$ denotes equality in $\Pi(p,p)$, i.e. the element in $\Pi(p,p)$ such that $\mathbf{1'}_{p,p}(i) = \text{odd}(i)$, $1 \le i \le p$.
If E is a set, $\mathcal{P}(E)$ denotes the set of subsets of E.

Proposition The following properties obtain, for any $\pi_1 \in \Pi(p,q)$, $\pi_2 \in \Pi(q,r)$ and $\pi_3 \in \Pi(r,s)$:
i) $((\pi_1 ; \pi_2) ; \pi_3) = (\pi_1 ; (\pi_2 ; \pi_3))$;
ii) $(\pi_1 ; \mathbf{1'}_{q,q}) = \pi_1$ and $(\mathbf{1'}_{p,p} ; \pi_1) = \pi_1$;
iii) $\mathbf{1'}_{p,p} \in (\pi_1 ; \pi_1^t)$ and $\mathbf{1'}_{q,q} \in (\pi_1^t ; \pi_1)$;
iv) $\pi \in (\pi_1;\pi_2)$ implies $\pi_1 \in (\pi; \pi_2^t)$ and $\pi_2 \in (\pi_1^t; \pi)$;
v) $(\pi_1 ; \pi_2)^t = (\pi_2^t ; \pi_1^t)$.

Let S be a non empty subset of the positive integers. In order to construct the relation algebra A(S) which describes the calculus of S-intervals, we proceed as follows:
- firstly, we take the union of all (p,q)-relations $\Pi(p,q)$, where $(p,q) \in S \times S$:
$\Pi(S) = \bigcup_{(p,q) \in S \times S} \Pi(p,q)$;
- composition extends to $\Pi(S)$, with values in $\mathcal{P}(\Pi(S))$, in the following way :

if $\pi_1 \in \Pi(p,q)$, $\pi_2 \in \Pi(p',q')$, then if q=p', $(\pi_1 ; \pi_2)$ is the same set of elements of $\Pi(p,q')$ as defined before, considered as elements of $\Pi(S)$; else, it is $\emptyset$ (the empty set);

- transposition is globally defined on $\Pi(S)$;
– the union of unit elements $\mathbf{1'}_{p,p}$, for $p \in S$, is an element $\mathbf{1'}_S$ of $\mathcal{P}(\Pi(S))$: $\mathbf{1'}_S = \{\mathbf{1'}_{p,p} \mid p \in S\}$.

The proposition implies that $\Pi(S)$, together with composition, transposition and $\mathbf{1'}_S$, is a connected groupoid in the sense of Comer [Com83, Def. 3.1]; if S has one element, this polygroupoid is in fact a polygroup. Still using [Com83], this can be expressed equivalently in terms of relation algebras in the sense of Tarski [JoTa52]:

Let $A(S) = \mathcal{P}(\Pi(S))$; it is a boolean algebra; besides, it inherits an unary operation of transposition, a binary operation of composition, and a distinguished element $\mathbf{1'}_S$. Then:

Theorem A(S) is a simple, complete, atomic relation algebra, with $0 \neq 1$. It is integral if and only $S = \{n\}$, for some $n \geq 1$.

Remark More generally, we can define $\Pi(S)$, hence A(S), in the case where S is an equivalence relation on a subset of N^+. In this way, we get algebras which are not necessarily simple. For instance, if S is the partition $\{1\},\{2\}$ of $\{1,2\}$, we get an algebra with 26 atoms which is defined in [LaMa88] in terms of constraint networks.

Examples

(i) A(1) has 8 elements, which can be identified with the subsets of $\Pi(1,1)$; identifying the elements 0, 1, 2 of $\Pi(1,1)$ with >, δ (equality), > resp., we see that the elements of A(1) are <, δ,>, $\leq$ (ie. < + δ),$\geq$ (ie. > + δ),$\neq$ (ie. < + >), 0 (the empty set), and 1 (ie. < + > + δ). The structure of A(1) is entirely determined by the effect of transposition on < (it exchanges < and >), and by the conditions $(< ; <) = <$, and $(< ; >) = 1$.

(ii) A(1,2) is an algebra with 26 atoms, which are the elements of

$\Pi(1,1) \cup \Pi(1,2) \cup \Pi(2,1) \cup \Pi(2,2)$.

Its structure is determined by the effect of transposition, which operates on $\Pi(1,1)$ as described above, on $\Pi(2,2)$ (which is the set of atoms of Allen's algebra) and exchanges $\Pi(1,2)$ and $\Pi(2,1)$, and by composition, which is as described by Allen's transition table and Vilain's extension.

