Reasoning about Bargaining Situations

Dongmo Zhang
Intelligent Systems Laboratory
University of Western Sydney, Australia
dongmo@scm.uws.edu.au

Abstract

This paper presents a logical axiomatization of bargaining solutions. A bargaining situation is described in propositional logic and the bargainers' preferences are quantified in terms of the logical structure of the bargaining situation. A solution to the n-person bargaining problems is proposed based on the maxmin rule over the degrees of bargainers' satisfaction. We show that the solution is uniquely characterized by four natural and intuitive axioms as well as three other fundamental assumptions. All the axioms and assumptions are represented in logical statements and most of them have a game-theoretic counterpart. The framework would help us to identify the logical and numerical reasoning behind bargaining processes.

Introduction

Traditionally, a bargaining situation is abstracted as a game (S, d) where $S \subseteq \Re^n$ represents the feasible set of alternatives and $d \in S$ stands for the disagreement point. A bargaining solution is a function that assigns to each bargaining game (S, d) a unique point of S. A bargaining theory is then an exploration of the relation between the solution and the bargaining situation(Osborne & Rubinstein 1990).

The game-theoretic bargaining theory provides "a 'clear-cut' numerical predication for a wide range of bargaining problems" ((Rubinstein 2000) p.81). Due to the purely numerical representation, however, the existing bargaining theory is confined to quantitative analysis of bargaining situations. As Rubinstein argues, "*the language of utility allows the use of geometrical presentations and facilitates analysis; in contrast, the numerical presentation results in unnatural statement of the axioms and the solution*" [(Rubinstein 2000)p.85]. Bargaining is actually intelligent rivalry between agents. Logical reasoning plays an essential role in most bargaining activities. A bargaining theory should be able to identify the logical reasoning behind bargaining processes.

This paper aims to develop a logical theory of bargaining in cooperative model. We shall represent bargainers' negotiation items in propositional logic and quantify the bargainers' preferences over their negotiation items in accordance with the logical structure of the items. We propose a solution to the n-person bargaining problems based on the degrees of bargainers' satisfaction from negotiation outcomes, which can be viewed as the logical version of Kalai-Smorodinsky solution (KS-solution in short) (Kalai & Smorodinsky 1975). We show that the solution is uniquely characterized by the four axioms: *inclusion*, *scale invariance*, *symmetry* and *mutually comparable monotonicity* as well as three other fundamental assumptions: *individual rationality*, *consistency* and *comprehensiveness*. We shall illustrate via an example how logical reasoning plays a central role in bargaining processes.

We will use a finite propositional language $\mathcal{L}$ to describe bargaining situations. The language consists of a finite set of propositional variables, the standard propositional connectives $\{\neg, \vee, \wedge, \rightarrow\}$ and two logical constants $\top$ **(true)** and $\bot$ **(false)**. The concepts of syntactic entailment $\vdash$, logical closure Cn and logical consistency are all in their standard meaning in the classical propositional logic. In addition, we will intensively use vectors. If x is a n-dimensional vector, x_i will indicate the ith component of x.

Logical Representation of bargaining situations

We shall represent a bargaining situation in terms of bargainers' belief states, in which the beliefs and demands of a bargainer are described in logical statements and quantified in terms of their logical structure. This approach is inspired by (Zhang 2005). We mix bargainer's beliefs and demands together and will just refer them to as negotiation items in the sense that a belief is something a bargainer wants to retain and a demand is a statement she wants the other parties to accept (See Labor Union vs Management Negotiation section for an example).

