Disjunctive Temporal Reasoning in Partially Ordered Models of Time

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Abstract

Certain problems in connection with, for example, cooperating agents and distributed systems require reasoning about time which is measured on incomparable or unsynchronized time scales. In such situations, it is sometimes approporiate to use a temporal model that only provides a partial order on time points. We study the computational complexity of partially ordered temporal reasoning in expressive formalisms consisting of point algebras extended with disjunctions. We show that the resulting algebra for partially ordered time contains four maximal tractable subclasses while the equivalent algebra for total-ordered time contains two.

Keywords: Temporal reasoning, constraint satisfaction, computational complexity.

Introduction

Many problems in Artificial Intelligence includes temporal reasoning in some form. Research on temporal reasoning has mainly focused on linear models of time, *cf.* Allen (1983). However, it is clear that more complex time models are needed in a variety of applications such as planning, cooperating robots, analysis of distributed systems etc. For analyzing such systems, more sophisticated time models such as *partially ordered* time (Anger 1989; Lamport 1986) have been proposed.

In Broxvall and Jonsson (1999) the satisfiability problem for the point algebra over partially ordered time is classified with respect to tractability; the result shows that there are three maximal tractable subclasses. Such classifications are feasible since the number of basic relations in point algebras are relatively small— the point algebra for partially-ordered time contains four. The number of basic relations increases when considering other representational entities, though. To exemplify, the interval algebra for total-ordered time contains 13 basic relations and 2^{8192} subclasses while the interval algebra for partially-ordered time contains 29 basic relations and $2^{2^{29}}$ subclasses. Needless to say, attempts to completely classify such algebrae have so far failed.

Instead of considering all possible subclasses of a temporal algebra, one can concentrate on subclasses that are "interesting" in some sense. We will study subclasses arising from extending point algebrae with disjunctions. Whether these subclasses are interesting or not is of course a matter of taste but we can provide some evidence that they are worth studying. First, simple constraint languages extended with disjunctions have historically proved to have interesting properties. For instance, the Horn DLRs (Jonsson & Bäckström 1998) which subsumes almost all previously presented temporal languages for total-ordered time can be viewed as a point algebra extended with disjunctions. Several other similar examples are given in Cohen *et al.* (1997). Secondly, disjunctions can compactly describe complex relations. Consider for example the ORD-Horn algebra (Nebel & Bürckert 1995) which is a tractable subclass of Allen's algebra. It contains 868 different relations and is consequently quite difficult to remember. Defining the ORD-Horn with the aid of disjunctions is much easier: ORD-Horn contains exactly the Allen relations which can be expressed by disjunctions of the form $x_1 \leq y_1 \vee x_2 \neq y_2 \vee \ldots \vee x_n \neq y_n$.

The main result of this paper is a total classification of tractability in the point algebra for partially ordered time extended with disjunctions. Our results show that there exists four maximal tractable subclasses. As a spin-off effect, we also get a total classification of the point algebra for total-ordered time extended with disjunctions; in this case, there are two maximal tractable subclasses.

The paper has the following organization: We begin by defining the basic concepts such as point algebrae and disjunctions. Thereafter we introduce the four tractable subclasses and provide a number of NP-completeness results that are needed in the classification. Finally, the classifications for partial-ordered and total-ordered time are carried out in the two last sections.

The point algebra

The point algebra is based on the notion of *relations* between pairs of variables interpreted over a partially-ordered set. In this paper we consider four *basic relations* which we denote by $<, >, =$ and $||$. If x, y are points in a partial order $\langle T, \leq \rangle$ then we define these relations in terms of the partial ordering $\leq$ as follows:

1. $x < y$ iff $x \leq y$ and not $y \leq x$
2. $x > y$ iff $y \leq x$ and not $x \leq y$
3. $x = y$ iff $x \leq y$ and $y \leq x$

4. $x||y$ iff neither $x \leq y$ nor $y \leq x$

The relations in a point algebra are always disjunctions of basic relations and they are represented as sets of basic relations. Since we have 4 different basic relations we get $2^4 = 16$ possible disjunctive relations. The set of basic relations is denoted $\mathcal{B}$ and the set of all 16 relations is denoted by $\mathcal{PA}$. Sometimes we use a short-hand notation for certain relations, for example, $\{<, =\}$ is sometimes written as $\leq$ and $\{=, ||\}$ as $\underline{||}$. The empty relations is denoted by $\perp$ and it is always unsatisfied. We will implicitly assume that the converse of relations are available. For instance, if $(<)$ is in a set of relations Γ, then the converse $(>)$ is also in Γ.

The basic computational problem of the point algebra is the satisfiability problem where we have a set of variables and a set of constraints over the variables and the question is whether there exists a mapping from the variables to some partial order such that all constraints are satisfied.

Definition 1 Let $\Re \subseteq \mathcal{PA}$ be a set of point relations and P a class of partial orders. A problem instance of $\mathrm{SAT_p}(\Re)$ is a set of variables V and a set of binary constraints of the form xRy where $x, y \in V$ and $R \in \Re$. A tuple $\langle f, \langle T, \leq \rangle \rangle$ where $f : V \to T$ is a total function and $\langle T, \leq \rangle \in P$ is called an interpretation of Π.

