

# An Egalitarian Fusion of Incommensurable Ranked Belief Bases under Constraints

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## Abstract

In the last decade, several approaches have been proposed for merging multiple and potentially conflicting pieces of information. Egalitarian fusion modes privilege solutions that minimize the (local) dissatisfaction of each agent (source, expert) who is involved in the fusion process. This paper proposes useful strategies for an egalitarian fusion of incommensurable ranked belief bases under constraints. We show that the fusion process can equivalently be characterized either by means of the notion of compatible ranked bases, or by means of a Pareto-like ordering on the set of possible solutions. Lastly, rational postulates for our merging operator are studied.

## Introduction

In many situations, relevant informations are provided by different sources. Taking advantage of the different sources of information usually requires to perform some combination operation on the pieces of information, and leads to the data fusion problem.

Recently, several approaches have been proposed for merging possibly contradictory belief bases (Konieczny & Pino Pérez 2002; 1998; Lin 1995; Liberatore & Schaerf 1995; Revesz 1993; 1997). Belief bases can be either flat (no priority relation is provided between different formulas of belief bases) or ranked. A ranked belief base (or a stratified belief base, a weighted belief base) is a set of well-founded formulas, each associated with a rank (assumed here to be an integer). The higher is the rank associated with a formula, the more important is the formula. In fact, ranked belief bases are convenient representations of what is usually known as epistemic states (Darwiche & Pearl 1997). Namely, each ranked belief base induces a ranking on set of possible interpretations (or solutions). Interpretations assigned to lower ranks are considered to be more plausible than interpretations assigned to higher ranks. In particular, interpretations with the rank '0' are the most preferred ones and represent agent's current beliefs.

In this paper, the term beliefs is used since pieces of information provided by sources are uncertain. We will reserve the term constraints to completely sure and consistent pieces

of information. Constraints should be present in the result of the fusion process, while beliefs can be accepted, weakened or if necessary ignored in the fusion process.

Among existing merging approaches, we can distinguish two important ones: utilitarian approaches (or majority approaches) and egalitarian approaches. Examples of majority approaches are those based on the "sum" operator, and examples of egalitarian approaches are those based on the maximum operator (and its extension Gmax), see (Konieczny & Pino Pérez 2002) for more details. Given a set of  $n$  consistent belief bases (or preferences sets), provided by  $n$  sources (or agents, or experts), a majority approach tries to minimize a global dissatisfaction. In particular, if a given belief base is supported by a large number of agents or sources involved in the fusion, then this belief base will be in the result of the fusion. Majority fusion operators make sense if all sources are considered to be independent. Egalitarian fusion modes behave differently since they try to minimize agent's individual dissatisfaction. In egalitarian approaches, which are for instance majority-independent merging operators, a repetition of a same piece of information has no impact on the result of fusion.

Both utilitarian and egalitarian approaches, when applied to merging ranked belief bases, are based on the assumption that belief bases to merge are commensurable. Namely all sources share a same common scale to order their information. This commensurability assumption may make sense in some applications, but can appear to be too strong for other applications.

This paper deals with egalitarian-based fusion modes, which are majority independent, of incommensurable ranked bases. We use the maximum-based merging operator as an example of egalitarian-based information fusion.

One way to deal with incommensurability problem is to use a variant of Pareto ordering (Moulin 1988). Interpretations (or solutions) are compared with respect to their compatibility with each individual bases. Namely, given a multi-set of consistent ranked bases  $E$ , a solution  $s$  is strictly preferred to another solution  $s'$ , if  $i$ )  $s'$  is not a model of each base in  $E$ , and  $ii$ ) for each base  $B_i \in E$ , either  $s$  is strictly preferred in  $s'$  with respect to  $B_i$ , or  $s$  and  $s'$  are both models of  $B_i$ .

Another natural way to define merging operator is to consider all compatible (or "uniform") scales. A compatible

scale is simply a re-assignment of ranks associated with beliefs in bases, such that the original relative ordering between beliefs is preserved. Then a solution  $s$  is said to be strictly preferred to another solution  $s'$ , if  $s$  is preferred to  $s'$  in each compatible scale using maximum-based merging operator.

This paper shows the surprising result, that using all compatible scales is equivalent to considering a Pareto-like ordering. It also shows that the merging operation obtained by dropping the commensurability assumption is still compatible with most of rational postulates proposed for merging operations.

The rest of this paper is organized as follows. First we present the concept of ranked belief bases, and maximum-based merging of commensurable bases. Then we present the two ways to deal with incommensurability assumption (compatible scales and Pareto-like ordering) and show their equivalence. Lastly, we discuss rational postulates for our merging operator.