In the general case, A(S) describes the calculus on n-intervals, where $n \in S$.

5.2. Using the algebra: weak representations

Temporal reasoning in AI deals with temporal databases consisting of sets of time objects; a basic problem consists in proving and maintaining the consistency of such a database; we now sketch how the algebraic machinery allows us to define such databases as algebraic structures.

Definition A weak representation of a relation algebra A is a map Φ of A into a direct product of algebras of the form $\mathcal{P}(U \times U)$, such that:

(a) Φ defines a homomorphism of boolean algebras;
(b) $\Phi(1') = \{(u,u) \mid u \in U\}$.
(c) $\Phi(^t) = {}^t$ (transposition of a binary relation)
(d) $\Phi(\alpha ; \beta) \supseteq \Phi(\alpha) \circ \Phi(\beta)$.

This notion is an extension of the classical notion of a representation, which is defined as a weak representation satisfying :

(e) Φ is one-to-one;
(f) $\Phi(\alpha ; \beta) = \Phi(\alpha) \circ \Phi(\beta)$.

If A is a relation algebra, we shall say that a weak representation of A into $\mathcal{P}(U \times U)$ is connected if $\Phi(1) = U \times U$ (1 denotes the greatest element in the underlying boolean algebra).

Examples

(i) A connected weak representation of A(1) is defined by the following data :

- a non empty set U (call its elements time points);
- a binary relation $R = \Phi(<)$ on U.

In fact, because of (b), the image of equality has to be $\Delta = \{(u,u) \mid u \in U\}$; because of (c), $\Phi(>)$ is the transpose R^t of R.

These data are subject to the following conditions:

by (a) and connectedness, $\{R, R^t, \Delta\}$ is a partition of $U \times U$; hence R is irreflexive and total; by (d), we have $R \supseteq R \circ R^t$. But this is just transitivity for R.

In other words, R defines a linear order on U. This shows that the notion of a connected weak representation of A(1) is equivalent to that of a linear order.

Representations of A(1) have to satisfy the further conditions :

$$R \circ R \supseteq R;$$
$$R \circ R^t \supseteq R;$$
$$R^t \circ R \supseteq R.$$

It is easily verified that the first condition expresses the fact that R is dense; the second one, that U is unlimited on the right; the third one, that U is unlimited on the left. Hence a representation of A(1) is a linear order which is dense and unlimited in both directions. By a theorem of Cantor, any denumerable order having these properties is isomorphic to the order of the rational numbers **Q**. We show in [Lig90] that this result generalizes to representations of A(n) for any $n \geq 1$.

(ii) Weak representations of A({1,2}) Consider such a representation Φ, with underlying set U. Then $\Phi(\mathbf{1'}_{1,1}) = \{(u,u) \mid u \in U^1\}$, $\Phi(\mathbf{1'}_{2,2}) = \{(u,u) \mid u \in U^2\}$. The sets U^1 and U^2 are time points and time intervals respectively.

For example, Fig. 9 represents a weak representation with $U^1 = \{u_1\}$, $U^2 = \{u_2,u_3\}$ which corresponds to the specifications:

$\Phi(o) = \{(u_2,u_3)\}$ (o denotes (0,2) in $\Pi(2,2)$);
$\Phi(\underline{.during}) = \{(u_1,u_2)\}$ ($\underline{.during}$ denotes 2 in $\Pi(1,2)$);
$\Phi(\underline{.before}) = \{(u_1,u_3)\}$ ($\underline{.before}$ denotes 0 in $\Pi(1,2)$).

Fig 9 : A weak representation of A({1,2})

In applications, we have sets of constraints which can be represented as networks with arcs labeled by elements of A(S), where S is some set of integers. The preceding discussion gives a framework for maintaining and checking the consistency of such a network. It also implies that restricting the labels to the subsets of intervals guarantees the completeness of the constraint propagation algorithm.

6. Conclusion

We have introduced a calculus of (p,q)-relations which provides a framework subsuming a number of formalisms used in temporal reasoning. Investigating the partial order structure of the set of relations allowed us to characterize in a simple way an important subset of relations: the subset of intervals; we related this subset to the operations of composition and transition, and obtained a simple explicit expression for the composition of two intervals. We also gave a precise definition of the algebraic basis of this generalized calculus, and showed how results on the computational feasability of consistency checking can be extended to this wider framework.

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