A problem instance Π is satisfiable iff there exists an interpretation $M = \langle f, \langle T, \leq \rangle \rangle$ such that $f(x)\ R\ f(y)$ holds for every constraint xRy. Such an M is called a *model* of Π.

Given an instance Π of $\mathrm{SAT_p}(\Re)$ and two variables x, y we write $x \leq^+ y$ to say that there exists zero or more variables $z_1, \ldots, z_n$ such that

$$x \leq z_1 \wedge z_1 \leq z_2 \wedge \cdots \wedge z_{n-1} \leq z_n \wedge z_n \leq y$$

and we write $x \leq^* y$ to say $x \leq^+ y$ or $x = y$.

We will consider two classes of partial orders: *po* which is the class of all partial orders and *to* $\subset$ *po* which is the class of all total orders.

Disjunctions

We will now extend the point algebra with disjunctions.

Definition 2 Let R_1, R_2 be relations of arity i, j and define the disjunction $R_1 \vee R_2$ of arity $i + j$ as follows:

$$R_1 \vee R_2 = \{(x_1, \ldots, x_{i+j}) \in D^{i+j} | (x_1, \ldots, x_i) \in R_1 \vee (x_{i+1}, \ldots, x_{i+j}) \in R_2\}$$

Thus, the disjunction of two relations with arity i, j is the relation with arity $i + j$ satisfying either of the two relations.

Let Γ_1, Γ_2 be sets of relations and define the disjunction $\Gamma_1 \overset{\times}{\vee} \Gamma_2$ as follows:

$$\Gamma_1 \overset{\times}{\vee} \Gamma_2 = \Gamma_1 \cup \Gamma_2 \cup \{R_1 \vee R_2 | R_1 \in \Gamma_1, R_2 \in \Gamma_2\}$$

The disjunction of two sets of relation $\Gamma_1 \overset{\times}{\vee} \Gamma_2$, first introduced by Cohen *et al.*(Cohen, Jeavons, & Koubarakis 1997), is the set of disjunctions of each pair of relations in Γ_1, Γ_2 plus the sets Γ_1, Γ_2. It is a natural to include Γ_1 and Γ_2 since one wants to have the choice of using the disjunction or not. The fact that if $R_1 \vee R_2$ is included in a set of relations, then both R_1 and R_2 are in the set is an property which we refer to as the $\overset{\times}{\vee}$-closure property. In the sequel, we will tacitly assume that all sets of relations that we consider can be constructed from $\mathcal{PA}$ and $\overset{\times}{\vee}$ and that they, consequently, satisfy the $\overset{\times}{\vee}$-closure property. In many cases we shall be concerned with constraints that are specified by disjunctions of an arbitrary number of relations. Thus, we make the following definition: for any set of relations, Δ, define $\Delta^* = \bigcup_{i=0}^{\infty} \Delta^{\vee i}$ where $\Delta^{\vee 0} = \{\perp\}$ and $\Delta^{\vee i+1} = \Delta^{\vee i} \overset{\times}{\vee} \Delta$.

The previously defined concepts of problem instances, interpretations, models and so on can obviously be extended to disjunctions in a natural way. It is worth noting that the problem $\mathrm{SAT_{po}}$ is in NP even if we allow the use of disjunctions.

We say that a set of constraints Γ is *maximal tractable* iff $\mathrm{SAT_p}(\Gamma)$ is tractable and for every set $X \not\subseteq \Gamma$ of relations which can be constructed by the relations in $\mathcal{PA}$ and the $\overset{\times}{\vee}$ operator, $\mathrm{SAT_p}(\Gamma \cup X)$ is not tractable.

Finally, we introduce the independence property as defined by Cohen *et al.* (1997). This concept will be used extensively for showing tractability results.

Definition 3 For any sets of relations Γ and Δ, we say that Δ is independent with respect to Γ if for any set of constraints C in $\mathrm{SAT_{po}}(\Gamma \cup \Delta)$, C has a solution whenever every $C' \subseteq C$, which contains at most one constraint whose constraint relation belongs to Δ, has a solution.

Theorem 4 For any sets of relations Γ and Δ, if $\mathrm{SAT_p}(\Gamma \cup \Delta)$ is tractable and Δ is independent with respect to Γ, then $\mathrm{SAT_p}(\Gamma \overset{\times}{\vee} \Delta^*)$ is tractable.

Gadgets

Gadgets allow us to concentrate on small sets of relations while proving tractability for large sets.

Definition 5 Let Ψ be a satisfiable instance of $\mathrm{SAT_{po}}(\Gamma)$ containing the variables $x, y, z_1, \ldots, z_n$ and assume R to be a relation not in Γ. We say that Γ *implements* R iff the following holds: in every problem instance Θ containing a constraint xRy the following holds: Θ is satisfiable iff Θ with xRy replaced by a fresh copy of the gadget Ψ is satisfiable.

