An Algebra for Cyclic Ordering of 2D Orientations

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Abstract

We define an algebra of ternary relations for cyclic ordering of 2D orientations. The algebra (1) is a refinement of the CYCORD theory; (2) contains 24 atomic relations, hence 2^{24} general relations, of which the usual CYCORD relation is a particular relation; and (3) is NP-complete, which is not surprising since the CYCORD theory is. We then provide: (1) a constraint propagation algorithm for the algebra, which we show is polynomial, and complete for a subclass including all atomic relations; (2) a proof that another subclass, expressing only information on parallel orientations, is NP-complete; and (3) a solution search algorithm for a general problem expressed in the algebra.

Introduction

Qualitative spatial reasoning (QSR) has become an important and challenging research area of Artificial Intelligence. An important aspect of it is topological reasoning (e.g. (Cohn 1997)). However, many applications (e.g., robot navigation (Levitt & Lawton 1990), reasoning about shape (Schlieder 1994)) require the representation and processing of orientation knowledge. A variety of approaches to this have been proposed: the CYCORD theory (Megiddo 1976; Röhrig 1994; 1997), Frank's (1992) and Hernández's (1991) sector models and Schlieder's (1993) representation of a panorama.

In real applications, CYCORDs may not be expressive enough. For instance, one may want to represent information such as "objects A, B and C are such that B is to the left of A; and C is to the left of both A and B, or to the right of both A and B", which is not representable in the CYCORD theory. This explains the need for refining the theory, which is what we propose in the paper. Before providing the refinement, which is an algebra of ternary relations, we shall define an algebra of binary relations which is much less expressive (it cannot represent the CYCORD relation). Among other things, we shall provide a composition table for the algebra of binary relations. One reason for doing this first is that it will then become easy to understand how the relations of the refinement are obtained.

We first provide some background on the CYCORD theory; then the two algebras. Next, we consider CSPs on cyclic ordering of 2D orientations. We then provide (1) a constraint propagation algorithm for the algebra of ternary relations, which we show is polynomial, and complete for a subclass including all atomic relations; (2) a proof that another subclass, expressing only information on parallel orientations, is NP-complete; and (3) a solution search algorithm for a general problem expressed in the algebra. Before summarising, we shall discuss some related work.

CYCORDs

Given a circle centred at O, there is a natural isomorphism from the set of 2D orientations to the set of points of the circle: the image of orientation X is the point P_X such that the orientation of the directed straight line (OP_X) is X. A CYCORD X-Y-Z represents the information that the images P_X, P_Y, P_Z of orientations X, Y, Z, respectively, are distinct and encountered in that order when the circle is scanned clockwise starting from P_X.

We now provide a brief background on the CYCORD theory, taken from (Megiddo 1976; Röhrig 1994; 1997). We consider a set $S = \{X_0, \ldots, X_n\}$:

Two linear orders $(X_{i_0}, \ldots, X_{i_n})$ and $(X_{j_0}, \ldots, X_{j_n})$ on S are called cyclically equivalent if: $\exists m \forall k (j_k = (i_k + m) mod(n+1))$. A total cyclic order on S is an equivalence class of linear orders on S modulo cyclic equivalence; X_{i_0}-…-X_{i_n} denotes the equivalence class containing $(X_{i_0}, \ldots, X_{i_n})$. A closed partial cyclic order on S is a set T of cyclically ordered triples such that:

(1)	X-Y-$Z \in T \Rightarrow X \neq Y$	(*irreflexivity*)
(2)	X-Y-$Z \in T \Rightarrow Z$-Y-$X \notin T$	(*asymmetry*)
(3)	$\{X$-Y-Z, X-Z-$W\} \subseteq T \Rightarrow X$-$Y$-$W \in T$	(*transitivity*)
(4)	X-Y-$Z \notin T \Rightarrow Z$-Y-$X \in T$	(*closure*)
(5)	X-Y-$Z \in T \Rightarrow Y$-Z-$X \in T$	(*rotation*)

The algebra of binary relations

The algebra is very similar to Allen's (1983) temporal interval algebra. We describe briefly its relations and its three operations.

Given an orientation X of the plane, another orientation Y can form with X one of the following qualitative configurations: (1) Y is equal to X (the angle (X, Y) is equal to 0); (2) Y is to the left of X (the angle (X, Y) belongs to $(0, \pi)$); (3) Y is opposite to X (the angle (X, Y) is equal to π); (4) Y is to the right of X (the angle (X, Y) belongs to $(\pi, 2\pi)$). The configurations, which we denote by *(Y e X)*, *(Y l X)*, *(Y o X)* and (Y r X), respectively, are Jointly Exaustive and Pairwise Disjoint (JEPD): given any two 2D orientations, they stand in one and only one of the configurations.

The algebra contains four atomic relations: e, l, o, r. A (general) relation is any subset of the set *BIN* of all four atomic relations (when a relation is a singleton set (atomic), we omit the braces in its representation). A relation $B = \{b_1, \ldots, b_n\}, n \leq 4$, between orientations X and Y, written $(Y\ B\ X)$, is to be interpreted as $(Y b_1 X) \vee \ldots \vee (Y b_n X)$.

The converse of an atomic relation b is the atomic relation $b^\smile$ such that: $\forall X, Y((Y\ b\ X) - (X\ b^\smile\ Y))$. The converse $B^\smile$ of a general relation B is the union of the converses of its atomic relations: $B^\smile = \bigcup_{b \in B}\{b^\smile\}$.

The intersection of two relations B_1 and B_2 is the relation B consisting of the set-theoretic intersection of B_1 and B_2: $B = B_1 \cap B_2$.

