A Non-Well-Founded Approach to Terminological Cycles

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Abstract

In this paper, we propose a new approach to intensional semantics of term subsumption languages. We introduce *concept algebras*, whose signatures are given by sets of primitive concepts, roles, and the operations of the language. For a given set of variables, standard results give us free algebras. We next define, for a given set of concept definitions, a term algebra, as the quotient of the free algebra by a congruence generated by the definitions. The ordering on this algebra is called *descriptive subsumption* ($\sqsupseteq_\Delta$). We also construct a *universal concept algebra*, as a non-well-founded set given by the greatest fixed point of a certain equation. The ordering on this algebra is called *structural subsumption* ($\succeq_\Delta$). We prove there are unique mappings from the free algebras, to each of these, and establish that our method for classifying cycles in a term subsumption language, K-REP, consists of constructing accessible pointed graphs, representing terms in the universal concept algebra, and checking a simulation relation between terms.

Introduction

Classification of cycles in term subsumption languages has thus far been avoided, for a variety of sound and perhaps not so sound reasons. In this paper we will discuss how cycle classification is handled in K-REP [Mays *et al.*, 1991], a KL-ONE [Brachman and Schmolze, 1985] style of language. Using ideas from universal algebra and the theory of non-well-founded sets, a new framework, that of *concept algebras* is presented. These algebras elucidate the structural comparisons that are actually made when testing subsumptions. The motivation is to view terms (referred to as concepts) in these languages as intensional descriptions, and to view subsumption as a process of structural comparison between terms. In this sense this framework differs from existing treatments that make use of model theory, in that it closely corresponds to the actual implementation of the classifier in K-REP.

In a recent paper Bill Woods [Woods, 1991] has suggested that a more intensional view of concept descriptions should be taken. Concepts might describe things that may or may not exist in the world. Different concept descriptions can have the same *meanings*, yet still may be regarded as distinct concepts ("the morning star" vs. "the evening star"). To date most of the work in semantics of term subsumption languages takes an extensional view. Concepts are interpreted as sets of objects from some universe. Roles of concepts are interpreted as binary relations over the universe. These languages are thus seen as some subset of first order logic. Concept descriptions are complex predicates and one asks whether or not given instances satisfy those predicates.

In this paper we'd like to pursue Woods' suggestion and try to look at concepts intensionally. We will consider a small subset of the K-REP language, that has primitive concepts, concepts formed by conjunctions, and roles of concepts whose value restrictions are other concepts. This subset is chosen not only to simplify the presentation, but because it is not clear yet how to extend these ideas to more complex constructs like disjunctions and negations. This subset is roughly the same subset handled in [Baader, 1990]. A knowledge base is seen as a set of possibly mutually recursive equations, involving terms of this concept language.

The outline of this paper is as follows. The next section is a brief overview of the K-REP language. Next will be a general discussion of cycles, the type that are of use, how they arise, and how they are handled in K-REP. As pointed out by Bernhard Nebel [Nebel, 1990], the type of cycles of interest appear through role chains. What he refers to as descriptive semantics comes closest to capturing our intuitive understanding of them. Descriptive semantics, as well as least and greatest fixed point semantics, are all based on the view of modeling concept descriptions as subsets of some universe. This is very appealing as we think of more general concepts as describing larger classes of objects. However, implementations reason with the descriptions of the classes, subsumption is determined by structural comparison and not by subset inclusion of sets of objects.

We then present a new model for this subset of the K-REP language. It makes use of basic tools from universal algebra [Jacobson, 1989], and Aczel's theory of non-well-founded sets [Aczel, 1988]. What will be modeled are the descriptions of the concepts. First we give a signature and a set of axioms, for which we can discuss a class of algebras called *concept algebras.* Standard results will give us free algebras generated from a given set of variables. Terms in the algebras will correspond to concept descriptions. Next a Knowledge Base (hereafter KB) is defined as a set of possibly mutually recursive definitions over terms of a free algebra. These equations generate a congruence on the free algebra and the quotient algebra is then the concept algebra generated from this given KB. Any KB thus gives rise to a quotient algebra in this way. We will refer to the ordering on terms of this algebra as *descriptive subsumption.* This algebra can be uniquely mapped to a certain non-well-founded set C that arises as the greatest fixed point solution of a certain equation. Suitable operators are defined on C to make it, too, into a concept algebra. We will see that in some sense C is the most abstract model for our language in that it captures concepts as intensional descriptions. The subsumption ordering on this algebra will be called *structural subsumption.* The reason that C is interesting is that its ordering captures the essence of the implementation of subsumption in K-REP, even when cycles occur. We are hopeful that we can extend this model to also include disjunctions of concepts, since they can be viewed as sets of descriptions, one of which a given instance might satisfy.