We assume that the variables $z_1, \ldots, z_n$ does not appear in the instance Θ. It is then easy to prove the following lemma.

Lemma 6 Let Γ be a set of relations such that $\mathrm{SAT_p}(\Gamma)$ is tractable and assume there exists a gadget Ψ from Γ implementing γ, then $\mathrm{SAT_p}(\Gamma \cup \{\gamma\})$ is tractable.

We also show that independence is preserved under the introduction of gadgets.

Theorem 7 Let Δ be a set of relations independent of Γ and assume there exists a gadget Ψ from Δ implementing the relation δ. Then, $\Delta \cup \{\delta\}$ is independent of Γ.

Proof: Let Π be an arbitrary instance of $\Gamma \cup \Delta \cup \{\delta\}$ and let S be the subinstance of Π only containing relations from $\Gamma - \Delta$. Assume that every subset $S \cup \{x_1 \delta_1 y_1\}, \ldots, S \cup \{x_n \delta_n y_n\}$ of Π where $\delta_i \in \Delta \cup \{\delta\}$ is satisfiable. Each such set of constraints can be rewritten on the form $S \cup \Pi_i$

1 **algorithm** A
2 Input: An instance Π of $\mathrm{SAT_{po}}(\Gamma_A)$ or $\mathrm{SAT_{po}}(\Gamma_B)$
3 **repeat**
4 $\Pi' \leftarrow \Pi$
5 **for** each pair of nodes $n_1, n_2 \in \Pi$ **do**
6 **if** $n_1 || n_2$ and $n_1 \leq^* n_2$ **then reject**
7 **if** $n_1 \underline{||} n_2$ and $n_1 \leq^* n_2$ **then**
8 **if** $n_1 \neq n_2$ **then reject**
9 **else** $\Pi' \leftarrow$ contract(Π, n_1, n_2)
10 **elsif** $n_1 \leq^* n_2$ and $n_2 \leq^* n_1$ **then**
11 **if** $n_1 \neq n_2 \in \Pi$ **then reject**
12 **else** $\Pi' \leftarrow$ contract(Π, n_1, n_2)
13 **end if**
14 **end for**
15 **until** $\Pi' = \Pi$
16 **accept**

Figure 1: The algorithm for solving $\mathrm{SAT_{po}}(\Gamma'_A)$ and $\mathrm{SAT_{po}}(\Gamma'_B)$

where Π_i is either a constraint from Δ or a set of constraints resulting from replacing the δ_i constraint by the gadget Ψ. Obviously each set $S \cup \Pi_i$ is also satisfiable as well as all the relaxed sets $S \cup \{\pi_{i,1}\}, \ldots, S \cup \{\pi_{i,m}\}$, where $\{\pi_{i,1}, \ldots, \pi_{i,m}\}$ are the constraints in Π_i. Since Δ is independent of Γ and Π_i only contains relations from Δ, we know that $S \cup H$ where $H = \{\pi_{i,j} | i \leq n, j \leq m\}$ is satisfiable. Since $S \cup H$ is the result of replacing each δ constraint in Π by the gadget Ψ, we have shown that Π is satisfiable and hence, that $\Delta \cup \{\delta\}$ is independent of Γ. $\Box$

Tractability Results

We continue by defining four classes of disjunctive relations $\mathcal{T}_A, \ldots, \mathcal{T}_D$ and show that their corresponding satisfiability problems are tractable. The classes are defined as follows: $\mathcal{T}_A = \Gamma_A \overset{\times}{\vee} \Delta^*_A$, $\mathcal{T}_B = \Gamma_B \overset{\times}{\vee} \Delta^*_B$, $\mathcal{T}_C = \Gamma_C \overset{\times}{\vee} \Delta^*_C$ and $\mathcal{T}_D = \Delta^*_D$ and the exact definitions of the sets of relations can be found in Table 1. The classes $\mathcal{T}_A$ and $\mathcal{T}_B$ are extensions of the algebras $\mathcal{A}_{14}$ and $\mathcal{A}_{10}$ as defined by Broxvall and Jonsson (1999). It should be noted that if a problem instance of $\mathcal{T}_B$ has a model, then it has a model that is a total order. The class $\mathcal{T}_D$ is trivial since if an instance has a model, then it has a one-point model. The class $\mathcal{T}_C$ is quite obscure but note that its basic relations are a subset of the basic relations in $\mathcal{T}_A$.

Contractions are needed in order to understand the algorithms presented in this section. Let n_1, n_2 be two variables in a problem instance Π with variable set V and constraints C and let Π' be a problem instance with variables $V - \{n_2\}$ and constraints $(C - \{n_2 R x, x R n_2 \mid x \in V\}) \cup \{n_1 R x \mid n_2 R x \in C\} \cup \{x R n_1 \mid x R n_1 \in C\}$. We say that Π' is obtained by *contracting* n_1, n_2, that is, we identify the nodes n_1, n_2 by n_1. Note that this operation may introduce constraints of the form $n_1\ R\ n_1$.