Definition 1 A *belief state* is a pair (ρ, e), where ρ is a function that assigns to each sentence in the language $\mathcal{L}$ a real number and e is a real number. We pre-assume that any belief state satisfies the following conditions:

(LR) If $\varphi_1, \cdots, \varphi_n \vdash \psi, \min\{\rho(\varphi_1), \cdots, \rho(\varphi_n)\} \leq \rho(\psi)$. *(Logical rationality)*

(NV) $\rho(\top) > e$. *(Non-vacuity)*

(DB) $\rho(\psi) = e$ for some $\psi \in \mathcal{L}$. *(Descriptive cut-off)*

ρ is called the *entrenchment measure* of the belief state[1]. It measures the strength with which the agent entrenches its negotiation items. Let

$$Bel(\rho, e) = \{\varphi \in \mathcal{L} : \rho(\varphi) > e\} \quad (1)$$

We call $Bel(\rho, e)$ the the negotiation set of the agent, which contains all the negotiation items the agent is entrenching in a negotiation. It is easy to see that $Bel(\rho, e)$ can never be empty due to (NV), and it is always consistent and logically closed because of (LR) and (DB). Moreover, for any $\varphi \in Bel(\rho, e)$, since $\varphi \vdash \top$, we have $e < \rho(\varphi) \leq \rho(\top)$.

To avoid too many notations, we overload the function ρ with the argument of sets of formulas: for any consistent set of sentences, X,

$$\rho(X) = \max\{\rho(\varphi) : \varphi \in \mathcal{L} \setminus X\}. \quad (2)$$

The definition is well founded because of the finiteness of the language $\mathcal{L}$. It is easy to know that (DB) implies $\rho(Bel(\rho, e)) = e$.

The following concept is an analogue of the concept of comprehensiveness in game theory[2]:

Definition 2 A set of sentences, X, is *comprehensive* w.r.t. ρ if $\psi \in X$ & $\rho(\varphi) \geq \rho(\psi)$ implies $\varphi \in X$.

Interestingly, a similar concept had been used in the belief revision literature, known as *"cut"* (Grove 1988) or *EE-cut* (Rott 1991).

We let

$$Cut(\rho, \eta) = \{\varphi \in \mathcal{L} : \rho(\varphi) > \eta\} \quad (3)$$

where η is a real number, called the *cut point*. The following lemma tells us that a comprehensive set is actually a "cut".

Lemma 1 *A set of sentences X is comprehensive w.r.t. ρ if and only if $X = Cut(\rho, \rho(X))$.*

This result will play an important role in mapping a logically represented bargaining game to a numerically represented bargaining game.

Bargaining games

We let $N = \{1, 2, \cdots, n\}$ stand for a set of players. Each player $i \in N$ is characterized by her belief state (ρ_i, e_i). $Bel(\rho_i, e_i)$ then represents her negotiation items. An *n-person bargaining game* is a vector in the form of $((\rho_1, e_1), \cdots, (\rho_n, e_n))$, denoted by $((\rho_i, e_i))_{i \in N}$. $\mathcal{B}^{n,\mathcal{L}}$ denotes the class of all n-person bargaining games described in $\mathcal{L}$.

A bargaining game is *compatible* if there is no conflict of negotiation items between players, that is, $\bigcup_{i \in N} Bel(\rho_i, e_i)$ is consistent.

A bargaining game is *mutually comparable* if $\rho_i(\top) = \rho_j(\top)$ and $e_i = e_j$ for any i and j.

Given two bargaining games $B = ((\rho_i, e_i))_{i \in N}$ and $B' = ((\rho'_i, e'_i))_{i \in N}$, B' is a *subgame* of B, denoted by $B' \sqsubseteq B$, if, for each $i \in N$,

1. $e'_i \geq e_i$;
2. $\rho'_i(\varphi) = \rho_i(\varphi)$ for all $\varphi \in \mathcal{L}$ such that $\rho_i(\varphi) > e'_i$.

It is easy to see that $B' \sqsubseteq B$ implies $Bel(\rho'_i, e'_i) \subseteq Bel(\rho_i, e_i)$ for all i.

Bargaining solutions

The major concern of negotiation is obviously how many negotiation items of each player can be included in the final agreement. We consider that a possible agreement of negotiation is a collection of subsets of negotiation items from each player.

Definition 3 Let $B = ((\rho_i, e_i))_{i \in N}$ be a bargaining game. A *deal* of B is a vector $D = (D_1, \cdots, D_n)$ satisfying:

1. $D_i \subseteq Bel(\rho_i, e_i)$ for all $i \in N$.
2. $\bigcup_{i \in N} D_i$ is consistent.
3. D_i is comprehensive w.r.t. ρ_i for all $i \in N$.