The composition of two relations B_1 and B_2, written $B_1 \otimes_2 B_2$, is the strongest relation B such that: $\forall X, Y, Z((Y B_1 X) \wedge (Z B_2 Y) \Rightarrow (Z B X))$.

The following tables give the converse $b^\smile$ of an atomic relation b (left), and the composition for atomic relations (right):

b	$b^\smile$
e	e
l	r
o	o
r	l

$\otimes_2$	e	l	o	r
e	e	l	o	r
l	l	$\{l,o,r\}$	r	$\{e,l,r\}$
o	o	r	e	l
r	r	$\{e,l,r\}$	l	$\{l,o,r\}$

Given three atomic binary relations b_1, b_2, b_3, we define the induced ternary relation $b_1 b_2 b_3$ as follows (see Figure 1(Top)(I)): $\forall X, Y, Z(b_1 b_2 b_3(X, Y, Z) - (Y b_1 X) \wedge (Z b_2 Y) \wedge (Z b_3 X))$. The composition table above has 12 entries consisting of atomic relations, the remaining four consisting of three-atom relations. Therefore any three 2D orientations stand in one of the following 24 JEPD configurations: *eee, ell, eoo, err, lel, lll, llo, llr, lor, lre, lrl, lrr, oeo, olr, ooe, orl, rer, rle, rll, rlr, rol, rrl, rro, rrr*.

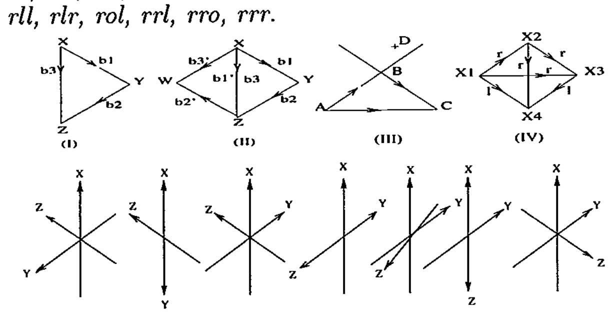

Figure 1: Illustrations.

Refining the CYCORD theory: The algebra of ternary relations

The algebra of binary relations introduced above cannot represent a CYCORD. However, if we use what we have called an *"induced ternary relation"*, we can easily define an algebra of ternary relations of which the CYCORD relation will be a particular relation.

Definition 1 (the relations) *An atomic ternary relation is any of the 24 JEPD configurations a triple of 2D orientations can stand in. We denote by TER the set of all atomic ternary relations: TER = {eee, ell, eoo, err, lel, lll, llo, llr, lor, lre, lrl, lrr, oeo, olr, ooe, orl, rer, rle, rll, rlr, rol, rrl, rro, rrr}. A (general) ternary relation is any subset* T *of TER:* $\forall X, Y, Z(T(X, Y, Z) - \bigvee_{t \in T} t(X, Y, Z))$.

As an example, a CYCORD X-Y-Z can be represented by the relation $CR = \{lrl, orl, rll, rol, rrl, rro, rrr\}$ (see Figure 1(Bottom)): $\forall X, Y, Z(X\text{-}Y\text{-}Z - CR(X, Y, Z))$.

Definition 2 (the operations) *The converse of an atomic ternary relation* t *is the atomic ternary relation* $t^\smile$ *such that:* $\forall X, Y, Z(t(X,Y,Z) - t^\smile(X, Z, Y))$. *The converse* $T^\smile$ *of a general ternary relation* T *is* $T^\smile = \bigcup_{t \in T}\{t^\smile\}$.

The rotation of an atomic ternary relation t *is the atomic ternary relation* $t^\frown$ *such that:* $\forall X, Y, Z(t(X,Y,Z) - t^\frown(Y, Z, X))$. *The rotation* $T^\frown$ *of a general ternary relation* T *is* $T^\frown = \bigcup_{t \in T}\{t^\frown\}$.

The following three tables provide the converse $t^\smile$ *and the rotation* $t^\frown$ *of an atomic ternary relation* t*:*

t	$t^\smile$	$t^\frown$
eee	eee	eee
ell	lre	lre
eoo	ooe	ooe
err	rle	rle
lel	lel	err
lll	lrl	lrr
llo	orl	lor
llr	rrl	llr

t	$t^\smile$	$t^\frown$
lor	rol	olr
lre	ell	rer
lrl	lll	rrr
lrr	rll	rlr
oeo	oeo	eoo
olr	rro	llo
ooe	eoo	oeo
orl	llo	rro

t	$t^\smile$	$t^\frown$
rer	rer	ell
rle	err	lel
rll	lrr	lrl
rlr	rrr	lll
rol	lor	orl
rrl	llr	rrl
rro	olr	rol
rrr	rlr	rll

The intersection of two ternary relations T_1 *and* T_2 *is the ternary relation* T *consisting of the set-theoretic intersection of* T_1 *and* T_2*:* $\forall X, Y, Z(T(X, Y, Z) - T_1(X, Y, Z) \wedge T_2(X, Y, Z))$.

The composition of two ternary relations T_1 *and* T_2*, written* $T_1 \otimes_3 T_2$*, is the most specific ternary relation* T *such that:* $\forall X, Y, Z, W(T_1(X, Y, Z) \wedge T_2(X, Z, W) \Rightarrow T(X, Y, W))$.