The Representation Language

K-REP is in the class of languages known as term subsumption languages. Terms in K-REP are called concepts. Concepts are meant to describe classes of objects in some universe, both by defining them in terms of other concepts, and by describing the attributes a given class has. There are two types of concepts, primitive and defined. Primitive concepts are like natural kinds, their attributes are necessarily true of instances belonging to the concept, but are not sufficient to determine membership in the class represented by the concept. Defined concepts on the other hand are normally defined in terms of other more general primitive concepts. Their attributes are both necessary and sufficient for instances belonging to the class. Attributes are called roles, and since they have values, they can be viewed as binary relations over the classes. Terms are constructed by a few concept-forming operators. Using set-theoretic semantics, concepts are interpreted as subsets of some universe $\mathcal{U}$, and roles as binary relations on $\mathcal{U}$. Table 1 shows the operators together with their abstract form and semantics for the subset of K-REP we will be considering in this paper. This is the standard set-theoretic semantics often seen in the literature. It is included here for purposes of comparison with the semantics we propose in this paper.

In order to simplify the technical details of our model introduced in the next section, we will assume that introduced primitives are defined only in terms of top. Since other defined primitives can be expressed in terms of these and role definitions, this results in no loss of expressivity in our language. A knowledge base (KB) is a collection of concept terms. Since we are not considering instances, we make no distinctions between T-boxes and A-boxes. A given set of concept definitions will define a KB. Consider the following KB:

$$C \equiv (\text{and } P\ (R\ D))$$
$$D \equiv (\text{and } P\ (S\ C))$$

where P is a primitive concept, R and S are roles and C and D are defined concepts. Notice that these two concepts contain a simple cycle. In the first implementation of K-REP forward references were not allowed, so that these could not be defined without some hackery. We will write $(R\ D)$ as an abbreviation for $(\forall\exists R : D)$, in order to motivate the view of roles as meet-homomorphisms on concepts, which we will see in our model. Since conjunction in the language is just set-theoretic intersection in the semantics, this is just the statement that $(\text{and } (R\ C)\ (R\ D)) \equiv (R\ (\text{and } C\ D))$.

The subsumption relation produces an ordering on the concepts of a given KB. One concept subsumes another written $C_1 \succeq C_2$ if all of C_1's primitives subsume a primitive of C_2, and for each role R of C_1, C_2 has that role and the value restriction of R on C_1 subsumes the value restriction of R on C_2. Note that if $C_1 \succeq C_2$, then (and C_1 C_2) must be equivalent to C_2, in the sense that each subsumes the other.

Cycles

When cycles are allowed through role chains, one can see that a straightforward approach to subsumption testing can lead to infinite looping. In the original implementation of K-REP, concepts were classified one at a time and forward references were not allowed. This prevented the occurrences of cycles, other than those of length one, which caused no harm and were more or less ignored. A first attempt at handling cycles was to classify all the concepts involved in a cycle at the same time. Each concept in the cycle is classified as far as possible, and one loops through them all until no more further relationships are discovered. This appeared to work well and only required that given a collection of concepts, one can detect the cycles syntactically before they are classified. However, it missed the following case:

$$C \equiv (\text{and } P\ (R_1\ D))$$
$$D \equiv (\text{and } P\ (S_1\ C))$$
$$C' \equiv (\text{and } P\ (R_1\ D')\ (R_2\ P'))$$
$$D' \equiv (\text{and } P\ (S_1\ C')\ (S_2\ P'))$$

Table 1: K-REP Language

Concrete Form	Abstract Form	Semantics
Concept Forming Operators		
top	$\top$	$\mathcal{U}$
(and $C_1 \ldots C_n$)	$C_1 \wedge \ldots \wedge C_n$	$C_1^{\mathcal{I}} \cap \ldots \cap C_n^{\mathcal{I}}$
(allsome R C)	$\forall\exists R : C$	$\{d \in \mathcal{U} \mid R^{\mathcal{I}}(d) \subseteq C^{\mathcal{I}} \wedge R^{\mathcal{I}}(d) \neq \emptyset\}$ where $R^{\mathcal{I}} \subseteq \mathcal{U} \times \mathcal{U}$
Terminological Axioms		
(defconcept N C)	$N \equiv C$	$N^{\mathcal{I}} = C^{\mathcal{I}}$
(defprimconcept N C)	$N \sqsubseteq C$	$N^{\mathcal{I}} \subseteq C^{\mathcal{I}}$

C' and D' are the same as C and D except that they have additional roles R_2 and S_2 and therefore C should subsume C' and D should subsume D'. If we were to classify C and D together and then C' and D' we would not detect that C subsumes C'. That test would involve checking if D subsumes D', which would not be known at that point.