The set of all deals of B is denoted by $\Omega(B)$, *i.e.*, the feasible set of B.

The first two requirements in the definition are purely logical. The third one relies on entrenchment measure of each player. In order to reach an agreement, a player might have to make a concession, *i.e.*, to give up some negotiation items she originally holds. It is reasonable to assume that when a player does so, she always abandons those less entrenched items and tries to keep the higher entrenched items as many as possible. Therefore a concession is actually a contraction of negotiation items with a higher cut-off. This idea is exactly the same as "cut revision" (Rott 1991) and is also implicitly assumed by the AGM theory (Gärdenfors 1988). Note that $(\emptyset, \cdots, \emptyset)$ belongs to any feasible set of bargaining games. Therefore $\Omega(B)$ is always non-empty and finite for any $B \in \mathcal{B}^{n,\mathcal{L}}$.

Definition 4 A *bargaining solution* on $\mathcal{B}^{n,\mathcal{L}}$ is a function f that assigns to a bargaining game $B \in \mathcal{B}^{n,\mathcal{L}}$ a unique deal of the game, *i.e.*, $f(B) \in \Omega(B)$.

Since $\Omega(B)$ can never be empty, a solution always exists. If you prefer to represent an agreement as a set of sentences (a contract), we can simply define it as:

$$A(B) = Cn(\bigcup_{i \in N} f_i(B)) \quad (4)$$

Note that a bargaining solution is purely represented by logical statements. It actually returns the real agreement rather than its utility. This is the main difference of our approach from the game-theoretic approach.

There are a few ways to compare different possible agreements. The first way is based on set-inclusion over vectors. For any two deals $D, D' \in \Omega(B)$, we write

- $D \succeq D'$ if $D_i \supseteq D'_i$ for all i;
- $D \succcurlyeq D'$ if $D \succeq D'$ but $D \neq D'$;
- $D \succ D'$ if $D_i \supset D'_i$ for all i.

[1] **LR** implies that any entrenchment measure uniquely determines an epistemic entrenchment ordering (Gärdenfors 1988).

[2] $S \subseteq \Re^n$ is d-comprehensive if $d \leq x \leq y$ and $y \in S$ implies $x \in S$ (Thomson 1994).

A deal can also be measured in size. Given a deal D, for each player i, we know that $\rho_i(D_i)$ is the cut point of ρ_i for set D_i (see Lemma 1). Therefore the lower $\rho_i(D_i)$ is, the bigger D_i will be. We can then use the value of $\rho_i(\top) - \rho_i(D_i)$ to measure D_i. However, such a value relies on the individual scale of entrenchment measure, so it is not interpersonally comparable. A better measure to assess the gain of a player is $\frac{\rho_i(\top)-\rho_i(D_i)}{\rho_i(\top)-e_i}$. We call it the *degree of satisfaction* of player i.

Numerical mapping

In game theory, the feasible set of a bargaining game is a subset of $\Re^n$ while in our framework, the feasible set is represented in terms of logic statements. However, it is possible to map the logically represented feasible set into a numerical domain. This can be done according to the following observation, which is a direct corollary of Lemma 1 and Definition 3.

Observation 1 *$D \in \Omega(B)$ if and only if there exists $x \in \Re^n$ such that for all i, $x_i \geq e_i$, $D_i = Cut(\rho_i, x_i)$ and $\bigcup_{i \in N} D_i$ is consistent.*

Thus, for each deal D, we can find a point $x \in \Re^n$ (not necessarily unique) that generates D through a "cut". Note that x is not the utility of D. In fact x_i is even not in direct proportion to the size of D_i. In such a case, we let

$$S(B) = \{x \in \Re^n : \exists D \in \Omega(B) \forall i.\, D_i = Cut(\rho_i, \rho_i(\top) - x_i)\}$$

We call $S(B)$ the *numerical feasible set* of game B. It is not hard to see that each point in $S(B)$ uniquely determines a deal in $\Omega(B)$ (not inversely). In particular, the point $\mathbf{0} = (0, \cdots, 0)$ corresponds to the empty deal, which can be viewed as the disagreement point of the game.