Given four 2D orientations X, Y, Z, W *and two atomic ternary relations* $t_1 = b_1 b_2 b_3$ *and* $t_2 = b'_1 b'_2 b'_3$*, the conjunction* $t_1(X, Y, Z) \wedge t_2(X, Z, W)$ *is inconsistent if* $b_3 \neq b'_1$ *(see Figure 1(Top)(II)). Stated otherwise, when* $b_3 \neq b'_1$ *we have* $t_1 \otimes_3 t_2 = \emptyset$*. Therefore, in defining composition for atomic ternary relations, we have to consider four cases: Case 1:* $b_3 = b'_1 = e$ *(*$t_1 \in \{eee, lre, ooe, rle\}$ *and* $t_2 \in \{eee, ell, eoo, err\}$*); Case 2:* $b_3 = b'_1 = l$*; Case 3:* $b_3 = b'_1 = o$*; Case 4:* $b_3 = b'_1 = r$*. The corresponding composition tables are given below (case* 1*, case* 3*, case* 2 *and case* 4*, respectively, from top to bottom, left to right). The entries* E_1, E_2, E_3, E_4 *stand for the relations* $\{lel, lll, lrl\}$*,* $\{llr, lor, lrr\}$*,* $\{rer, rlr, rrr\}$*,* $\{rll, rol, rrl\}$*, respectively. If* T_1 *and* T_2 *are general ternary relations:* $T_1 \otimes_3 T_2 = \bigcup_{t_1 \in T_1, t_2 \in T_2} t_1 \otimes_3 t_2$.

Definition 3 (projection and cross product) *The 1st, 2nd and 3rd projections of a ternary relation* T*,* *which we shall refer to as* $proj_1(T), proj_2(T), proj_3(T)$, *respectively, are the binary relations* $proj_1(T) = \{b_1 \in BIN|(\exists b_2, b_3 \in BIN|b_1 b_2 b_3 \in T)\}, proj_2(T) = \{b_2 \in BIN|(\exists b_1, b_3 \in BIN|b_1 b_2 b_3 \in T)\}, proj_3(T) = \{b_3 \in BIN|(\exists b_1, b_2 \in BIN|b_1 b_2 b_3 \in T)\}$. *The cross product of three binary relations* B_1, B_2, B_3, *written* $B_1 \times B_2 \times B_3$, *is the ternary relation* $B_1 \times B_2 \times B_3 = \{b_1 b_2 b_3|(b_1 \in B_1, b_2 \in B_2, b_3 \in B_3)\} \cap TER$.

$\otimes_3$	*eee*	*ell*	*eoo*	*err*
eee	*eee*	*ell*	*eoo*	*err*
lre	*lre*	E_1	*llo*	E_2
ooe	*ooe*	*orl*	*oeo*	*olr*
rle	*rle*	E_4	*rro*	E_3

$\otimes_3$	*oeo*	*olr*	*ooe*	*orl*
eoo	*eoo*	*err*	*eee*	*ell*
llo	*lre*	E_1	*llo*	E_2
oeo	*orl*	*olr*	*oor*	*orr*
rro	*rro*	E_3	*rle*	E_4

$\otimes_3$	*lel*	*lll*	*llo*	*llr*	*lor*	*lre*	*lrl*	*lrr*
ell	*ell*	*ell*	*eoo*	*err*	*err*	*eee*	*ell*	*err*
lel	*lel*	*lll*	*llo*	*llr*	*lor*	*lre*	*lrl*	*lrr*
lll	*lll*	*lll*	*llo*	E_2	*lrr*	*lre*	E_1	*lrr*
lrl	*lrl*	E_1	*llo*	*llr*	*llr*	*lre*	*lrl*	E_2
orl	*orl*	*orl*	*oeo*	*olr*	*olr*	*ooe*	*orl*	*olr*
rll	*rll*	E_4	*rro*	*rrr*	*rrr*	*rle*	*rll*	E_3
rol	*rol*	*rrl*	*rro*	*rrr*	*rer*	*rle*	*rll*	*rlr*
rrl	*rrl*	*rrl*	*rro*	E_3	*rlr*	*rle*	E_4	*rlr*

$\otimes_3$	*rer*	*rle*	*rll*	*rlr*	*rol*	*rrl*	*rro*	*rrr*
err	*err*	*eee*	*ell*	*err*	*ell*	*ell*	*eoo*	*err*
llr	*llr*	*lre*	*lrl*	E_2	*lrl*	E_1	*llo*	*llr*
lor	*lor*	*lre*	*lrl*	*lrr*	*lel*	*lll*	*llo*	*llr*
lrr	*lrr*	*lre*	E_1	*lrr*	*lll*	*lll*	*llo*	E_2
olr	*olr*	*ooe*	*orl*	*olr*	*orl*	*orl*	*oeo*	*olr*
rer	*rer*	*rle*	*rll*	*rlr*	*rol*	*rrl*	*rro*	*rrr*
rlr	*rlr*	*rle*	E_4	*rlr*	*rrl*	*rrl*	*rro*	E_3
rrr	*rrr*	*rle*	*rll*	E_3	*rll*	E_4	*rro*	*rrr*

CSPs of 2D orientations

A CSP of 2D orientations (henceforth 2D-OCSP) consists of (a) a finite number of variables ranging over the set 2DO of 2D orientations[1]; and (b) relations on cyclic ordering of these variables, standing for the constraints of the CSP. A binary (resp. ternary) 2D-OCSP is a 2D-OCSP of which the constraints are binary (resp. ternary). We shall refer to binary 2D-OCSPs as BOCSPs, and to ternary 2D-OCSPs as TOCSPs.

We now consider a 2D-OCSP P (either binary or ternary) on n variables $X_1, \ldots, X_n$.