## Background on ranked bases and Maximum-based fusion

Let  $\mathcal{L}$  be a *finite* propositional language. We denote by  $\Omega$  the set of interpretations of  $\mathcal{L}$  and by  $\omega$  an element of  $\Omega$ . Greek letters  $\phi, \psi$  denote propositional formulas.  $Mod(\phi)$  represents the set of models of  $\phi$ , namely  $Mod(\phi) = \{\omega \in \Omega : \omega \models \phi\}$ .

### Ranked bases

Ranked belief bases are convenient frameworks for representing uncertain (or prioritized) pieces of information. Ranked belief bases are used in different frameworks, such as possibility theory (Dubois, Lang, & Prade 1994; Guilin, Liu, & Bell 2006) or ordinal conditional functions (OCF) (Williams 1994; 1995; Meyer 2001; Spohn 1988).

In this paper, ranked belief bases are simply multi-sets of ranked propositional formulas. Namely:

**Definition 1 (Ranked bases)** A ranked base  $B_i$  is a multi-set of ranked propositional formulas,

$$B_i = \{(\phi_{ij}, R_{B_i}(\phi_{ij})), j \in \{1, \dots, m_i\}\},$$

where  $\phi_{ij}$  is a propositional formula, and  $R_{B_i}(\phi_{ij}) \in \mathbb{N}^*$ .

Intuitively,  $(\phi_{ij}, R_{B_i}(\phi_{ij}))$  asserts that  $\phi_{ij}$  has a priority rank of at least  $R_{B_i}(\phi_{ij})$  (where a higher rank is better). Only strictly positive ranks are represented. Moreover, we reserve the infinity symbol  $+\infty$  for integrity constraints.

From a ranked belief base  $B$ , we can define  $B^*$ , which is the set of propositional formulas simply obtained by ignoring weights in  $B$ . More formally,

$$B^* = \{\phi_{ij} : (\phi_{ij}, R_{B_i}(\phi_{ij})) \in B\}.$$

There exists different ways to induce an ordering on possible interpretations from a given ranked belief base. In this paper, we use the so-called best-out ordering, which is defined as: an interpretation  $\omega$  is preferred to another interpretation  $\omega'$ , if and only if the highest belief falsified by  $\omega$  is less important than the highest belief falsified by  $\omega'$ . Hence,

each interpretation is associated with the ranks of highest formula that it falsifies. Interpretations which are models of  $B_i^*$  have a rank equal to 0 and are the preferred ones. More precisely:

**Definition 2 (Ranking functions)** A ranking function  $\kappa_{B_i}$  associated with a ranked belief base  $B_i$  is a function that maps each interpretation  $\omega \in \Omega$  to a natural number  $\kappa_{B_i}(\omega)$  such that:

$$\kappa_{B_i}(\omega) = \begin{cases} 0 & \text{if } \forall (\phi_{ij}, R_{B_i}(\phi_{ij})) \in B_i, \omega \models \phi_{ij} \\ \max\{R_{B_i}(\phi_{ij}) : \omega \not\models \phi_{ij}, (\phi_{ij}, R_{B_i}(\phi_{ij})) \in B_i\} & \text{otherwise.} \end{cases}$$

The degree  $\kappa_{B_i}(\omega)$  will be called degree of surprise (or consistency degree) following Spohn's terminology (Spohn 1988).

Best-out ordering is the basis of possibilistic logic semantics (Dubois, Lang, & Prade 1994) and adjustment revision (Williams 1995).

**Example 3** Consider a ranked belief base  $B = \{(\neg a \vee b, 8), (a \vee b, 5), (a, 2)\}$ . The following table gives the ranking function  $\kappa_B$  associated with  $B$ .

$\omega_i \in \Omega$	a	b	$\kappa_B(\omega_i)$
$\omega_0$	0	0	5
$\omega_1$	0	1	2
$\omega_2$	1	0	8
$\omega_3$	1	1	0

Table 1: An example of ranking function

### Egalitarianist-based fusion

Let  $E = \{B_1, \dots, B_n\}$  be a multi-set of  $n$  ranked bases issued from  $n$  sources, and let  $\mu$  be a propositional formula representing integrity constraints to be satisfied. We suppose in this section that all the sources share the same meaning of ranks assigned to formulas. We also suppose that each ranked belief base is consistent (but of course, their union can be inconsistent).