Ultimately we can map each *logically represented* bargaining game B to a normalized *numerically represented* bargaining game $(S(B), \mathbf{0})$. It is easy to prove that such a numerical feasible set is always $\mathbf{0}$-*comprehensive* and *compact* but, unfortunately, does not satisfy the expected utility assumptions and is not necessarily convex due to the discrete nature of logical representation. Figure 1 shows a typical example of such a domain.

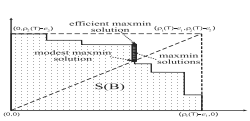


Figure 1: *Numerical map of a bargaining game.*

Characterization of bargaining solutions

One may think that the logically represented bargaining problem can be easily solved by applying one of game-theoretic bargaining solutions to its corresponding numerical domains. Unfortunately it is not such easy. As mentioning above, the corresponding numerical feasible set of a logical game is not necessarily convex. Therefore the classical solutions are not applicable (Thomson 1994). Even worse, most existing solutions to non-convex problems, such as (Herrero 1989; Conley & Wilkie 1991; Mariotti 1998; Hougaard & Tvede 2003), fail to apply to our problem because these solutions require to extend the feasible sets by using convex hull or set operations which would result in numerical feasible sets without logical representation. More importantly, the numerical mapping loses the most important information that is used in bargaining reasoning.

Axioms

We first investigate what are the properties we expect a bargaining solution to hold and then try to find the solution that satisfies these properties. Before we start, let us review the properties which have been built in the definition of bargaining solution (see Definition 3):

- Individual Rationality(**IR**): For each i, $f_i(B) \subseteq Bel(\rho_i, e_i)$.
- Consistency(**Con**): $\bigcup_{i \in N} f_i(B)$ is consistent.
- Comprehensiveness(**Com**): For each i, $f_i(B)$ is comprehensive w.r.t ρ_i.

The first two are purely logical requirements which are intuitive. We remark that the requirement for *comprehensiveness* is more technical than realistic. In game theory, comprehensiveness is a fundamental assumption for most existing bargaining solutions (Thomson 1994). A few basic axioms are incompatible without the assumption. We have seen in last section that the logical comprehensiveness (Definition 2) guarantees the comprehensiveness of its corresponding numerical domain.

We consider that the following properties are desirable for a bargaining solution:

- Inclusion(**Inc**): *If B is compatible, $f_i(B) = Bel(\rho_i, e_i)$ for all i.*
- Scale Invariance(**Inv**): *For any positive affine transformations $\tau = (\tau_1, \cdots, \tau_n)$[3], $f(\tau(B)) = f(B)$, where $\tau(B) = ((\tau_1 \circ \rho_1, \tau_1(e_1)), \cdots, (\tau_n \circ \rho_n, \tau_n(e_n)))$.*
- Symmetry(**Sym**): *For any mutually comparable bargaining game B, there exists a real number η such that $f_i(B) = Cut(\rho_i, \eta)$ for all i.*
- Mutually Comparable Monotonicity(**MCM**): *For any mutually comparable bargaining games B and B', $B' \sqsubseteq B$ implies $f(B') \preceq f(B)$.*

Inc says that if there is no conflict between players' negotiation items, then no concession is needed from any player. This is a fundamental assumption for any conflict resolving formalism, such as belief revision and belief merging (Gärdenfors 1988; Konieczny & Pino-Pérez 1998).

Inv is an analogue of the game-theoretic axiom with the same name. It says that scaling players' entrenchment measures with positive affine transformations does not affect the bargaining outcome. Note the subtle difference between this

[3]A positive affine transformation τ_i is a linear function $\tau_i(x_i) = a_i x_i + b_i$, where $a_i > 0$.

axiom and its game-theoretic counterpart: the assumption applies to players' entrenchment measures, which is interpersonally independent, while its game-theoretic counterpart applies to possible outcomes, which may be mutually related.