The solution to this is to begin testing the first concept in a cycle. If any subsumption questions arise that involve other concepts in that cycle then recursively invoke the next subsumption question. At some point this process terminates, or a subsumption question arises that one has already visited, at which point one stops. This is tantamount to just assuming it is true. Applying this technique to the previous example we see that the question $C \succeq C'$ leads to $D \succeq D'$ which leads back to $C \succeq C'$.

In order to see this more closely one can draw certain graphs to represent the concepts of a given KB. These graphs are actually encoded forms of accessible pointed graphs *(apg's)*, used in non-well-founded set theory. Each one has a root node labelled by the concept name, an epsilon labelled arc that points to the conjunction of the primitives in the concept, and an R labelled arc for each role R in the definition. Let us call the collection of these $\mathcal{D}$. Using these graphs we check if $C_1 \succeq C_2$ by first checking that the primitive pointed to by the epsilon arc of C_1 subsumes the primitive pointed to by the epsilon arc of C_2. Then for each R labelled arc of C_1 we check that C_2 has an arc of the same label, and that the nodes pointed to by them are also in the subsumption relation. If either of these nodes is in a cycle, we arrive at the recursive step. Take the graphs corresponding to the concept names of these two nodes and paste them down at these nodes. However, change any labelled arcs that point to concepts whose graphs are already present, to point to those nodes. This may introduce loops in the graphs. Continue testing the nodes where the new graphs were pasted on. Since the number of concepts is finite, the number of cycles must be also, and this process terminates. Recognition of previously asked subsumptions questions, occurs when loops with the same labels are introduced that point to pairs of previous nodes. This process actually constructs a Hoare simulation between the two apgs, from the top down. We are certain that it terminates because our KBs are finite.

The Model

Consider the signature Σ containing a set P of constants, a set $\Re$ of unary operators, a binary operator $\wedge$, and a constant $\top$. Let E be the following set of axioms with respect to this signature:

$$\begin{array}{rcll} x \wedge x &=& x & (\textit{idempotence}) \\ x \wedge y &=& y \wedge x & (\textit{commutativity}) \\ (x \wedge y) \wedge z &=& x \wedge (y \wedge z) & (\textit{associativity}) \\ x \wedge \top &=& x & (\top \text{ is a unit}) \\ R(x \wedge y) &=& R(x) \wedge R(y) & \forall R \in \Re \end{array}$$

We now consider the class of algebras for this signature Σ, that satisfy the axioms E. We will call such algebras *concept algebras*. Using $\wedge$, a partial order $\geq$ can be imposed on a concept algebra ($p \geq q$ iff $p \wedge q = q$). The constants in P are meet-irreducible primitives and each $R \in \Re$ defines a meet homomorphism on the algebra.

Given $X = \{x_1, \ldots, x_n\}$, $A[X] = A[x_1, \ldots, x_n]$ is the free concept algebra generated by X. $A[] = A[\emptyset]$ is the initial concept algebra.

A KB is a set of n possibly mutually recursive definitions

$$\Delta = \{x_1 \equiv t_1, \ldots, x_n \equiv t_n\}$$

where $t_i \in A[x_1, \ldots, x_n]$. Δ gives rise to $\equiv_\Delta$, the least congruence on $A[x_1, \ldots, x_n]$ containing Δ. For each KB, Δ, we will be interested in three relations on $A[X]$:

$$s_1 \sqsupseteq_\Delta s_2 \text{ (descriptive subsumption)}$$

$$s_1 \succeq_\Delta s_2 \text{ (structural subsumption)}$$

$$s_1 \supseteq_\Delta s_2 \text{ (extensional subsumption)}$$

A congruence on $A[X]$ is an equivalence relation that is also a subalgebra of $A[X] \times A[X]$. For any relation T on $A[X]$, let $E(T)$ be the smallest equivalence relation on $A[X]$ that contains T, and let $F(T)$ be the smallest subalgebra on $A[X] \times A[X]$ that contains T. Let $S_0 = \{(x_i, t_i) | x_i \equiv t_i \in \Delta\}$ and let $S_{k+1} = F(E(S_k))$ for $k > 0$. This generates an increasing sequence of algebras

$$S_0 \subseteq S_1 \subseteq S_2 \ldots$$

and we define $\equiv_\Delta$ as $\bigcup_n S_n$. This is a congruence on $A[X]$ that contains Δ, and if B is any other algebra that contains Δ, then an induction on k shows that B contains S_k and therefore contains $\equiv_\Delta$, so it is the smallest.