Remark 1 (normalised 2D-OCSP) *If P is a BOCSP, we assume that for all* i, j, *at most one constraint involving* X_i *and* X_j *is specified. The network representation of P is the labelled directed graph defined as follows: (1) the vertices are the variables of P; (2) there exists an edge* (X_i, X_j), *labelled with B, if and only if a constraint of the form* $(X_j B X_i)$ *is specified. If P is a TOCSP, we assume that for all* i, j, k, *at most one constraint involving* X_i, X_j, X_k *is specified.*

Definition 4 (matrix representation) *If P is a BOCSP, it is associated with an* $n \times n$*-matrix, which we shall refer to as P for simplicity, and whose elements will be referred to as* $P_{ij}, i, j \in \{1, \ldots, n\}$. *The matrix P is constructed as follows: (1) Initialise all entries of P to the universal relation BIN:* $P_{ij} := BIN, \forall i, j$; *(2)* $P_{ii} := e, \forall i$; *(3)* $\forall i, j$ *such that P contains a constraint of the form* $(X_j\ B\ X_i)$: $P_{ij} := P_{ij} \cap B; P_{ji} := P_{ij}^{\smile}$.

If P is a TOCSP, it is associated with an $n \times n \times n$*-matrix, which we shall refer to as P, and whose elements will be referred to as* $P_{ijk}, i, j, k \in \{1, \ldots, n\}$. *The matrix P is constructed as follows: (1) Initialise all entries of P to the universal relation TER:* $P_{ijk} := TER, \forall i, j, k$; *(2)* $P_{iii} := eee, \forall i$; *(3) For all* i, j, k *such that P contains a constraint of the form* $T(X_i, X_j, X_k)$: $P_{ijk} := P_{ijk} \cap T; P_{ikj} := P_{\overset{\smile}{ijk}}; P_{jki} := P_{\overset{\frown}{ijk}}; P_{jik} := P_{\overset{\smile}{jki}}; P_{kij} := P_{\overset{\frown}{jki}}; P_{kji} := P_{\overset{\smile}{kij}}$; *(4) For all* $i, j, i < j$: *(a)* $B := \bigcap_{k=1}^{n} proj_1(P_{ijk})$; *(b)* $P_{iij} := e \times B \times B; P_{iji} := P_{\overset{\smile}{iij}}; P_{jii} := P_{\overset{\frown}{iji}}$; *(c)* $P_{jji} := e \times B^{\smile} \times B^{\smile}; P_{jij} := P_{\overset{\smile}{jji}}; P_{ijj} := P_{\overset{\frown}{jij}}$.

Without loss of generality, we make the assumption that a *TOCSP* is closed under projection: $\forall i, j, k, l (proj_1(P_{ijk}) = proj_1(P_{ijl}))$.

Definition 5 *(Freuder 1982) An instantiation of* P *is any n-tuple of* $[0, 2\pi)^n$, *representing an assignment of an orientation value to each variable. A consistent instantiation, or solution, is an instantiation satisfying all the constraints. A sub-CSP of size k,* $k \leq n$, *is any restriction of* P *to k of its variables and the constraints on the k variables.* P *is k-consistent if every solution to every sub-CSP of size* $k - 1$ *extends to every k-th variable; it is strongly k-consistent if it is j-consistent, for all* $j \leq k$.

1-, 2- and 3-consistency correspond to node-, arc- and path-consistency, respectively (Mackworth 1977; Montanari 1974). Strong n-consistency of P corresponds to global consistency (Dechter 1992). Global consistency facilitates the exhibition of a solution by backtrack-free search (Freuder 1982).

Remark 2 *A BOCSP is strongly* 2*-consistent. A TOCSP is strongly* 3*-consistent.*

We now assume that to the plane is associated a reference system (O, x, y); and refer to the circle centred at O and of unit radius as $\mathcal{C}_{O,1}$. Given an orientation z, we denote by $rad(z)$ the radius $(O, P_z]$ of $\mathcal{C}_{O,1}$, excluding the centre O, such that the orientation of the directed straight line (OP_z) is z. An orientation z can be assimilated to $rad(z)$.

Definition 6 (sector of a binary relation) *The sector,* $sect(z, B)$, *determined by an orientation z and a binary relation B is the sector of circle* $\mathcal{C}_{O,1}$, *excluding O, representing the set of orientations* z' *related to z by relation B:* $sect(z, B) = \{rad(z')|z'\ B\ z\}$.

Definition 7 *The projection,* $proj(P)$, *of a TOCSP P is the BOCSP* P' *having the same set of variables and such that:* $\forall i, j, k, P'_{ij} = proj_1(P_{ijk})$. *A ternary relation, T, is projectable if* $T = proj_1(T) \times proj_2(T) \times proj_3(T)$. *A TOCSP is projectable if for all* i, j, k, P_{ijk} *is a projectable relation.*

Definition 8 *Let B be a binary relation. The dimension of B is the dimension of its sector. B is convex if its sector is a convex part of the plane; it is holed if (1) it is equal to BIN, or (2) the difference* $BIN \setminus B$ *is a binary relation of dimension* 1 *(is equal to e, o or* $\{e, o\}$*). The subclass of all binary relations which are either convex or holed will be referred to as BCH.*

[1] The set 2DO is isomorphic to the set $[0, 2\pi)$.

A ternary relation is {convex,holed} if (1) it is projectable, and (2) each of its projections belongs to BCH. The subclass of all {convex,holed} ternary relations will be referred to as TCH.

Example 1 (the 'Indian tent') *The 'Indian tent' consists of a clockwise triangle (ABC), together with a fourth point* D *which is to the left of each of the directed lines (AB) and (BC) (see Figure 1(Top)(III)).*

The knowledge about the 'Indian tent' can be represented as a BOCSP on four variables, X_1, X_2, X_3 *and* X_4, *representing the orientations of the directed lines (AB), (AC), (BC) and (BD), respectively. From (ABC) being a clockwise triangle, we get a first set of constraints:* $\{(X_2 r X_1), (X_3 r X_1), (X_3 r X_2)\}$. *From D being to the left of each of the directed lines (AB) and (BC), we get* $\{(X_4 l X_1), (X_4 l X_3)\}$.