Sym is not a direct counterpart of symmetry in game theory. The original idea of the assumption is to impose fairness on bargaining processes (Nash 1950). This can be easily done in game theory because if we cannot differentiate players on the basis of the information contained in the numerical description, we can safely assume that all players gain the same. However, we cannot simply assume that all players have identical logical description because if it is so, there is no conflict between players. Our interpretation of fairness is that "*each player should reach the same degree of satisfaction at the termination of a negotiation*" (otherwise, the less satisfied player could reject the agreement). In other words, if the entrenchment measures are mutually comparable, each player should achieve the same (cut-off) in retaining their negotiation items.

MCM says that expanding negotiation items while keeping the entrenchment measures for the existing items will not affect the previously reached agreement provided the expansion is done evenly by all players. Obviously this axiom is the logical version of Roth's *Restricted Monotonicity* which is a weaker version of Kalai-Smorodinsky's *Individual Monotonicity* (Kalai & Smorodinsky 1975; Roth 1979).

All the above axioms are represented in logic with numerical representation of preferences over negotiation items. They are by no means less intuitive than their game-theoretic counterparts. The **Sym** and **MCM** are even more natural.

Maxmin solutions

We now try to find the solution which satisfies the above four axioms. Since we assume monotonicity, it is easy for us to associate it with KS-solution (Kalai & Smorodinsky 1975). This leads to the following concepts.

Given $B = ((\rho_i, e_i))_{i\in N}$, let

$$\Lambda(B) = arg \max_{D\in\Omega(B)} \min_{i\in N} \frac{\rho_i(\top) - \rho_i(D_i)}{\rho_i(\top) - e_i} \tag{5}$$

We call any solution f satisfying $f(B) \in \Lambda(B)$ a *maxmin solution*. "arg" means the argument(of max), *i.e*, the maximizers. In Figure 1, all the highlighted area on the Pareto frontier are the points that correspond to the elements of $\Lambda(B)$. Intuitively, a maxmin solution is a kind of Egalitarian solution which tries to balance the degrees of satisfaction of all players. Apparently maxmin solutions are not necessarily unique. However, the following function defines a unique solution:

$$F(B) = (\bigcap_{D\in\Lambda(B)} D_1, \cdots, \bigcap_{D\in\Lambda(B)} D_n) \tag{6}$$

We shall call F *the modest maxmin solution*.

The following lemma is needed for the proof of Theorem 1.

Lemma 2 *Given $B = ((\rho_i, e_i))_{i\in N}$, let*

$$\lambda = \max_{D\in\Omega(B)} \min_{i\in N} \frac{\rho_i(\top) - \rho_i(D_i)}{\rho_i(\top) - e_i}.$$

Then $F_i(B) = \{\varphi \in \mathcal{L} : \rho_i(\varphi) > \lambda e_i + (1-\lambda)\rho_i(\top)\}$ for all i.

Characterization

The following result shows that the modest maxmin solution is characterized exactly by the four axioms shown above.

Theorem 1 *A bargaining solution f satisfies inclusion, scale invariance, symmetry and mutually comparable monotonicity if and only if it is the modest maxmin solution F.*

Proof: "$\Leftarrow$" Lemma 2 implies that F is a bargaining solution. We are to prove that F satisfies the four axioms. The proof of **Inc**, **Inv** and **Sym** is elementary. We only present the proof of **MCM**. Consider two mutually comparable bargaining games $B = ((\rho_i, e_i))_{i\in N}$ and $B' = ((\rho'_i, e'_i))_{i\in N}$. Let

$\lambda = \max_{D\in\Omega(B)} \min_{i\in N} \frac{\rho_i(\top)-\rho_i(D_i)}{\rho_i(\top)-e_i}$

$\lambda' = \max_{D\in\Omega(B')} \min_{i\in N} \frac{\rho'_i(\top)-\rho'_i(D_i)}{\rho'_i(\top)-e'_i}$

Since both B and B' are mutually comparable, for each i, we have

$$\lambda e_i + (1-\lambda)\rho_i(\top) = \min_{D\in\Omega(B)} \max_{i\in N} \rho_i(D_i) \tag{7}$$

$$\lambda' e'_i + (1-\lambda')\rho'_i(\top) = \min_{D\in\Omega(B')} \max_{i\in N} \rho'_i(D_i) \tag{8}$$