Let $A_\Delta[x_1, \ldots, x_n]$ hereafter refer to the quotient algebra $A[x_1, \ldots, x_n]/\equiv_\Delta$. Given a set of concept definitions Δ we can view $A_\Delta[x_1, \ldots, x_n]$ as the algebra of congruence classes of concept terms with respect to this set of equations. Then

Theorem 1 *$A_\Delta[x_1, \ldots, x_n]$ is a* conservative extension *of $A[]$, i.e., the unique homomorphism $A[] \xrightarrow{\iota} A_\Delta[x_1, ..., x_n]$ is one-one.*

Proof. Since the only nontrivial identifications of terms of $A[x_1, \ldots, x_n]$ implied by $\equiv_\Delta$ involve the free variables $x_1, \ldots, x_n$, and since $A[]$ has no terms involving $x_1, \ldots, x_n$, the unique homomorphism from $A[]$ into $A_\Delta[x_1, \ldots, x_n]$ is one-to-one $\Box$.

We are now ready to define *descriptive subsumption.*

Definition 1 *Given two terms $s_1, s_2 \in A[X]$, we say s_1 descriptively subsumes s_2, written $s_1 \sqsupseteq_\Delta s_2$, iff $\pi_\Delta s_1 \geq_{A_\Delta[X]} \pi_\Delta s_2$ (π_Δ is the canonical projection map from $A[X]$ to $A_\Delta[X]$).*

We next wish to construct an algebra that provides semantics for these terms, so that each term can be mapped to an element of the semantic algebra. There will be no variables in this algebra, and we will show that even sets of equations with cycles in them have unique solutions in this semantic algebra. Keep in mind that the elements of this algebra, which we call the *universal concept* algebra, are modeling the concepts as descriptions or intensions, rather than extensions.

The "universal concept" algebra is the greatest fixed point solution of the equation

$$C = (\mathcal{P}_{<\omega} P) \times (\Re \xrightarrow[<\omega]{\cdot} C)$$

where $(\mathcal{P}_{<\omega} P)$ is the collection of finite subsets of P, and $(\Re \xrightarrow[<\omega]{\cdot} C)$ is the collection of partial functions from $\Re$ to C, with finite domain.

A concept definition is composed of a collection of primitives conjoined together with a collection of role definitions. Each element of C is an ordered pair whose first component is a set of primitives from P, and whose second component is a set of ordered pairs, each arising from one of the role definitions. This set can just be represented as a partial function on $\Re$, defined for each role in the concept's definition. Note that C is a not a set in the normal ZFC sense, but in the sense of Aczel's theory of non-well-founded sets (roughly sets that can contain themselves as members). This is how circularity of concept definitions can be allowed.

To see that C is a concept algebra:

$$\top_C = \langle \emptyset, \emptyset \rangle$$

$$P_{i_C} = \langle \{P_i\}, \emptyset \rangle$$

$$R_C(x) = \langle \emptyset, \{\langle R, x \rangle\} \rangle$$

Given $C_1 = \langle Q_1, f_{C_1} \rangle$ where $Q_1 \subseteq P$ and $f_{C_1} \in (\Re \xrightarrow[<\omega]{\cdot} C)$ and similarly for $C_2 = \langle Q_2, f_{C_2} \rangle$ we define:

$$C_1 \wedge C_2 \equiv \langle Q_1 \cup Q_2, f_{C_1 \wedge C_2} \rangle$$

where $f_{C_1 \wedge C_2} \in (\Re \xrightarrow[<\omega]{\cdot} C)$ is defined as $f_{C_1 \wedge C_2}(R) =$

$$\begin{cases} f_{C_1}(R) & \text{if } R \in dom(f_{C_1}) - dom(f_{C_2}) \\ f_{C_2}(R) & \text{if } R \in dom(f_{C_2}) - dom(f_{C_1}) \\ f_{C_1}(R) \wedge f_{C_2}(R) & \text{if } R \in dom(f_{C_1}) \cap dom(f_{C_2}) \end{cases}$$

Technically, to define the meet operation on C, one must do a bit more than write down this recursive definition, because it is not entirely clear that this definition leads to a well defined function. However, the recursive definition can readily be translated into a simultaneous system of equations, to which the Solution Lemma [Aczel, 1988] can be applied, and this allows us to assert the existence and uniqueness of the meet operation. With respect to Δ, let us define what is meant by a *good* set of definitions.