The aim of merging is, given  $n$  commensurable ranked bases, to compute  $\Delta(E)$ , a propositional formula representing the result of the fusion of these bases. In the literature, different methods for merging  $E$  have been proposed.

This paper focuses on an egalitarianist fusion, and uses the maximum operator to illustrate the fusion process.

But first, we need to introduce the notion of *profile* associated with an interpretation  $\omega$ , denoted by  $\nu_E(\omega)$ , and defined by

$$\nu_E(\omega) = \langle \kappa_{B_1}(\omega), \dots, \kappa_{B_n}(\omega) \rangle.$$

It represents the degree of surprise (or dissatisfaction) of an interpretation  $\omega$  with respect to the multi-set of ranked bases.

The computation of the result of merging  $\Delta(E)$  is done in two step: the surprise degrees  $\kappa_{B_i}(\omega)$ 's are first combined with a merging operator (here the maximum operator), and the interpretations with lowest ranks are selected. This leads to define a strict order, denoted by  $\triangleleft_{Max}$ , between interpretations as follows: an interpretation  $\omega$  is preferred to another interpretation  $\omega'$  if the maximum element of the profile of

$\omega$  is smaller than the maximum element of the profile of  $\omega'$ . More formally:

**Definition 4 (definition of  $\triangleleft_{Max}$ )** Let  $E$  be a multi-set of ranked bases. Let  $\omega$  and  $\omega'$  be two interpretations and  $\nu_E(\omega)$ ,  $\nu_E(\omega')$  be their associated profiles. Then:

$$\omega \triangleleft_{Max}^E \omega' \text{ iff } Max(\nu_E(\omega)) < Max(\nu_E(\omega'))$$

where

$$Max(\nu_E(\omega)) = Max\{\kappa_{B_i}(\omega) : i \in \{1, \dots, n\}\}.$$

The result of the merging  $\Delta_\mu^{max}(E)$  is a propositional formula whose models are interpretations which are models of a constraint  $\mu$  and which are minimal with respect to  $\triangleleft_{Max}$ . More formally:

**Definition 5 (Maximum-based merging operator)** Let  $E = \{B_1, \dots, B_n\}$  be a multi-set of ranked belief bases and  $\mu$  be an integrity constraint. The result of merging is a propositional formula, denoted by  $\Delta_\mu^{max}(E)$ , defined by:

$$Mod(\Delta_\mu^{max}(E)) = \{\omega \in Mod(\mu) : \nexists \omega' \in Mod(\mu), \omega' \triangleleft_{Max}^E \omega\}$$

Let us illustrate these definitions with the following example.

**Example 6** Let  $E = \{B_1, B_2\}$  be such that  $B_1 = \{(a, 8), (-b, 4)\}$  and  $B_2 = \{(b, 2), (-a, 1)\}$ . The profile of each interpretation is given in Table 2.

	a	b	$\kappa_{B_1}(\omega)$	$\kappa_{B_2}(\omega)$	$\nu_E(\omega)$	Max
$\omega_0$	0	0	8	2	$\langle 8, 2 \rangle$	8
$\omega_1$	0	1	8	0	$\langle 8, 0 \rangle$	8
$\omega_2$	1	0	0	2	$\langle 0, 2 \rangle$	2
$\omega_3$	1	1	4	1	$\langle 4, 1 \rangle$	4

Table 2: Profiles associated with interpretations

The result of the Max-merging, considering  $\mu \equiv a$  is such that:

$$Mod(\Delta_\mu^{max}(E)) = \{\omega_2\}.$$

### Fusion-based on compatible scalings

The merging operation defined above assume that the sources, who provide  $B_i$ 's, are commensurable. In example 6, it is assumed that the weight associated with  $-b$  in  $B_1$  (namely 4) can be compared to the weight associated with  $-a$  (namely 1) in  $B_2$ . Such an assumption is not always true. In the following, we drop this commensurability assumption.

We present in this section a strategy for an egalitarian fusion of incommensurable ranked belief bases. A natural way to make them commensurable is to apply a compatible scaling on existing ranks. A scaling is said to be compatible if it preserves original relative orders between beliefs of each ranked bases.

A scaling  $\mathcal{S}$  assigns new ranks to beliefs of each ranked bases from the multi-set  $E$ . Namely:

**Definition 7 (compatible scaling)** Let  $E = \{B_1, \dots, B_n\}$  where  $B_i = \{(\phi_{ij}, R_{B_i}(\phi_{ij}))\}$ .