If we add the constraint $(X_4 r X_2)$ *to the BOCSP, this clearly leads to an inconsistency. Röhrig (1997) has shown that using the CYCORD theory one can detect such an inconsistency, whereas this cannot be detected using classical constraint-based approaches such as those in (Frank 1992; Hernández 1991).*

The BOCSP is represented graphically in Figure 1(Top)(IV). The CSP is path-consistent; i.e.: $\forall i, j, k (P_{ij} \subseteq (P_{ik} \otimes_2 P_{kj}))$.[2] *However, as mentioned above, the CSP is inconsistent. Therefore:*

Theorem 1 *Path-consistency does not detect inconsistency even for BOCSPs of atomic relations.*

The algebra of ternary relations is NP-complete:

Theorem 2 *Solving a TOCSP is NP-complete.*

Proof: We shall show that a TOCSP of atomic relations is polynomial. So, we need to prove that there exists a deterministic polynomial transformation of an NP-complete problem, e.g. a problem expressed in the CYCORD theory (Galil & Megiddo 1977), to a TOCSP. Such a problem, i.e., a conjunction of CYCORD relations, is so transformed by the rule illustrated in Figure 1(Bottom) transforming a CYCORD relation into a relation of the ternary algebra. ■

A constraint propagation algorithm

A constraint propagation procedure, $s4c(P)$, for TOCSPs is given below. The input is a TOCSP P on n variables, given by its $n \times n \times n$-matrix. When the algorithm completes, P verifies: $\forall i, j, k, l (P_{ijk} \subseteq P_{ijl} \otimes_3 P_{ilk})$.

The algorithm makes use of a queue *Queue*. Initially, we can assume that all variable triples (X_i, X_j, X_k) such that $1 \leq i < j < k \leq n$ are entered into *Queue*. The algorithm removes one variable triple from *Queue* at a time. When a triple (X_i, X_j, X_k) is removed from *Queue*, the algorithm eventually updates the relations on the neighbouring triples (triples sharing two variables with (X_i, X_j, X_k)). If such a relation is successfully updated, the corresponding triple is sorted, in such a way to have the variable with smallest index first and the variable with greatest index last, and the sorted triple is placed in *Queue* (if it is not already there) since it may in turn constrain the relations on neighbouring triples: this is done by add-to-queue(). The process terminates when *Queue* becomes empty.

```
procedure s4c(P);
repeat{
  get next triple (X_i, X_j, X_k) from Queue;
  for m := 1 to n{
    Temp := P_ijm ∩ (P_ijk ⊗3 P_ikm);
    if Temp ≠ P_ijm
      {add-to-queue(X_i, X_j, X_m);change(i, j, m, Temp);}
    Temp := P_ikm ∩ (P_ikj ⊗3 P_ijm);
    if Temp ≠ P_ikm
      {add-to-queue(X_i, X_k, X_m);change(i, k, m, Temp);}
    Temp := P_jkm ∩ (P_jki ⊗3 P_jim);
    if Temp ≠ P_jkm
      {add-to-queue(X_j, X_k, X_m);change(j, k, m, Temp);}
    }
  }
until Queue is empty;
procedure change(i, j, k, T);
  P_ijk := T; P_jki := T⌢; P_kij := P_jki⌢;
  P_ikj := T⌣; P_kji := P_ikj⌢; P_jik := P_kji⌢;
```

Theorem 3 *The constraint propagation algorithm runs into completion in* $O(n^4)$ *time.*

Proof (sketch). The number of variable triples (X_i, X_j, X_k) is $O(n^3)$. A triple may be placed in Queue at most a constant number of times (24, which is the total number of atomic relations). Every time a triple is removed from Queue for propagation, the algorithm performs $O(n)$ operations. ■

Complexity classes

Theorem 4 *The propagation procedure s4c(P) achieves strong 4-consistency for the input TOCSP P.*

Proof. A *TOCSP* is strongly 3-consistent (Remark 2). The algorithm clearly ensures 4-consistency, hence it ensures strong 4-consistency. ■

We refer to the subclass of all 28 entries of the four composition tables of the algebra of ternary relations as *CT*. We show that the closure under strong 4-consistency, CT^c, of *CT* is tractable. We then show that the subclass $PAR = \{\{oeo, ooe\}, \{eee, oeo, ooe\}, \{eee, eoo, ooe\}, \{eee, eoo, oeo, ooe\}\}$, which expresses only information on parallel orientations, is NP-complete.

Definition 9 *Let S denote a subclass of the algebra of ternary relations. The closure of S under strong 4-consistency, or s4c-closure of S, is the smallest subclass* S^c *of the algebra such that: (1)* $S \subseteq S^c$; *(2)* $\forall T_1, T_2 \in S^c (T_1^{\smile}, T_1^{\frown}, T_1 \cap T_2 \in S^c)$; *and (3)* $\forall T_1, T_2, T_3 \in S^c (proj_3(T_1) = proj_1(T_2) \wedge proj_1(T_1) = proj_1(T_3) \wedge proj_3(T_2) = proj_3(T_3) \Rightarrow T_3 \cap (T_1 \otimes_3 T_2) \in S^c)$.

Theorem 5 *Let P be a TOCSP expressed in TCH. If P is strongly 4-consistent then it is globally consistent.*

The proof will use the specialisation to $n = 2$ of Helly's convexity theorem (Chvátal 1983):

Theorem 6 (Helly's Theorem) *Let S be a set of convex regions of the n-dimensional space* $\mathbb{R}^n$. *If every* $n+1$ *elements in S have a non empty intersection then the intersection of all elements of S is non empty.*

[2] This can be easily checked using the composition table for atomic binary relations.