Lemma 8 $\mathrm{SAT_{po}}(\Gamma'_A)$, $\mathrm{SAT_{po}}(\Gamma'_B)$ and $\mathrm{SAT_{po}}(\Gamma'_C)$ are tractable.

1 **algorithm** C
2 Input: An instance Π of $\mathrm{SAT_{po}}(\Gamma_C)$
3 **repeat**
4 $\Pi' \leftarrow \Pi$.
5 **if** exists n_1, n_2 such that $n_1 = n_2$ in Π **then** $\Pi \leftarrow$ contract(Π)
6 **until** $\Pi' = \Pi$
7 **if** exists node n such that $n \neq n$ or $n < n$ in Π **then reject**
8 **if** $n_1 <^* n_2$ and $n_1\ R\ n_2$ in Π where $< \notin R$ **then reject**
9 **accept**

Figure 2: The algorithm for solving $\mathrm{SAT_{po}}(\Gamma'_C)$

Proof: An algorithm for $\mathrm{SAT_{po}}(\Gamma'_A)$ and $\mathrm{SAT_{po}}(\Gamma'_B)$ which is a slightly modified version of the algorithm for $\mathcal{A}_{14}$ and $\mathcal{A}_{10}$ proven correct and polynomial by Broxvall and Jonsson (1999) is presented in figure 1.
A polynomial-time algorithm for $\mathrm{SAT_{po}}(\Gamma'_C)$ is presented in figure 2. Obviously algorithm C rejects only unsatisfiable instances of $\mathrm{SAT_{po}}(\Gamma'_C)$.

Assume algorithm C accepts a certain instance Π and let Π' be the instance resulting from the contractions made by the algorithm in lines 3-6. Observe that Π' does not contain the relation $(=)$ and if Π' is satisfiable, then Π is satisfiable. Let I be the interpretation given by the function $f(x) = x$ and the partial order $\langle V, \{\langle x, y\rangle | x <^* y \text{ in } \Pi'\}\rangle$ where V is the set of variables appearing in Π'.

Each $\neq$ and $<$ constraint is satisfied by I and if there exists a constraint $x \underline{||} < y$ in Π' then either $x < y$ or $x || y$ under I which guarantees that each such constraint is satisfied so I is a model and Π is satisfiable. Thus, $\mathrm{SAT_{po}}(\Gamma'_C)$ is tractable. $\Box$

Next, we show a number of independence results.

Lemma 9 $\Delta'_A, \Delta'_B, \Delta'_C$ are independent of Γ'_A, Γ'_B and Γ'_C, respectively.

Proof: We show that Δ'_A is independent of Γ'_A; the proofs of the other two cases are similar.

Let Π be an instance of $\mathrm{SAT_{po}}(\Gamma'_A \cup \Delta'_A)$ and assume that every instance Π' such that Π' only contains constraints present in Π and at most one constraint from Δ'_A is satisfiable. We prove Π to be satisfiable.

Assume to the contrary that Π is not satisfiable. Then, there exists at least one constraint $x\ R\ y \in \Pi$ where R is $\neq$ or $||$ which causes algorithm A to reject. Let Π' denote the problem instance containing all constraints from Π of the form $z\ R'\ w$ where R' is $\leq$ or $\underline{||}$ plus the constraint $x\ R\ y$. Note that Π' causes the algorithm to reject. Contradiction, since Π' is a problem instance previously assumed to be satisfiable and the algorithm correctly solves $\mathrm{SAT_{po}}(\Gamma'_A \cup \Delta'_A)$. Hence, Δ'_A is independent of Γ'_A. $\Box$

We can now show that the four classes are tractable.

Theorem 10 $\mathrm{SAT_{po}}(\mathcal{T}_A)$, $\mathrm{SAT_{po}}(\mathcal{T}_B)$, $\mathrm{SAT_{po}}(\mathcal{T}_C)$ and $\mathrm{SAT_{po}}(\mathcal{T}_D)$ are tractable.

Table 1: Tractable classes

	Γ'_A	Δ'_A	Γ_A	Δ_A	Γ'_B	Δ'_B	Γ_B	Δ_B	Γ'_C	Δ'_C	Γ_C	Δ_C	Δ_D
$<$			•				•		•		•		
$\leq$	•		•		•		•						•
$<>$							•	•					
$<=>$					•	•	•	•					•
$\|$	•	•	•	•							•	•	
$\underline{\|}$	•		•								•	•	•
$=$			•				•		•		•		•
$\neq$	•	•	•	•	•	•	•	•	•	•	•	•	
$<\|$			•	•							•	•	
$\leq\|$			•						•	•	•	•	•