Then a scaling  $\mathcal{S}$  is defined by:

$$\mathcal{S}: \begin{array}{l} B_1 \sqcup \dots \sqcup B_n \rightarrow \mathbb{N} \\ (\phi_{ij}, R_{B_i}(\phi_{ij})) \mapsto \mathcal{S}(\phi_{ij}) \end{array}$$

where  $\sqcup$  represents union of multi-sets.

A scaling  $\mathcal{S}$  is said to be compatible with  $R_{B_1}, \dots, R_{B_n}$  if and only if:

$$\forall B_i \in E, \forall (\phi, R_{B_i}(\phi)), (\phi', R_{B_i}(\phi')) \in B_i,$$

$$R_{B_i}(\phi) \leq R_{B_i}(\phi') \text{ iff } \mathcal{S}(\phi) \leq \mathcal{S}(\phi').$$

Clearly, a compatible scaling is not unique, as it is illustrated by the following example.

**Example 8 (continued)** Let us consider again  $B_1 = \{(a, 8), (-b, 4)\}$  and  $B_2 = \{(b, 2), (-a, 1)\}$ . Table 3 gives 2 scalings:  $\mathcal{S}_1$  and  $\mathcal{S}_2$ .

	$\phi_{ij}$	$R_{B_i}(\phi_{ij})$	$\mathcal{S}_1(\phi_{ij})$	$\mathcal{S}_2(\phi_{ij})$
$B_1$	a	8	5	4
	$-b$	4	2	7
$B_2$	b	2	7	3
	$-a$	1	1	2

Table 3: examples of scaling

The scaling  $\mathcal{S}_1$  is a compatible one, because it preserves the order inside each ranked base. However, the scaling  $\mathcal{S}_2$  is not a compatible one: it inverts the priorities inside  $B_1$ .

The set of compatible scalings with  $E$  is denoted by  $\mathbb{S}_E$ . Note that  $\mathbb{S}_E$  is never empty. A straightforward compatible scaling is the one simply obtained by letting  $\mathcal{S}(\phi_{ij}) = R_{B_i}(\phi_{ij})$ . It is compatible because it obviously preserves the relative ordering between beliefs of each base.

Given a compatible scaling  $\mathcal{S}$ , we denote by  $B_i^{\mathcal{S}}$  the ranked base obtained from  $B_i$  by replacing each pair  $(\phi_{ij}, R_i(\phi_{ij}))$  by  $(\phi_{ij}, \mathcal{S}(\phi_{ij}))$ . Similarly, we denote by  $E^{\mathcal{S}}$  the multi-set obtained from  $E$  by replacing each  $B_i$  in  $E$  by  $B_i^{\mathcal{S}}$ .

A natural question now is, given the set of all compatible scalings  $\mathbb{S}_E$ , how to define the result of merging? Different options exist, either we use some uncertainty measure to select one compatible scaling from  $\mathbb{S}_E$ , or we consider all compatible scalings. In this paper, we adopt a skeptical option and consider all compatible scalings, in order to avoid arbitrary choices. An interpretation  $\omega$  is then said to be preferred to  $\omega'$ , if for each compatible scaling  $\mathcal{S}$ ,  $\omega$  is preferred to  $\omega'$  using Definition 4 (namely,  $\omega \triangleleft_{Max}^{E^{\mathcal{S}}} \omega'$ ). More precisely,

**Definition 9 (Ordering between interpretations)** Let  $E$  be a multi-set of ranked belief bases,  $\mathbb{S}_E$  be the set of all compatible scalings associated with  $E$ . Let  $\omega, \omega'$  be two interpretations. Then:

$$\omega <_{\forall}^E \omega' \text{ iff } \forall \mathcal{S} \in \mathbb{S}_E, \omega \triangleleft_{Max}^{E^{\mathcal{S}}} \omega'$$

where  $\triangleleft_{Max}^{E^{\mathcal{S}}}$  is the result of applying Definition 4 on  $E^{\mathcal{S}}$ .

Models of  $\Delta_\mu^\forall(E)$  are those which are models of  $\mu$  and minimal for  $<_{\forall}^E$ , namely:

$$\text{Mod}(\Delta_\mu^\forall(E)) = \{\omega \in \text{Mod}(\mu) : \nexists \omega' \in \text{Mod}(\mu), \omega' <_{\forall}^E \omega\}.$$

Note that  $<_{\forall}^E$  is only a partial order.