Definition 2 *A set of definitions Δ is* good *if each x_i appears only once on the left hand side, and each equation $x_i = t_i$ is of the form $x_i = (\bigwedge_j P_{i,j}) \wedge (\bigwedge_k (R_{i,k}\ s_{i,k}))$ where $P_{i,j} \in P$ and $s_{i,k} \in A[X]\ \forall i, j, k$. In other words each equation is composed of a conjunction of a conjunction of primitives, and a conjunction of role terms that may or may not contain variables.*

Theorem 2 *There exists a unique concept algebra homomorphism $A_\Delta[x_1, \ldots, x_n] \xrightarrow{\varphi_\Delta} C$ if Δ is a* good set *of definitions*

Proof. Without loss of generality, we may take the generators $\{x_1, \ldots, x_n\}$, as atoms (or *Urelemente*), i.e., objects that are not sets and not in any way composed of sets. Using the interpretations in C, of the operators in Σ, the KB Δ can be interpreted as a system of equations for which we seek a solution,

$$x_1 = Y_1, \ldots, x_n = Y_n,$$

where each Y_i is a pure set (a set involving no atoms) in C. The existence and uniqueness of this solution follows immediately from Aczel's Solution Lemma [Aczel, 1988], p. 13. Hence there exists a unique homomorphism ρ from the free algebra $A[X]$ to C, such that $\rho(x_i) = Y_i\ \forall i$.

The fundamental homomorphism theorem then implies that there exists a unique mapping, call it ρ_A, from $A[X]/Ker\,\rho$ to C such that the following diagram commutes.

$$\begin{tikzcd} A[X] \arrow[r, "\rho"] \arrow[d] & C \\ A[X]/Ker\,\rho \arrow[ur, "\rho_A"'] & \end{tikzcd}$$

Provided that we can show that $Ker\,\rho$ contains $\equiv_\Delta$ we can apply Theorem 2.3 in [Jacobson, 1989] (page 64) to obtain a unique mapping ψ from $A_\Delta[X]$ to $A[X]/Ker\,\rho$ such that the following commutes.

$$\begin{tikzcd} A[X] \arrow[r] \arrow[d, "\pi_\Delta"'] & A[X]/Ker\,\rho \\ A_\Delta[X] \arrow[ur, "\psi"'] & \end{tikzcd}$$

But since ρ was defined on the generators x_i, we see that for $(x_i, t_i) \in \equiv_\Delta$, $\rho(x_i) = \rho(t_i)$, and therefore $\equiv_\Delta \subseteq Ker\,\rho$. The unique map from $A_\Delta[X]$ to C is the composition of ρ_A with ψ, i.e. $\varphi_\Delta = \rho_A \circ \psi$ $\Box$.

Now we have the following picture:

$$A[X] \xrightarrow{\pi_\Delta} A_\Delta[X] \xrightarrow{\varphi_\Delta} C$$

given two terms in $A[X]$, s_1 and s_2, *descriptive subsumption* says that

$$s_1 \sqsupseteq_\Delta s_2 \; iff \; \pi_\Delta s_1 \geq_{A_\Delta[X]} \pi_\Delta s_2$$

and now we can define *structural subsumption.*

Definition 3 *Given two terms $s_1, s_2 \in A[X]$, we say s_1 structurally subsumes s_2, written $s_1 \succeq_\Delta s_2$, iff $(\varphi_\Delta \circ \pi_\Delta)s_1 \geq_C (\varphi_\Delta \circ \pi_\Delta)s_2$.*

Structural subsumption is given by the ordering on C. We now explicitly state the ordering on C, to both relate it to $\wedge$ and the actual implementation in K-REP. Given $C_1 = \langle Q_1, f_{C_1}\rangle$ where $Q_1 \subseteq P$ and $f_{C_1} \in (\Re \xrightarrow[<\omega]{\bullet} C)$ and similarly for $C_2 = \langle Q_2, f_{C_2}\rangle$, we say that $C_1 \geq_C C_2$, if $Q_1 \subseteq Q_2$, and $dom(f_{C_1}) \subseteq dom(f_{C_2})$, and $\forall R \in dom(f_{c_1})$, $f_{C_1}(R) \geq_C f_{C_2}(R)$. This is exactly the test for subsumption of concepts stated in the previous section on the language. Notice also that when $C_1 \geq_C C_2$, if one inspects $C_1 \wedge C_2$, that $Q_1 \cup Q_2 = Q_2$ and since $dom(f_{C_1}) \subseteq dom(f_{C_2})$, that $f_{C_1 \wedge C_2}$ reduces to the second and third cases. The third case corresponds to $f_{C_1}(R) \geq_C f_{C_2}(R)$, and thus $C_1 \geq_C C_2$ iff $C_1 \wedge C_2 = C_2$.