Proof: That $\mathrm{SAT_{po}}(\Gamma'_A \overset{\times}{\vee} \Delta'^*_A)$ is tractable follows from Lemmata 8, 9 and Theorem 4. That Γ'_A implements Γ_A is proven by Broxvall and Jonsson (1999). Let Π be an arbitrary instance of $\mathrm{SAT_{po}}(\mathcal{T}_A)$. We will show how every $\| <$ relation can be replaced by relations in $\Gamma'_A \overset{\times}{\vee} \Delta'^*_A$ which implies the tractability of $\mathrm{SAT_{po}}(\mathcal{T}_A)$. Choose an arbitrary disjunction $\gamma = x\ R_1\ y \vee \ldots \vee x_n\ R_n\ y_n$ in Π where $R_1 = (\| <)$. Introduce a fresh variable $t_{x,y}$, add the constraint $x \leq t_{x,y}$ and replace γ by the disjunction $\gamma' = t_{x,y} \| y \vee \ldots \vee x_n\ R_n\ y_n$. Repeat this transformation until no $(\| <)$ remains—a process which clearly takes polynomial time. Let Π' denote the resulting instance. Since the gadget $x \leq t_{x,y}, t_{x,y} \| y$ implements the relation $< \|$, it follows that Π' is satisfiable iff Π is satisfiable. Hence, $\mathrm{SAT_{po}}(\mathcal{T}_A)$ is tractable.

The tractability of $\mathrm{SAT_{po}}(\Gamma'_B \overset{\times}{\vee} \Delta'^*_B)$ follows from Lemmata 8, 9 and Theorem 4. That Γ'_B implements Γ_B is proven by Broxvall and Jonsson (1999). The gadget $x \neq y$, $x <=> y$ implements $x <> y$ which shows that Δ'_B implements Δ_B. This fact combined with Theorem 7 proves that $\mathrm{SAT_{po}}(\mathcal{T}_B)$ is tractable.

To show that $\mathcal{T}_C$ is tractable we begin by noting that the following gadgets from Δ'_C implements Δ_C: $x \leq \|y, y \leq \|x$ implement $x \underline{\|} y$, $x \leq \|y, y \leq \|x, x \neq y$ implement $x \| y$ and $x \leq \|y, x \neq y$ implement $x < \|y$. Hence, Δ_C is independent of Γ'_C by Lemma 9 and Theorem 7. That $\mathrm{SAT_{po}}(\Gamma'_C \overset{\times}{\vee} \Delta^*_C)$ is tractable follows from Theorem 4. The theorem follows since $\Gamma_C = \Gamma'_C \cup \Delta_C$ which implies that $\mathcal{T}_C = \Gamma'_C \overset{\times}{\vee} \Delta^*_C = \Gamma_C \overset{\times}{\vee} \Delta^*_C$.

Finally, the tractability of $\mathrm{SAT_{po}}(\mathcal{T}_D)$ is easy to show since each relation present in Γ_D contains only the relations $=$ and/or $\perp$. Consequently, it is sufficient to check whether an instance Π contains a disjunction containing only the $\perp$ relation. If this is the case, the instance is not satisfiable. Otherwise, Π is satisfied by the model mapping every variable to a single node. ☐

Intractability Results

This section contains the NP-completeness results which are needed in the classification theorem.

Lemma 11 The following problems are NP-complete:

1. $\mathrm{SAT_{po}}(\{(<>), R\})$ if $R \in \{\|, < \|, \underline{\|}, \leq \|\}$;
2. $\mathrm{SAT_{po}}(\{<\} \vee \{<\})$;
3. $\mathrm{SAT_{po}}(\{R\} \cup \{\leq\} \vee \{\leq\})$ if $R \in \{\neq, <>\}$;
4. $\mathrm{SAT_{po}}(\{R\} \cup \{=\} \vee \{=\})$ if $R \in \{<, <>, \neq, \|, \| <\}$;
5. $\mathrm{SAT_{po}}(\{R_1, R_2\} \cup \{\underline{\|}\} \vee \{\underline{\|}\})$ if $R_1 \in \{\leq, <=>\}$ and $R_2 \in \{<, <>, \neq, \| <\}$.

Proof: Case 1 is shown in Broxvall and Jonsson (1999). We only prove case 2 since the proofs of the other cases are similar. The proof is by a reduction from the NP-complete problem MONOTONE 3SAT:

INSTANCE: Set U of variables, collection C of clauses over U such that for each $c \in C$ has $|c| = 3$ and each clause contain either only positive or negative literals.
QUESTION: Is there a satisfying truth assignment for C?

Given an arbitrary instance P of MONOTONE 3SAT with variables $U = \{u_i, \ldots, u_m\}$ and clauses $C = \{c_1, \ldots, c_n\}$, construct incrementally an instance Π of $\mathrm{SAT_{po}}(\{<\} \vee \{<\})$ by the following steps.