The following proposition shows that an interpretation  $\omega$  is a model of  $\Delta_\mu^\forall(E)$  if and only if there exists a compatible scaling where this interpretation belongs to the result fusion, namely it is a model of  $\Delta_{Max}^{E^S}(E)$ . More formally:

**Proposition 10** *Let  $E$  be a multi-set of ranked belief bases. Then  $\omega \in \text{Mod}(\Delta_\mu^\forall(E))$ , if and only if there exists a compatible scaling  $S$  such that  $\omega \in \text{Mod}(\Delta_{Max}^{E^S}(E^S))$ .*

Let us illustrate the fusion based on all compatible scalings with the following example.

**Example 11 (continued)** *Assume that  $\mu \equiv \top$ . Let us consider again  $B_1 = \{(a, 8), (-b, 4)\}$  and  $B_2 = \{(b, 2), (-a, 1)\}$ . Let us consider again  $S_1$  where  $B_1^{S_1} = \{(a, 8), (-b, 4)\}$  and  $B_2^{S_1} = \{(b, 2), (-a, 1)\}$  and a scaling  $S_2$ , where  $B_1^{S_2} = \{(a, 5), (-b, 4)\}$  and  $B_2^{S_2} = \{(b, 3), (-a, 2)\}$ . Both of them are compatible. Table 4 presents the profile of each interpretation for each scaling.*

	a	b	$\nu_{E^S_1}(\omega)$	Max	$\nu_{E^S_2}(\omega)$	Max
$\omega_0$	0	0	$\langle 8, 2 \rangle$	8	$\langle 5, 3 \rangle$	5
$\omega_1$	0	1	$\langle 8, 0 \rangle$	8	$\langle 5, 0 \rangle$	5
$\omega_2$	1	0	$\langle 0, 2 \rangle$	<b>2</b>	$\langle 0, 3 \rangle$	<b>3</b>
$\omega_3$	1	1	$\langle 4, 1 \rangle$	4	$\langle 4, 2 \rangle$	4

Table 4: Two equivalent compatible scalings

Note that in both compatible scalings  $S_1$  and  $S_2$ ,  $\omega_2$  is the preferred one. Table 5 shows six additional compatible scalings and Table 6 gives their associated profiles.

	$\phi_{ij}$	$R_{B_i}$	$S_3$	$S_4$	$S_5$	$S_6$	$S_7$	$S_8$
$B_1$	$a$	8	4	3	2	2	2	2
	$-b$	4	3	2	1	1	1	1
$B_2$	$b$	2	2	2	2	3	3	4
	$-a$	1	1	1	1	1	2	3

Table 5: Representative compatible scalings

	$\nu_{E^S_3}(\omega)$	$\nu_{E^S_4}(\omega)$	$\nu_{E^S_5}(\omega)$	$\nu_{E^S_6}(\omega)$	$\nu_{E^S_7}(\omega)$	$\nu_{E^S_8}(\omega)$
$\omega_0$	$\langle 4, 2 \rangle$	$\mathfrak{i}3,2_i$	$\mathfrak{i}2,2_i$	$\mathfrak{i}2,3_i$	$\mathfrak{i}2,3_i$	$\mathfrak{i}2,4_i$
$\omega_1$	$\langle 4, 0 \rangle$	$\mathfrak{i}3,0_i$	$\mathfrak{i}2,0_i$	$\mathfrak{i}2,0_i$	<b><math>\mathfrak{i}2,0_i</math></b>	<b><math>\mathfrak{i}2,0_i</math></b>
$\omega_2$	$\langle 0, 2 \rangle$	<b><math>\mathfrak{i}0,2_i</math></b>	$\mathfrak{i}0,2_i$	$\mathfrak{i}0,3_i$	$\mathfrak{i}0,3_i$	$\mathfrak{i}0,4_i$
$\omega_3$	$\langle 3, 1 \rangle$	<b><math>\mathfrak{i}2,1_i</math></b>	<b><math>\mathfrak{i}1,1_i</math></b>	<b><math>\mathfrak{i}1,1_i</math></b>	<b><math>\mathfrak{i}1,2_i</math></b>	$\mathfrak{i}1,3_i$

Table 6: Profiles of compatible scalings

In fact, it can be shown that these six compatible scalings given in Table 5 are enough to characterize the result of fusion. Namely, for each compatible scaling  $S$ , there exists a scaling  $S_i \in \{S_3, \dots, S_8\}$  given in Table 5, such that for all  $\omega$  and  $\omega'$ ,  $\omega <_{Max}^{E^S} \omega'$  iff  $\omega <_{Max}^{E^{S_i}} \omega'$ .