Proof of Theorem 5. Since P lies in the TCH subclass and is strongly 4-consistent, (1) it is equivalent to its projection, say P^{pr}, which is a $BOCSP$ expressed in BCH; and (2) P^{pr} is strongly 4-consistent.

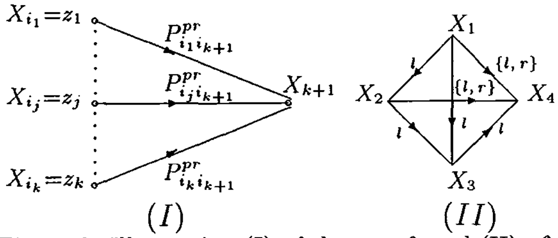

Figure 2: Illustration (I) of the proof; and (II) of non closure of TCH under strong 4-consistency.

So the problem becomes that of showing that P^{pr} is globally consistent. For this purpose, we suppose that the instantiation $(X_{i_1}, X_{i_2}, \dots, X_{i_k}) = (z_1, z_2, \dots, z_k), k \geq 4$, is a solution to a k-variable sub-CSP, say S, of P^{pr} whose variables are $X_{i_1}, X_{i_2}, \dots, X_{i_k}$. We need to prove that the partial solution can be extended to any $(k+1)$st variable, say $X_{i_{k+1}}$, of P^{pr}.[3] This is equivalent to showing that the following sectors have a non empty intersection (see Figure 2(I) for illustration): $sect(z_1, P^{pr}_{i_1 i_{k+1}}), sect(z_2, P^{pr}_{i_2 i_{k+1}}), \dots, sect(z_k, P^{pr}_{i_k i_{k+1}})$.

Since the $P^{pr}_{i_j i_{k+1}}, j = 1 \dots k$, belong to BCH, each of these sectors is (1) a convex subset of the plane, or (2) almost equal to the surface of circle $C_{O,1}$ (its topological closure is equal to that surface). We split these sectors into those verifying condition (1) and those verifying condition (2). We assume, without loss of generality, that the first m verify condition (1), and the last $k - m$ verify condition (2). We write the intersection of the sectors as $I = I_1 \cap I_2$, with $I_1 = \bigcap_{j=1}^{m} sect(z_j, P^{pr}_{i_j i_{k+1}}), I_2 = \bigcap_{j=m+1}^{k} sect(z_j, P^{pr}_{i_j i_{k+1}})$.

Due to strong 4-consistency, every three of these sectors have a non empty intersection. If any of the sectors is a radius (the corresponding relation is either e or o) then the whole intersection must be equal to that radius since the sector intersects with every other two.

We now need to show that when no sector reduces to a radius, the intersection is still non empty:

Case 1: m=k. This means that all sectors are convex. Since every three of them have a non empty intersection, Helly's theorem immediately implies that the intersection of all sectors is non empty.

Case 2: m=0. This means that no sector is convex; which in turn implies that each sector is such that its topological closure covers the whole surface of $C_{O,1}$. Hence, for all $j = 1 \dots k$: (1) $sect(z_j, P^{pr}_{i_j i_{k+1}})$ is equal to the whole surface of $C_{O,1}$ minus the centre ($P_{i_j i_{k+1}}$ is equal to BIN), or (2) $sect(z_j, P^{pr}_{i_j i_{k+1}})$ is equal to the whole surface of $C_{O,1}$ minus one or two radii ($P_{i_j i_{k+1}}$ is equal to $\{e,l,r\}$, $\{l,o,r\}$ or $\{l,r\}$). So the intersection of all sectors is equal to the whole surface of $C_{O,1}$ minus a finite number (at most $2k$) of radii. Since the surface is of dimension 2 and a radius is of dimension 1, the intersection must be non empty.

Case 3: $0 < m < k$. This means that some sectors (at least one) are convex, the others (at least one) are such that their topological closures cover the whole surface of $C_{O,1}$. The intersection I_1 is non empty due to Helly's theorem, since every three sectors appearing in it have a non empty intersection:

Subcase 3.1: I_1 is a single radius, say r. Since the sectors appearing in I_1 are less than π, there must exist two sectors, say s_1 and s_2, appearing in I_1 such that their intersection is r. Since, due to strong 4-consistency, s_1 and s_2 together with any sector appearing in I_2 form a non empty intersection, the whole intersection, i.e. I, must be equal to r.

Subcase 3.2: I_1 is a 2-dimensional (convex) sector. The intersection I_2 is the whole surface of $C_{O,1}$ minus a finite number (at most $2(k-m)$) of radii. Since a finite union of radii is of dimension 0 or 1, and that the intersection I_1 is of dimension 2, the whole intersection I must be non empty (of dimension 2).

The intersection of all sectors is non empty in all cases. The partial solution can hence be extended to variable $X_{i_{k+1}}$ (which can be instantiated with any orientation in the intersection of the k sectors). ■

It follows from Theorems 3, 4 and 5 that if the TCH subclass is closed under strong 4-consistency, it must be tractable. Unfortunately, as illustrated by the following example, TCH is not so closed.