Begin with the variables in the set U and a fresh variable a and add the constraints $(u_i < a) \vee (u_i > a)$, $1 \leq i \leq m$. For each positive clause $c_i \in C$ of the form $x \vee y \vee z$ add a fresh variable t_i and the constraints:

$$x > a \vee t_i > a,\ y > t_i \vee z > a.$$

For each negative clause $c_i \in C$ of the form $\neg x \vee \neg y \vee \neg z$ add a fresh variable t_i and the constraints:

$$x < a \vee t_i < a,\ y < t_i \vee z < a.$$

We show that the problem instance constructed is satisfiable iff the original MONOTONE 3SAT problem instance is satisfiable. For the if-direction, assume that P is satisfiable and that $\mathcal{I}$ is a truth assignment satisfying all clauses in C. Construct an interpretation $M = \langle f, \langle T, \leq \rangle \rangle$ of Π as follows: let $T = \{a_i \mid 1 \leq i \leq 6\}$, $\leq = \{\langle a_i, a_j \rangle | i \leq j\}$ and

$$
\begin{aligned}
f = \quad & \{\langle x, a_5\rangle | x \text{ true in } \mathcal{I}\} \cup \\
& \{\langle x, a_1\rangle | x \text{ false in } \mathcal{I}\} \cup \\
& \{\langle t_i, a_0\rangle | \, c_i = x \vee y \vee z \text{ and } \mathcal{I}(x)\text{=true}\} \cup \\
& \{\langle t_i, a_4\rangle | \, c_i = x \vee y \vee z \text{ and } \mathcal{I}(x)\text{= false}\} \cup \\
& \{\langle t_i, a_2\rangle | \, c_i = \neg x \vee \neg y \vee \neg z \text{ and } \mathcal{I}(x)\text{=true}\} \cup \\
& \{\langle t_i, a_6\rangle | \, c_i = \neg x \vee \neg y \vee \neg z \text{ and } \mathcal{I}(x)\text{=false}\} \cup \\
& \{\langle a, a_3\rangle\}
\end{aligned}
$$

That each constraint is satisfied by M can trivially be verified so M is a model of Π and Π is satisfiable if P is satisfiable. For the only-if direction, assume that Π is satisfiable and that M is a model of Π with mapping f. Construct a truth assignment $\mathcal{I}$ over the variables U assigning a variable x the value true if $f(x) > f(a)$ and otherwise the value false. Assume that there exists a clause c_i not satisfied by $\mathcal{I}$. The case when c_i is a positive clause $x \vee y \vee z$ leads to $f(x) < f(a), f(y) < f(a), f(z) < f(a)$ so $f(t_i) > f(a)$ and $f(y) > f(t_i)$ which contradicts that $f(y) < f(a)$. The case where u is a negative clause can be ruled out in a similar manner. Hence, no unsatisfied clause exists and $\mathcal{I}$ is a satisfying truth assignment. Since the reduction can be performed in polynomial time, the result follows. $\square$

Maximality

This section contains the main result of this paper, namely that $\mathcal{T}_A, \mathcal{T}_B, \mathcal{T}_C$ and $\mathcal{T}_D$ are the only maximal tractable sets of the disjunctive point algebra for partially ordered time. However, we need some auxiliary results first.

To reduce the number of NP-completeness results needed in the classification, we will employ a *closure* operator $\mathcal{C}(\cdot)$ which is defined with the aid of the standard operators converse, intersection ($\cap$) and composition ($\circ$) together with a number of rules for handling disjunctions. These rules are defined and their tractability-preserving properties are stated in the next lemma. The straightforward proof is omitted.

Lemma 12 Let $R_i \in \mathcal{PA}$ and $\Gamma_j \subseteq \mathcal{PA}$.
(1) if $\text{SAT}_{\text{po}}(\Gamma_1 \cup \{<\}\stackrel{\times}{\vee}\Gamma_2)$ is tractable, then $\text{SAT}_{\text{po}}(\Gamma_1 \cup \{<\}\stackrel{\times}{\vee}\Gamma_2 \cup \Gamma_2\stackrel{\times}{\vee}\Gamma_2)$ is tractable;
(2) assume $<, <>, \neq$ or $< ||$ is in Γ_1. If $\text{SAT}_{\text{po}}(\Gamma_1 \cup \{=\}\stackrel{\times}{\vee}\Gamma_2)$ is tractable, then $\text{SAT}_{\text{po}}(\Gamma_1 \cup \{=\}\stackrel{\times}{\vee}\Gamma_2 \cup \Gamma_2\stackrel{\times}{\vee}\Gamma_2)$ is tractable.
(3) assume $\oplus \in \{\cap, \circ\}$. If $\text{SAT}_{\text{po}}(\Gamma_1 \cup \{R_1 \vee R_3, R_2 \vee R_3\})$ is tractable, then $\text{SAT}_{\text{po}}(\Gamma_1 \cup \{R_1 \vee R_3, R_2 \vee R_3, (R_1 \oplus R_2) \vee R_3\})$ is tractable.
(4) if $\text{SAT}_{\text{po}}(\Gamma_1 \cup \{R_1 \vee R_2\})$ is tractable, then $\text{SAT}_{\text{po}}(\Gamma_1 \cup \{R_1, R_2, R_1 \vee R_2\})$ is tractable.