Bold elements in Table 6 represent models of  $\Delta_\mu^{max}$  for a given scaling. For instance, the interpretations  $\omega_1$  and  $\omega_3$  are models of  $\Delta_\mu^{max}$  for the compatible scaling  $S_5$  from Tables 5.

Finally, the strict partial order between interpretations is only defined by  $\omega_3 <_{\forall}^E \omega_1$ .

Hence, models of  $\Delta_\mu^\forall(E)$  are  $\{\omega_1, \omega_2, \omega_3\}$ , and  $\Delta_\mu^\forall(E) \equiv a \vee b$ .

As every skeptical approaches, conclusions obtained using all compatible scalings are safe.

Moreover, considering all possible compatible scales does not mean that the approach is too cautious and for instance, only tautologies can be derived from the result of merging. In particular, if the union of bases is consistent, then the result of merging is simply the conjunct of the bases. More formally:

**Proposition 12** *Let  $E = \{B_1, \dots, B_n\}$ . Then if  $\bigwedge_{B_i \in E} (B_i^*) \wedge \mu$  is consistent, then*

$$\Delta_\mu^\forall(E) \equiv \bigwedge_{B_i \in E} (B_i^*).$$

**Proof 13** *The proof is immediate. Let  $\omega$  be a model of  $\bigwedge_{B_i \in E} (B_i^*)$ . For each compatible scaling  $S$ , it can be checked that its associated profile is  $\nu_{E^S}(\omega) = \langle 0, \dots, 0 \rangle$ , namely  $Max(\nu_{E^S}(\omega)) = 0$ . Hence,  $\omega$  is minimal. Now let  $\omega'$  be such that it falsifies at least one belief of some base in  $E$ , then for each compatible scaling  $Max(\nu_{E^S}(\omega')) > 0$ . Therefore, for each compatible scale,  $\omega$  is minimal iff  $\omega$  is a model of  $\bigwedge_{B_i \in E} (B_i^*)$ .*

Of course in extreme situations, where sources are strongly conflicting, only tautologies can be derived. For instance, let  $B_1 = \{(a, 1), (b, 4)\}$  and  $B_2 = \{(-a, 2), (-b, 3)\}$ . This example represents an extreme situation, where the two sources strongly disagree. It is then hard to make decisions. Since, our motivation is that if there is no additional information, we prefer to avoid inferring arbitrary conclusions.

## A characterization of compatible-based fusion with Pareto-like ordering

This section describes how identifying preferred interpretations from  $\Omega$  (or from  $\text{Mod}(\mu)$ ) for merging  $E$  without computing  $\mathbb{S}_E$ , using a strict Pareto-like ordering.

Intuitively, an interpretation  $\omega$  is strictly preferred to  $\omega'$  if:

- i)  $\omega'$  is not a model of each base  $B_i^*$ , and
- ii) for each  $B_j$ , either  $\omega$  and  $\omega'$  are models of  $B_j^*$ , or  $\kappa_{B_j}(\omega) < \kappa_{B_j}(\omega')$  (namely,  $\omega$  is preferred to  $\omega'$  with respect to  $B_j$ ).

More formally:

**Definition 14 (Pareto-like ordering)** *Let  $E = \{B_1, \dots, B_n\}$ .  $\omega$  is pareto-preferred to  $\omega'$ , denoted by  $\omega \triangleleft_{Pareto} \omega'$ , iff the two following conditions are satisfied:*

- $\exists i \in \{1, \dots, n\}, \kappa_{B_i}(\omega') \neq 0$ ,

- $\forall i \in \{1, \dots, n\}$ ,  $\kappa_{B_i}(\omega) = \kappa_{B_i}(\omega') = 0$ , or  $\kappa_{B_i}(\omega) < \kappa_{B_i}(\omega')$ .

Let us illustrate this definition by the following example.

**Example 15 (continued)** Let us consider again  $B_1 = \{(a, 8), (-b, 4)\}$  and  $B_2 = \{(b, 2), (-a, 1)\}$ . Table 7 presents the profile of each interpretation of  $\Omega$ .

	a	b	$\kappa_{B_1}(\omega)$	$\kappa_{B_2}(\omega)$	$\nu_E(\omega)$
$\omega_0$	0	0	8	2	<del>8, 2</del>
$\omega_1$	0	1	8	0	<del>8, 0</del>
$\omega_2$	1	0	0	2	<del>0, 2</del>
$\omega_3$	1	1	4	1	<del>4, 1</del>

Table 7: Profiles associated with interpretations

From Table 7 we have  $\omega_3$  strictly preferred to  $\omega_0$  using  $\triangleleft_{\text{Pareto}}$ , since for each belief base, the rank associated with  $\omega_3$  is strictly lower than the rank associated with  $\omega_0$ .