Example 2 *The $BOCSP$ depicted in Figure 2(II) can be represented as the projectable $TOCSP$ P whose matrix representation verifies: $P_{123} = lll, P_{124} = l \times \{l,r\} \times \{l,r\}, P_{134} = P_{234} = l \times l \times \{l,r\}$. Applying the propagation algorithm to P leaves unchanged $P_{123}, P_{134}, P_{234}$, but transforms P_{124} into the relation $\{lll, llr, lrr\}$, which is not projectable: this is done by the operation $P_{124} := P_{124} \cap (P_{123} \otimes_3 P_{134})$.*

$\emptyset$	$l \times r \times r$	$r \times r \times r$	$\{l,r\} \times l \times l$
$e \times e \times e$	$o \times e \times o$	$l \times \{e,l,r\} \times l$	$r \times l \times \{e,l,r\}$
$e \times l \times l$	$o \times l \times r$	$l \times \{l,o,r\} \times r$	$r \times r \times \{l,o,r\}$
$e \times o \times o$	$o \times o \times e$	$r \times \{e,l,r\} \times r$	$l \times l \times \{l,o,r\}$
$e \times r \times r$	$o \times r \times l$	$r \times \{l,o,r\} \times l$	$r \times \{l,r\} \times r$
$l \times e \times l$	$r \times e \times r$	$\{e,l,r\} \times r \times r$	$l \times l \times \{l,r\}$
$l \times l \times l$	$r \times l \times e$	$\{l,o,r\} \times l \times r$	$l \times r \times \{e,l,r\}$
$l \times l \times o$	$r \times l \times l$	$\{e,l,r\} \times l \times l$	$r \times \{l,r\} \times l$
$l \times l \times r$	$r \times l \times r$	$\{l,r\} \times l \times r$	$l \times \{l,r\} \times l$
$l \times o \times r$	$r \times o \times l$	$\{l,r\} \times r \times r$	$r \times r \times \{l,r\}$
$l \times r \times e$	$r \times r \times l$	$\{l,o,r\} \times r \times l$	$l \times \{l,r\} \times r$
$l \times r \times l$	$r \times r \times o$	$\{l,r\} \times r \times l$	$r \times l \times \{l,r\}$
			$l \times r \times \{l,r\}$

Enumerating CT^c leads to 49 relations (see table above) all of which lie in TCH. Therefore:

Corollary 1 *CT^c is tractable.*

Proof. Immediate from Theorems 3, 4 and 5. ■

Example 3 *Transforming the $BOCSP$ of the 'Indian tent' into a $TOCSP$, say P', leads to $P'_{123} = rrr, P'_{124} = rrl, P'_{134} = rll, P'_{234} = rlr$. P' lies in CT^c, hence the*

[3]Since the $TOCSP$ P is projectable, any solution to any sub-CSP of the projection P^{pr} is solution to the corresponding sub-CSP of P. This would not be necessarily the case if P were not projectable.

propagation algorithm must detect its inconsistency. Indeed, the operation $P'_{124} := P'_{124} \cap (P'_{123} \otimes_3 P'_{134})$ leads to the empty relation, since $rrr \otimes_3 rll = rll$.

Theorem 7 *The subclass PAR is NP-complete.*

Proof. The subclass *PAR* belongs to NP, since solving a TOCSP of atomic relations is polynomial. We need to prove that there exists a (deterministic) polynomial transformation of an NP-complete problem (we consider 3-SAT: a SAT problem of which every clause contains exactly three literals) into a TOCSP expressed in *PAR* in such a way that the former is satisfiable if and only if the latter is consistent.

Suppose that S is a 3-SAT problem, and denote by: (1) $Lit(S) = \{\ell_1, \ldots, \ell_n\}$ the set of literals appearing in S; (2) $Cl(S)$ the set of clauses of S; (3) $BinCl(S)$ the set of binary clauses which are subclauses of clauses in $Cl(S)$. The TOCSP, P_S, we associate with S is as follows. Its set of variables is $V = \{X(c) | c \in Lit(S) \cup BinCl(S)\} \cup \{X_0\}$. X_0 is a truth determining variable: all orientations which are equal to X_0 correspond to elements of $Lit(S) \cup BinCl(S)$ which are true, the others (those which are opposite to X_0) to elements of $Lit(S) \cup BinCl(S)$ which are false. The constraints of P_S are constructed as follows: (a) for all pair $(X(p), X(\overline{p}))$ of variables such that $\{p, \overline{p}\} \subseteq Lit(S)$, p and $\overline{p}$ should have complementary truth values; hence $X(p)$ and $X(\overline{p}))$ should be opposite to each other in P_S: $\{oeo, ooe\}(X(p), X(\overline{p}), X_0)$; (b) for all variables $X(c_1), X(c_2)$ such that $c_1 \vee c_2$ is a clause of S, c_1 and c_2 cannot be simultaneously false; translated into P_S, $X(c_1)$ and $X(c_2)$ should not be both opposite to X_0: $\{eee, oeo, ooe\}(X(c_1), X(c_2), X_0)$; (c) for all variables $X(\ell_1 \vee \ell_2), X(\ell_1)$, if ℓ_1 is true then so is $(\ell_1 \vee \ell_2)$: $\{eee, eoo, ooe\}(X(\ell_1 \vee \ell_2), X(\ell_1), X_0)$; (d) for all other triple $(X, Y, Z) \in V^3$ of variables, add to P_S the constraint $\{eee, eoo, oeo, ooe\}(X, Y, Z)$.

The transformation is deterministic and polynomial. If M is a model of S, it is mapped to a solution of P_S as follows. X_0 is assigned any value of $[0, 2\pi)$. For all $\ell \in Lit(S)$, $X(\ell)$ is assigned the same value as X_0 if M assigns the value true to literal ℓ, the value opposite to that of X_0 otherwise. For all $(\ell_1 \vee \ell_2) \in BinCl(S)$, $X(\ell_1 \vee \ell_2)$ is assigned the same value as X_0 if either $X(\ell_1)$ or $X(\ell_2)$ is assigned the same value as X_0, the opposite value otherwise. On the other hand, any solution to P_S can be mapped to a model of S by assigning to every literal ℓ the value true if and only if the variable $X(\ell)$ is assigned the same value as X_0. ■