If Γ is a set of relations, we define $\mathcal{C}(\Gamma)$ as the least set X such that $\Gamma \subseteq X$ and X is closed under converse, intersection and composition on point relations and the expansion rules in the previous lemma. A composition table for the point relations can be found in Broxvall and Jonsson (1999). By using the lemma, the $\stackrel{\times}{\vee}$-closure property and the properties of converse, intersection and composition, it is a routine verification to show that if $\text{SAT}_{\text{po}}(\Gamma)$ is tractable, then $\text{SAT}_{\text{po}}(\mathcal{C}(\Gamma))$ is also tractable.

Now, we introduce a construction which simplifies the proof by allowing us to only consider a small number of disjunctions; this is shown in the next lemma.

Definition 13 Let $\mathcal{T} = \Gamma\stackrel{\times}{\vee}\Delta^*$ and $\Delta \subseteq \Gamma$. We define $\overline{\mathcal{T}}$ as $(\mathcal{PA} - \Gamma) \cup (\Gamma - \Delta)\stackrel{\times}{\vee}(\Gamma - \Delta)$.

Lemma 14 If $\mathcal{T} = \Gamma\stackrel{\times}{\vee}\Delta^*$, $\Delta \subseteq \Gamma$ and $\mathcal{T}' \not\subseteq \mathcal{T}$, then there exists $C \in \overline{\mathcal{T}}$ such that $C \in \mathcal{T}'$.

Proof: Arbitrarily choose a $C \in \mathcal{T}'$ such that $C \not\in \mathcal{T}$ and choose $C_1, \cdots, C_n \in \mathcal{PA}$ such that $C_1 \vee \cdots \vee C_n = C$. Assume first that there exists some i such that $C_i \not\in \Gamma$. The definition of $\overline{\mathcal{T}}$ implies that $C_i \in \overline{\mathcal{T}}$ and the $\stackrel{\times}{\vee}$-closure property implies that $C_i \in \mathcal{T}'$.

Assume instead that $C_1, \cdots, C_n \in \Gamma$. Since $\Gamma \subseteq \mathcal{T}$ and $C \not\in \mathcal{T}$ we know that $n > 1$. If all or all but one $C_i \in \Delta$, then we know that $C \in \mathcal{T}$. Hence, there exists at least two i, j such that $C_i, C_j \not\in \Delta$. Now, $C_i \vee C_j \in \mathcal{T}'$ by the $\stackrel{\times}{\vee}$-closure property and the definition of $\overline{\mathcal{T}}$ gives that $C_i \vee C_j \in \overline{\mathcal{T}}$. $\square$

Using the tractability and NP-completeness results given in previous sections as well as the previous definition and results we can now by a simple computer assisted case analysis prove the main result of this paper.

Theorem 15 $\mathcal{T}_A, \mathcal{T}_B, \mathcal{T}_C$ and $\mathcal{T}_D$ are the only maximal tractable subclasses of SAT_{po}.

Proof: Suppose to the contrary that there exists another maximal tractable algebra $\mathcal{T}$. From the previous lemma it follows that there exists $\gamma_A, \cdots, \gamma_D$ in $\mathcal{T}$ such that $\gamma_A \in \overline{\mathcal{T}}_A, \cdots, \gamma_D \in \overline{\mathcal{T}}_D$. Note that there exists only a finite number of possible values for $\gamma_A, \ldots, \gamma_D$.

To prove the result, a machine-assisted case analysis of the following form was performed: each admissible choice of $\gamma_A, \ldots, \gamma_D$ was generated and $X = \mathcal{C}(\gamma_{\text{A}}, \ldots, \gamma_{\text{D}})$ was computed. Each such set X was examined and it was found that at least one of the NP-complete sets of Lemma 11 was a subset of X. Thus, $\text{SAT}_{\text{po}}(\mathcal{T})$ is NP-complete and the theorem follows. $\square$

Totally Ordered Time

We will now identify the maximal tractable subclasses of SAT_{to}, *i.e.*, SAT_{po} restricted to total orders. The classification will rely on the results in the previous sections but the main theorem is shown without the use of a computer-assisted case analysis.

Note that the basic relation $||$ is unnecessary when dealing with total orders so we only have three basic relations ($<, =$ and $>$) and eight possible disjunctions of these relations. Let $\mathcal{PA}_{\text{to}}$ denote the set of these eight relations and define $\mathcal{X}_1 = \mathcal{PA}_{\text{to}}\stackrel{\times}{\vee}\{\neq\}^*$ and $\mathcal{X}_2 = \Delta^*$ where $\Delta = \{r \in \mathcal{PA}_{\text{to}} \mid r = \perp \text{ or } \{=\} \subseteq r\}$. As we will see later on, $\mathcal{X}_1$ and $\mathcal{X}_2$ are the only maximal tractable subclasses of SAT_{to}.

Lemma 16 $\text{SAT}_{\text{to}}(\mathcal{X}_i)$, $1 \leq i \leq 2$, are tractable problems.

Proof: Tractability of $\mathcal{X}_1$ has been proved by Jonsson and Bäckström (1998) and Koubarakis (1996) while the tractability of $\mathcal{X}_2$ is shown in Theorem 10. $\square$

The NP-completeness results for $\mathrm{SAT_{to}}$ are all based on the previously presented NP-completeness results; interestingly, many of these results hold even when restricted to total orders.