The following proposition contains one of main results of the paper and shows the equivalence between Pareto-like ordering and compatible scaling ordering:

**Proposition 16 (Equivalence between the orders)** Let  $E$  be a multi-set of ranked belief bases,  $<_{\nabla}^E$  and  $\triangleleft_{\text{Pareto}}$  be the two partial orders defined respectively by Definition 9 and Definition 14. Then:

$$\forall \omega, \omega' \in \Omega, \omega <_{\nabla}^E \omega' \text{ iff } \omega \triangleleft_{\text{Pareto}} \omega'.$$

Thus, this Pareto-like ordering allows us to merge incommensurable ranked bases, without computing all compatible scalings.

$$\text{Mod}(\Delta_{\mu}^{\text{Pareto}}(E)) = \{\omega \models \mu : \nexists \omega' \models \mu, \omega' \triangleleft_{\text{Pareto}} \omega\}.$$

**Example 17 (continued)** Let us consider again  $B_1 = \{(a, 8), (-b, 4)\}$  and  $B_2 = \{(b, 2), (-a, 1)\}$ . From Table 7,  $\omega_1$ ,  $\omega_2$  and  $\omega_3$  are models of  $\Delta_{\mu}^{\text{Pareto}}$  because they are minimal with respect to  $\triangleleft_{\text{Pareto}}$  ordering defined in Definition 14.

Hence,  $\Delta_{\mu}^{\text{Pareto}}(E) \equiv a \vee b$  for  $\mu = \top$ . This is exactly the same result as the one given in Example 11.

## Logical properties

We provide in this section some logical properties to our merging operator.

### Rational postulates

Rational postulates have been proposed for characterizing fusion operators (Konieczny & Pino Pérez 1998) under integrity constraints. These postulates are defined when the belief base  $B_i$  is represented by a propositional formula.

- (IC0)  $\Delta_{\mu}(E) \models \mu$ ;
- (IC1) If  $\mu$  is consistent, then  $\Delta_{\mu}(E)$  is consistent;
- (IC2) If  $\bigwedge_{B \in E} B$  is consistent with  $\mu$ , then  $\Delta_{\mu}(E) \equiv \bigwedge_{B \in E} B \wedge \mu$ ;
- (IC3) If  $E_1 \equiv E_2$  and  $\mu_1 \equiv \mu_2$ , then  $\Delta_{\mu_1}(E_1) \equiv \Delta_{\mu_2}(E_2)$ ;

- (IC4) If  $B_1 \models \mu$  and  $B_2 \models \mu$ , then  $\Delta_{\mu}(\{B_1, B_2\}) \wedge B_1$  is consistent iff  $\Delta_{\mu}(\{B_1, B_2\}) \wedge B_2$  is consistent;
- (IC5)  $\Delta_{\mu}(E_1) \wedge \Delta_{\mu}(E_2) \models \Delta_{\mu}(E_1 \sqcup E_2)$ ;
- (IC6) If  $\Delta_{\mu}(E_1) \wedge \Delta_{\mu}(E_2)$  is consistent, then  $\Delta_{\mu}(E_1 \sqcup E_2) \models \Delta_{\mu}(E_1) \wedge \Delta_{\mu}(E_2)$ ;
- (IC7)  $\Delta_{\mu_1}(E) \wedge \mu_2 \models \Delta_{\mu_1 \wedge \mu_2}(E)$ ;
- (IC8) If  $\Delta_{\mu_1}(E) \wedge \mu_2$  is consistent, then  $\Delta_{\mu_1 \wedge \mu_2}(E) \models \Delta_{\mu_1}(E) \wedge \mu_2$ .

A complete description of these postulates can be found in (Konieczny & Pino Pérez 2002). In our framework, (IC2), (IC3) and (IC4) need to be adapted. For instance,  $B_1 \models \mu$  should be replaced by  $B_1^* \models \mu$ . Moreover (IC3) needs to be adapted by defining the equivalence between two multi-set of ranked knowledge base, denoted by  $\equiv_r$ . Two multisets of ranked belief bases  $E_1$  and  $E_2$  are equivalent if for each ranked base of  $E_1$ , there exists a ranked base in  $E_2$  inducing the same ranking function  $\kappa$ , given by Definition 2. More formally  $E_1 \equiv_r E_2$  if and only if  $\forall B \in E_1$  (resp.  $E_2$ ),  $\exists B' \in E_2$  (resp.  $E_1$ ) such that  $\kappa_B = \kappa_{B'}$ .