As $B' \sqsubseteq B$ implies $\Omega(B') \subseteq \Omega(B)$, it is not hard to prove that for any $D \in \Omega(B')$ and $i \in N$, $\rho'_i(D_i) \geq \rho_i(D_i)$. It then follows that $\min_{D\in\Omega(B')} \max_{i\in N} \rho'_i(D_i) \geq \min_{D\in\Omega(B)} \max_{i\in N} \rho_i(D_i)$. By using (7) and (8), we yield that $\lambda' e'_i + (1-\lambda')\rho'_i(\top) \geq \lambda e_i + (1-\lambda)\rho_i(\top)$. According to Lemma 2, we conclude that $F(B') \preceq F(B)$.

"$\Rightarrow$" Given a bargaining game $B = ((\rho_i, e_i))_{i\in N}$, by **Inv**, we can simply assume that B itself is mutually comparable. According to **Sym**, there exists a real number η such that

$$f_i(B) = \{\varphi : \rho_i(\varphi) > \eta\} \quad \text{for each } i \in N. \tag{9}$$

Let $\check{\lambda}_i = \frac{\rho_i(\top)-\rho_i(f_i(B))}{\rho_i(\top)-e_i}$. Thus $\rho_i(f_i(B)) = \check{\lambda}_i e_i + (1-\check{\lambda}_i)\rho_i(\top)$. It follows from (9) that $\check{\lambda}_i e_i + (1-\check{\lambda}_i)\rho_i(\top) \leq \eta$. Let $\check{\lambda} = \min\{\check{\lambda}_1, \cdots, \check{\lambda}_n\}$. By mutual comparability, we have $\check{\lambda} e_i + (1-\check{\lambda})\rho_i(\top) \leq \eta$ for all i. It then follows from (9) again that

$$f_i(B) \subseteq \{\varphi : \rho_i(\varphi) > \check{\lambda} e_i + (1-\check{\lambda})\rho_i(\top)\}. \tag{10}$$

Let

$$\lambda = \max_{D\in\Omega(B)} \min_{i\in N} \frac{\rho_i(\top) - \rho_i(D_i)}{\rho_i(\top) - e_i} \tag{11}$$

By Lemma 2, we have $F_i(B) = \{\varphi : \rho_i(\varphi) > \lambda e_i + (1-\lambda)\rho_i(\top)\}$ for all i. Since $\lambda \geq \check{\lambda}$, we know $\lambda e_i + (1-\lambda)\rho_i(\top) \leq \check{\lambda} e_i + (1-\check{\lambda})\rho_i(\top)$. Equation (10) then implies that $f_i(B) \subseteq \{\varphi : \rho_i(\varphi) > \lambda e_i + (1-\lambda)\rho_i(\top)\} = F_i(B)$ for all i. Therefore we yield $f(B) \preceq F(B)$.

To show $F(B) \preceq f(B)$, let $B' = ((\rho'_i, e'_i))_{i\in N}$ be a bargaining game such that, for each $i \in N$,

1. $e'_i = \lambda e_i + (1-\lambda)\rho_i(\top)$;
2. $\rho'_i(\varphi) = \begin{cases} \rho_i(\varphi), & \text{if } \varphi \in F_i(B); \\ \lambda e_i + (1-\lambda)\rho_i(\top), & \text{otherwise.} \end{cases}$

where λ is defined by (11). It is easy to see that $B' \sqsubseteq B$ and B' is mutually comparable. Moreover, B' is compatible. According to **Inc** and Lemma 2, $f(B') = F(B)$. By **MCM**, we have $F(B) \preceq f(B)$. Therefore $f(B) = F(B)$. □

You might have noticed that we did not require weak Pareto optimality(WPO) which is necessary for KS-solution to avoid the solution to degenerate to the disagreement point. In fact, the purely logical axiom *inclusion* acts the role of WPO. More interestingly, we have

Proposition 1 ***Inc***, ***Inv***, ***Sym*** *and* ***MCM*** *implies the following two properties:*

- Weak Pareto Optimality(**WPO**): *There does not exist a deal* $D \in \Omega(B)$ *such that* $D \succ f(B)$.
- Balanced Contraction(**BC**): *For any mutually comparable bargaining games* B *and* B'*, if* $B' \sqsubseteq B$ *and* $f(B) \in \Omega(B')$*, then* $f(B') = f(B)$.