Lemma 17 $\mathrm{SAT_{to}}(\mathcal{N}_i)$, $1 \leq i \leq 5$, is NP-complete where

$$\begin{array}{ll} \mathcal{N}_1 = \{(< \vee <)\} & \mathcal{N}_2 = \{\neq,\ (\leq \vee \leq)\} \\ \mathcal{N}_3 = \{<,\ (= \vee =)\} & \mathcal{N}_4 = \{\neq,\ (= \vee =)\} \\ \mathcal{N}_5 = \{<,\ (\leq \vee \leq)\}. & \end{array}$$

Proof sketch: We begin by examining the proof of Lemma 11 which shows that $\mathrm{SAT_{po}}(\{<\} \vee \{<\})$ is NP-complete. Obviously, the proof holds even if we restrict the possible partial orders to total orders which implies that $\mathrm{SAT_{to}}(\{<\}\overset{\times}{\vee}\{<\})$ is NP-complete. By analyzing the proofs of the other cases, it follows that $\mathrm{SAT_{to}}(\mathcal{N}_i)$, $2 \leq i \leq 4$, is NP-complete. The NP-completeness of $\mathrm{SAT_{to}}(\mathcal{N}_5)$ follows from the NP-completeness of $\mathrm{SAT_{to}}(\mathcal{N}_3)$ plus the fact that $\{\leq\}\overset{\times}{\vee}\{\leq\}$ implements $\{=\}\overset{\times}{\vee}\{=\}$ □

We are now ready to prove the main theorem for the point algebra for totally ordered time.

Theorem 18 $\mathcal{X}_1$ and $\mathcal{X}_2$ are the only maximal tractable subclasses of $\mathrm{SAT_{to}}$.

Proof: Assume that there exists a maximal tractable algebra $\mathcal{X}$ such that $\mathcal{X} \not\subseteq \mathcal{X}_1$ and $\mathcal{X} \not\subseteq \mathcal{X}_2$. By Lemma 14, there exists γ_1, γ_2 in $\mathcal{T}$ such that $\gamma_1 \in \overline{\mathcal{X}_1}$ and $\gamma_2 \in \overline{\mathcal{X}_2}$. It is easy to see that $\overline{\mathcal{X}_1} \subseteq \{r_1, \ldots, r_6\}$ where

$$\begin{array}{lll} r_1 = (< \vee <) & & r_2 = (\leq \vee \leq) \\ r_3 = (= \vee =) & & r_4 = (\leq \vee =) \\ r_5 = (< \vee =) & & r_6 = (< \vee \leq) \end{array}$$

and $\overline{\mathcal{X}_2} \subseteq \{<, \neq, (< \vee <), (< \vee \neq), (\neq \vee \neq)\}$. By Lemma 11, we know that $\mathrm{SAT_{to}}\{(< \vee <)\}$ is NP-complete so the disjunction $r_1 = (< \vee <)$ can be excluded from further consideration. We will show that if γ_1 equals one of $r_2, \ldots, r_6$ and γ_2 equals $<$ or $\neq$, then $\mathrm{SAT_{to}}(\{\gamma_1, \gamma_2\})$ is NP-hard. By noting that $(< \vee \neq)$ implements the relation $<$ (by the construction $x < y \vee x \neq x$) and $(\neq \vee \neq)$ implements $\neq$ (by $x \neq y \vee x \neq y$), this result implies that $\mathrm{SAT_{to}}(\{\gamma_1, \gamma_2\})$ is NP-hard for all $\gamma_1 \in \overline{\mathcal{X}_1}$ and $\gamma_2 \in \overline{\mathcal{X}_2}$. This contradicts our initial assumptions and proves the theorem.

1. $\gamma_1 \in \{r_2, r_3\}$. Then, the NP-completeness of $\mathrm{SAT_{to}}(\{\gamma_1, \gamma_2\})$ is an immediate consequence of Lemma 6.2.
2. $\gamma_1 = r_4$. Note that r_4 implements r_3 so $\mathrm{SAT_{to}}(\{\gamma_1, \gamma_2\})$ is NP-complete by case 1.
3. $\gamma_1 = r_5$ and $\gamma_2 = \{<\}$. We show how to implement the disjunction $a < b \vee c < d$ with γ_1 and γ_2: introduce an auxiliary variable t and the following two relations: $a < t$ and $t = b \vee c < d$. NP-completeness follows from case 1.
4. $\gamma_1 = r_5$ and $\gamma_2 = \{\neq\}$. The relation $a < b$ can be implemented by the relations $t_1 \neq t_2$ and $t_1 = t_2 \vee a < b$ where t_1 and t_2 are auxiliary variables. NP-completeness follows from the previous case.
5. $\gamma_1 = r_6$. The NP-completeness of $\mathrm{SAT_{to}}(\{\gamma_1, \gamma_2\})$ is a consequence of cases 3 and 4 and the observation that r_6 implies r_5. □

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