We finally rewrite (IC2), (IC3) and (IC4) as follows:

- (IC2') If  $\bigwedge_{B \in E} B^*$  is consistent with  $\mu$ , then  $\Delta_{\mu}(E) \equiv \bigwedge_{B \in E} B^* \wedge \mu$ ;
- (IC3') If  $E_1 \equiv_R E_2$  and  $\mu_1 \equiv \mu_2$ , then  $\Delta_{\mu_1}(\{E_1\}) \equiv \Delta_{\mu_2}(\{E_2\})$ ;
- (IC4') If  $B_1^* \models \mu$  and  $B_2^* \models \mu$ , then  $\Delta_{\mu}(\{B_1, B_2\}) \wedge B_1^*$  is consistent iff  $\Delta_{\mu}(\{B_1, B_2\}) \wedge B_2^*$  is consistent.

The following proposition shows that even if we drop the commensurability assumption, our merging operator is very compatible with most natural postulates for merging operators.

**Proposition 18** Let  $E = \{B_1, \dots, B_n\}$  be a multi-set of ranked belief bases. If all  $B_i^*$  are coherent then  $\Delta_{\mu}^{\text{Pareto}}(E)$  satisfies (IC0), (IC1), (IC2'), (IC3'), (IC4'), (IC5), (IC7) and (IC8).

However,  $\Delta_{\mu}^{\text{Pareto}}(E)$  falsifies (IC6). For the counter example, consider  $\mu = \top$ ,  $E_1 = \{B_1 = \{(a, 1)\}, B_2 = \{(-a, 1)\}\}$  and  $E_2 = \{B_3 = \{(a, 1)\}\}$ . We have  $\Delta_{\mu}(E_1) \equiv \top$  and  $\Delta_{\mu}(E_2) \equiv a$ . Furthermore, we have  $\Delta_{\mu}(E_1, E_2) \equiv \top$ , but  $\Delta_{\mu}(E_1) \wedge \Delta_{\mu}(E_2) \equiv a$ .

It is very important to note that this is not due to the incommensurability assumption. Indeed, the maximum-based merging operator defined for merging commensurable belief bases falsifies also (IC6).

The following postulate has also been proposed in (Konieczny & Pino Pérez 2002) for characterizing the notion of majority independence.

$$(MI) \forall n \Delta_{\mu}(E_1 \sqcup E_2^n) \equiv \Delta_{\mu}(E_1 \sqcup E_2).$$

This property states that the result of merging is fully independent of the repetition of the beliefs: it only takes into account each different view.

**Proposition 19**  $\Delta_{\mu}^{\text{Pareto}}(E)$  satisfies MI.

At corollary (also pointed in (Konieczny & Pino Pérez 2002)) of this proposition, we do not need to consider multi-sets of ranked belief bases, but only sets.

## Related works

There are some approaches that merge stratified beliefs bases without commensurability assumptions. For instance, the approach proposed in (Benferhat, Dubois, & Prade 1999) in possibility theory framework, indeed drops the commensurability but assumes the existence of an ordering between stratified bases to merge. Our approach does not require such assumption.

Recently, (Guilin, Liu, & Bell 2006) have proposed an approach to merge stratified belief bases. Basically, their approach can be described as follows: each stratified belief base, on the basis of same ordering strategy, induces a ranking between interpretations. Namely, possible interpretations are associated with vectors of priority levels in all the original belief bases. The result of merging is obtained by only considering interpretations whose associated vectors are minimal with respect to lexicographical ordering. The main problem with such an approach is that priorities issued from different sources are considered to be comparable. Hence, incommensurability assumption is not satisfied. In the sense that if some source assigns a rank  $i$  to some formula  $\phi$ , and another source assigns a rank  $j$  to another formula  $\psi$ , if both sources use the same ordering strategy, then  $\phi$  and  $\psi$  can be compared. In our approach,  $\phi$  and  $\psi$  are not assumed to be comparable.

## Conclusion

This paper has addressed an issue which is not widely considered in belief fusion. It concerns the problem of merging incommensurable ordered belief bases. We have proposed a natural definition based on the idea of compatible rankings.

We have also proposed a characterization of inference based on the family of compatible rankings. More precisely, it is possible to define equivalently the result of merging by ordering interpretations with respect to a variant of Pareto criteria.

This paper has also studied rational postulate for our merging operator, and we showed that it satisfies most of rational postulates proposed for merging operators, even if commensurability is not assumed.

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