It is easy to see that BC is the logical counterpart of Nash's Independence of Irrelevant Alternatives (IIA). This seemly suggests that we can never characterize Nash's solution unless we allow to play lotteries over possible agreements(Zhang 2005). In fact, BC is "slightly" weaker than IIA due to the requirement of mutually comparability (balanced contraction). Nevertheless, Nash's solution seems not the most intuitive one to the logically represented bargaining situations. See more discussion later.

Labor union vs. management negotiation

For a better understanding of our solution, let us consider a simple bargaining scenario which was originally presented in (Kraus, Sycara, & Evenchik 1998).

A labor union negotiates with the management for a wage increase. The management claims that if it grants the increase, it will have to lay off employees.

We formalize the scenario in logical bargaining games and depict them with the following figures.

In the figures, *layoff* $\rightarrow \neg$*jobs* represents a belief of the union, meaning that "if the management lays off employees, its members will lose jobs". The threat of the management is expressed by *wage* $\rightarrow$ *layoff*, saying that "if the wage increase is executed, a layoff will be enforced". $jobs$ (for "*keeping jobs*") and $wage$ (for "*wage increase*") are two demands of the union.

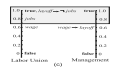
Figure (a) shows a particular situation where the entrenchment measure for the union is $\rho_u(\top) = \rho_u(\textit{layoff} \rightarrow \neg\textit{jobs}) = 1.0$, $\rho_u(\textit{jobs}) = 0.8$, $\rho_u(\textit{wage}) = 0.6$, $\rho_u(\bot) = 0.0$ while for the management $\rho_m(\textit{wage} \rightarrow \textit{layoff}) = 0.7$[4]. In this situation, the union values its members jobs more than a wage increase; therefore the union loses the battle due to a deep worry of unemployment. More precisely, the modest maxmin solution, which is also the unique maxmin solution, is (see the shaded part):

$$(Cn(\{\textit{layoff} \rightarrow \neg\textit{jobs}, \textit{jobs}\}), Cn(\{\textit{wage} \rightarrow \textit{layoff}\})$$

[4]Note that we only give the entrenchment values to the statements that directly affect the bargaining outcomes. The values for other sentences can be generated by using *logical rationality* **LR**.

It is easy to imagine that the result would be totally different if the union had more entrenched the demand for wage raise than the request for securing jobs.

Figure (b) shows another situation in which the management hesitates to turn down the union's appeal possibly because the reduction of productivity caused by laying off employees would surpass the increased labor costs. In such a situation, the union wins.



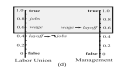


Figure (c) shows a situation where $\rho_u(\textit{wage}) = \rho_m(\textit{wage} \rightarrow \textit{layoff}) = 0.6$. In such a case there are two maxmin solutions:

$$(Cn(\{\textit{layoff} \rightarrow \neg\textit{jobs}, \textit{jobs}, \textit{wage}\}, Cn(\{\top\})) \text{ and}$$

$$(Cn(\{\textit{layoff} \rightarrow \neg\textit{jobs}, \textit{jobs}\}), Cn(\{\textit{wage} \rightarrow \textit{layoff}\}))$$

The modest maxmin solution will be:

$$(Cn(\{\textit{layoff} \rightarrow \neg\textit{jobs}, \textit{jobs}\}), Cn(\{\top\})).$$

More interestingly, if *layoff* $\rightarrow \neg$*jobs* is not a hard rule as it is shown in Figure (d) (possibly because the job market is quite optimistic), then the negotiation outcome will also be related to the ranking of this rule. We might notice that the situation here is exactly the same as what we have encountered in non-monotonic reasoning. The statements *layoff* $\rightarrow \neg$*jobs* and *wage* $\rightarrow$ *layoff* act as soft rules (or default rules) in the determination of bargaining solution. This seemly suggest that the reasoning behind bargaining would be non-monotonic. We believe that this will lead to a promising research in the future.