The algorithm discussed in the previous section for coping with cycles, is essentially constructing objects in C, by first constructing an apg-like object in $\mathcal{D}$, for each term of $A[X]$ in which we are interested. This construction corresponds to the map Imp_Δ in the diagram below. Each apg in $\mathcal{D}$ then describes a unique set in C via the unnamed arrow. The testing of subsumption is actually testing the presence of a Hoare simulation between the objects in C. With respect to our implementation we now have the following commutative diagram.

$$\begin{tikzcd} & \mathcal{D} \arrow[dd] \\ A[X] \arrow[ur, "Imp_\Delta"] \arrow[dr, "\varphi_\Delta \circ \pi_\Delta"'] & \\ & C \end{tikzcd}$$

(Our new implementation of K-REP actually creates two spaces of objects representing concepts. Our *definition space* corresponds to $A_\Delta[X]$, and our *semantic space* corresponds to C. Thus the definition space allows for multiple definitions that might map to the same object in the semantic space).

The fact that φ_Δ preserves order proves the following claim:

Theorem 3 *Descriptive subsumption implies structural subsumption*

This shows us that descriptive subsumption is weaker than structural subsumption. Proposition 5.2 (page 133) of [Nebel, 1990], states that subsumption with respect to descriptive semantics is weaker than subsumption with respect to a least or greatest fixed point semantics. Let us now define *extensional subsumption.*

Definition 4 *Given two terms $s_1, s_2 \in A[X]$, we say s_1 extensionally subsumes s_2, written*

$$s_1 \supseteq_\Delta s_2, \; iff \; s_1^{\mathcal{M}} \supseteq s_2^{\mathcal{M}},$$

$\forall$ *extensional greatest fixed point models $\mathcal{M}$.*

It appears that our *structural subsumption* is equivalent to subsumption with respect to all extensional greatest fixed point models. Stated more succintly,

Conjecture 1

$$s_1 \succeq_\Delta s_2 \; iff \; s_1 \supseteq_\Delta s_2.$$

Conclusion

In this paper we have developed concept algebras as a new approach to semantics for term subsumption languages. We have shown that for a subset of the K-REP language, these algebras accurately model the process of subsumption testing, even in the presence of cycles. We feel this approach is somewhat simpler as these algebras model the concepts as descriptions, without reference to subsets of some universe. These algebras also allow for multiple definitions, i.e., concepts with different names that semantically represent the same concept. We are incorporating both these ideas into a new version of K-REP. Though we have only dealt with a subset of K-REP it appears that cardinality restrictions do not add much more complexity to the model. However role value maps certainly do and they warrant further investigation.

One immediate extension appears to be disjunction. Since one can view a disjunction as a collection of concept descriptions, one of which a given instance might satisfy, it seems that the finite power set of the algebra C under a suitable ordering (perhaps the Hoare), could provide an algebra that also allows for disjunctions.

References

Aczel, Peter 1988. *Non-Well-Founded Sets*, volume 14 of *CSLI Lectures Notes.* CSLI/Standford.

Baader, Franz 1990. Terminological cycles in kl-one-based knowledge representation languages. In *Proceedings of AAAI*, Boston, Mass.

Brachman, R. J. and Schmolze, J. G. 1985. *An Overview of the KL-One Knowledge Representation System*, volume 9. 171–216.

Jacobson, Nathan 1989. *Basic Algebra II.* W. H. Freeman and company.

Mays, E.; Dionne, R.; and Weida, R. 1991. K-rep system overview. *SIGART Bulletin* 2.

Nebel, Bernhard 1990. *Reasoning and Revision in Hybrid Representation Systems*, volume 422 of *Lecture Notes in Artificial Intelligence.* Springer-Verlag.

Woods, William A. 1991. *Understanding Subsumption and Taxonomy: A Framework for Progress.* Morgan Kaufman.