A solution search algorithm

Since the constraint propagation procedure *s4c* is complete for the subclass of atomic ternary relations (Corollary 1), it is immediate that a general *TOCSP* can be solved using a solution search algorithm such as the one below, which is similar to the one provided by Ladkin and Reinefeld (1992) for temporal interval networks, except that (1) it instantiates triples of variables at each node of the search tree, instead of pairs of variables, and (2) it makes use of the procedure *s4c*, which achieves strong 4-consistency, in the preprocessing step and as the filtering method during the search, instead of a path consistency procedure. The other details are similar to those of Ladkin and Reinefeld's algorithm.

```
Input: the matrix representation of a TOCSP P;
Output: true if and only if P is consistent;
function consistent(P);
  s4c(P);
  if(P contains the empty relation)return false;
  else
    if(P contains triples labelled with non atomic relations){
      choose such a triple, say (X_i, X_j, X_k);
      T := P_ijk;
      for each atomic relation t in T{
        instantiate triple (X_i, X_j, X_k) with t (P_ijk := t);
        if(consistent(P))return true;
      }
      P_ijk := T;
      return false;
    }
    else return true;
```

Related work

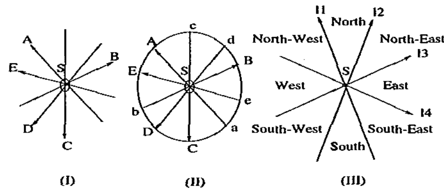

Figure 3: (I-II) The panorama of a location; and (III) Frank's system of cardinal directions.

Representing a panorama. Figure 3(I-II) illustrates the panorama of an object S with respect to five reference objects (landmarks) A, B, C, D, E in Schlieder's system (1993) (page 527). The panorama is described by the total cyclic order of the five directed straight lines $(SA), (SB), (SC), (SD), (SE)$, and the lines which are opposite to them, namely $(Sa), (Sb), (Sc), (Sd), (Se)$: (SA)-(Sc)-(Sd)-(SB)-(Se)-(Sa)-(SC)-(SD)-(Sb)-(SE). By using the algebra of binary relations, only the five straight lines joining S to the landmarks are needed to describe the panorama: $\{(SB)r(SA), (SC)r(SB), (SD)r(SB), (SD)r(SC), (SE)l(SB), (SE)l(SA)\}$; using the algebra of ternary relations, the description can be given as a 2-relation set: $\{rll((SA), (SB), (SE)), rrr((SB), (SC), (SD))\}$.

Because the algebra of binary relations contains the relations e(qual) and o(pposite), its use rules out an implicit assumption which seems to be made in Schlieder's system, which is that the object to be located (i.e., S) is supposed not to be on any of the straight lines joining pairs of the reference objects. Finally, note that Schlieder does not describe how to reason about a panorama description.

Sector models for reasoning about orientations. These models use a partition of the plane into sectors determined by straight lines passing through the reference object, say S. The sectors are generally equal, and the granularity of a sector model is determined by the number of sectors, therefore by the number of straight lines (n straight lines determine $2n$ sectors). Determining the relation of another object relative to the reference object becomes then the matter of giving the sector to which the object belongs.

Suppose that we consider a model with $2n$ sectors, determined by n (directed) straight lines $\ell_1, \ldots, \ell_n$ which we shall refer to as reference lines. We can assume without loss of generality that (the orientations of) the reference lines verify: ℓ_j is to right of ℓ_i (i.e., $(\ell_j r \ell_i)$), for all $j \in \{2, \ldots, n\}$, for all $i \in \{1, \ldots, j-1\}$. We refer to the sector determined by ℓ_i and ℓ_{i+1}, $i = 1 \ldots n-1$, as s_i, to the sector determined by ℓ_n and the directed line opposite to ℓ_1 as s_n. For each sector $s_i, i = 1 \ldots n$, the opposite sector will be referred to as s_{n+i}. Figure 3(III) illustrates these notions for Frank's (1992) system of cardinal directions, for which $n = 4$: $\ell_1, \ldots, \ell_4$ are as indicated in the figure; $s_1, \ldots, s_{2\times4}$ are North,North-East,East,South-East,South,South-West,West and North-West, respectively. Hernández's (1991) sector models can also benefit from this representation.

Suppose that a description is provided, consisting of qualitative positions of objects relative to the reference object S. S may be a robot for which the current panorama has to be given; the description may consist of sentences such as "landmark 1 is north-east, and landmark 2 south of the robot". Such a description can be translated into a *BOCSP P* in the following natural way. P includes all the relations described above on pairs of the reference lines. For each sentence such as the one above, the relations $(X_{(rob,1)}\ r\ \ell_2)$, $(X_{(rob,1)}\ l\ \ell_3)$, $(X_{(rob,2)}\ l\ \ell_1)$ and $(X_{(rob,2)}\ r\ \ell_2)$ are added to P. $X_{(rob,1)}$, for instance, stands for the orientation of the directed straight line joining the reference object 'robot' to landmark 1.

Summary and future work

We have provided a refinement of the theory of cyclic ordering of 2D orientations, known as CYCORD theory (Megiddo 1976; Röhrig 1994; 1997). The refinement has led to an algebra of ternary relations, for which we have given a constraint propagation algorithm and shown several complexity results.

A discussion of some related work in the literature has highlighted the following: (1) Existing systems for reasoning about 2D orientations are covered by the presented approach (CYCORDs (Megiddo 1976; Röhrig 1994; 1997) and sector models (Frank 1992; Hernández 1991)); (2) The presented approach seems more adequate than the one in (Schlieder 1993) for the representation of a panorama.

There has been much work on Allen's interval algebra (1983), e.g. Nebel and Bürckert's (1995) maximal tractable subclass. Most of this work could be adapted for the two algebras of 2D orientations we have defined.

Finally, a calculus of 3D orientations, similar to what we have presented for 2D orientations, might be developed.

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