We would like to emphasize again here that in our framework both the bargaining situations and their solutions are represented in logical statements. The conflicts between bargainers are explicitly expressed and clearly identified. The process of solution searching is exactly the process of conflict resolving. We have seen from the example that logical reasoning plays the key role in this process while the numerical preferences assist us in picking up the right rules in the reasoning.

Discussion and Conclusion

We have defined and characterized a logical solution to n-person bargaining problems. The solution is constructed based on the idea of maximizing satisfaction of all players and is represented in logic. We have shown that the solution is uniquely characterized by four natural and intuitive axioms in conjunction with three fundamental assumptions. These axioms clearly show the logical (so qualitative) and quantitative properties of bargaining reasoning. The main trick of the paper is that the concept of comprehensiveness

in game theory is associated to the concept of cut sets in belief revision. This allows us to map the logical solution to a point in n-dimensional Euclidean space though it is not one-to-one. We then can work in logic and think in both logic and game theory.

One might be curious why the natural and intuitive assumptions do not lead to the Nash solution. Technically, as we have mentioned earlier, our axioms are weaker than their game-theoretic counterparts in such a way that we cannot arbitrarily generate a numerical domain with particular mathematical properties, say $\{x \in \Re^n : \Sigma x_i \leq n\}$, because we may be unable to find its logical representation. Therefore most tricks that are used in the proof of Nash solution fail to apply to our problem (Conley & Wilkie 1996; Zhou 1996). In practice, we may even doubt whether the Nash solution is still a reasonable candidate if the "inter-agent" arithmetic operations on different agent's utilities are not allowed. For instance, if the utilities represent player's degree of satisfaction, it is hard to accept that $(0.2, 0.9)$ would be a better solution than $(0.4, 0.4)$. In general, the Nash solution is more suitable for continuous, even differential, domains. However, most computer science and AI applications are discrete. Therefore the maxmin solutions are more applicable to these applications. Nevertheless, if we allow to randomize possible deals, the natural axioms will lead to the Nash's bargaining solution (see (Zhang 2005)).

Another interesting question is "what is the nature of bargaining reasoning: logical or numerical?". The answer should be both. Although all the axioms and assumptions are represented in logical form, only **IR**, **Con** and **Inc** are purely in logic without numerical components. The other four more or less rely on the numerical measure of entrenchment. Nevertheless, different axioms play different roles in the characterization of the bargaining solution. **IR** confines the bargaining theme to the participants' interests (*logical*). **Con** prevents inconsistent agreements (*logical*). **Inc** sets the termination condition for the bargaining procedure (*logical*). **Sym** balances the gains of all players (*both*). **MCM** allows bargaining to be proceeded in a progressive procedure so that conflicts can be gradually resolved until an consistent agreement is reached (both). **Inv** allows each agent to individually adjust their entrenchment measures after withdrawing or introducing demands (*numerical*). **Com** links the quantitative measure (entrenchment) with the qualitative measure (set-inclusion). All in all, neither the pure logical axioms nor the numerical conditions can stand alone in the characterization of the solution. The axiomatization seemly shows a natural combination and a nice cooperation between quantitative and qualitative methods.

This work is closely related to (Zhang 2005), (Conley & Wilkie 1991) and (Hougaard & Tvede 2003). We use a similar approach to represent bargaining situations as in (Zhang 2005) but, and most importantly, we do not allow to play lotteries over possible agreements. Therefore our solution and characterization can stay in its logical representation. Our axiomatization can be viewed as the logical version of (Conley & Wilkie 1991)'s characterization even though the implications of the axioms are quite different[5]. The idea of maxmin solution is inspired by (Hougaard & Tvede 2003). That work provides a non-convex extension of KS-solution which is represented as the maximizers of the least normalized utilities among agents. The solution is characterized by four restricted version of standard KS axioms, which are weaker than other axiomatizations of KS-solution (Herrero 1989; Conley & Wilkie 1991). However, their axioms are much less intuitive than ours and are purely quantitative.

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[5]We have shown in the extended version of the paper that without **Inc**, the other three axioms plus **WPO** are not enough to characterize the modest maxmin solution.