

Algebraic Approach to Specifying Part-Whole Associations

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Introduction

Expressive power and deductive power are two critical characteristics of knowledge representation languages. They capture respectively, what information can be explicitly stated, and what information can be deduced. Object (class) based representations have gained almost universal acceptance because of their ability to capture inter-class associations and their implied ability to reason about these associations. While this is true for taxonomical relations (generalizations, specializations), this is far from being true for structural associations relating a whole to its parts. Most graphical and formal languages provide constructs for stating part-whole associations, but most languages have limited or no support for making inferences from them.

This shortcoming is not a new revelation. Extensive research has been ongoing in philosophy, linguistics, logic, artificial intelligence, and software engineering with a focus to formalize the semantics of the part-whole association. This research resulted in a diverse pool of formalisms, some deemed too weak and thus not very useful, and some deemed too strong and thus not very usable; and most deemed both too weak and too strong because they do not capture all the properties of interest to some application domain, and capture properties that do not hold in the same application domain. This paradoxical state of affairs is in fact a reflection of the nature of the part-whole association. While there is an intuitive universal understanding of what the association means, the specific properties that one needs to reason about vary from one domain to the next, and from one application to the next.

In this paper, we take an approach to defining part-whole associations that account both for the universality and the variability:

- We account for the “universality” by defining all part-whole associations in terms of a common set of primitive associations.
- We account for the “variability” by the fact that each part-whole association may be a different combination of primitive associations.

We define an algebra of associations that serves as a basis for the deductive power of languages capturing the part-whole association.

Most researchers focusing on the representation of part-whole relations base their work on Description Logics (DL) [1]. We use a Tarski-like algebra of binary relations which has a similar expressive power but presents the convenience of an algebra.

A *relation* on a set Σ is a subset of $\Sigma \times \Sigma$. Constant relations on set Σ include: the Universal relation $L = \Sigma \times \Sigma$, the *identity* relation $I = \{(s,s) | s \in \Sigma\}$ and the empty relation $\Phi = \{\}$. Given A , a subset of Σ , we define $I(A)$ as $\{(s,s) | s \in A\}$. In addition, given two sets A and B subsets of Σ , we define the relation $D(A,B) = A \times B = \{(s,s') | s \in A \text{ and } s' \in B\}$. The intersection of two relations R and R' is defined by: $R \cap R' = \{(s,s') | (s,s') \in R \text{ and } (s,s') \in R'\}$. The composition of two relations R and R' is denoted by $R \circ R'$ and defined by: $R \circ R' = \{(s,s') | \exists s'' : (s,s'') \in R \text{ and } (s'',s') \in R'\}$. The inverse of a relation R is denoted by R^\wedge and defined by $R^\wedge = \{(s,s') | (s',s) \in R\}$. The nucleus of a relation R is denoted by $\nu(R)$ and defined by $\nu(R) = R \circ R^\wedge$. The co-nucleus of a relation R is denoted by $\gamma(R)$ and defined by $\gamma(R) = R^\wedge \circ R$.

Representing Basic Object-Oriented Constructs

Consider the following Object-Oriented schema represented graphically in Figure 1.

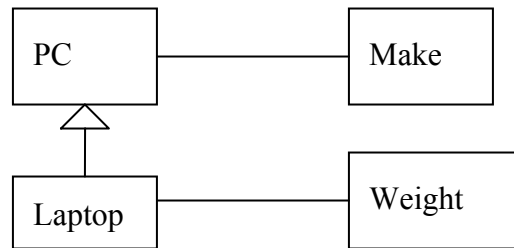


Figure 1

Let Σ be the set of all instances of interest. The schema introduces subsets PC, Make, Laptop, and Weight, with Laptop subset of PC.

The schema also introduces two binary relations: Has-Make $\subseteq D(PC, Make)$, and Has-Weight $\subseteq D(Laptop, Weight)$.

Capturing Part-Whole Association

We augment the schema introduced in Figure 1 by introducing two parts of PC: Motherboard and keyboard.

To capture the part-whole relationship, we introduce three (categories of) relations:

1. *Part-Whole relation* \diamond , defined on Σ by:

$$\diamond = \{(s, s') \mid s \text{ is part of the whole } s'\}.$$

Relation \diamond is anti-reflexive, asymmetric, and transitive.

2. *Same-Property relation* $\xi(P)$. Given a set P of interest (e.g. Make), we denote by $\zeta(P)$ —that we pronounce has- P —the deterministic relation from Σ to P that associates elements from the universe Σ with their P property values (e.g. has-Make). $\xi(P)$ —that we pronounce same- P —(e.g. sameMake) is the nucleus of the relation $\zeta(P)$, i.e. $\xi(P) = \zeta(P) \circ \zeta(P)^\wedge$. $\xi(P)$ contains the pairs of instances that have the same P value. It is reflexive, symmetric, and transitive.

3. *Spatial Containment relation* $\diamond(S)$. Given a set S of interest (e.g. volume= \langle length, width, depth \rangle), and the associated deterministic relation $\zeta(S)$ from Σ to S (e.g. has-Volume), given an ordering relation \leq on S (e.g. \leq defined on the Volume domain), we denote by $\diamond(S)$ —pronounced within S —the relation defined by $\diamond(S) = \zeta(S) \circ \leq \circ \zeta(S)^\wedge$. $\diamond(S)$ is reflexive, anti-symmetric, and transitive. It contains the pairs of instances (s, s') where the S property of s is contained in the S property of s' as defined by the ordering relation \leq . We will use this relation to capture the fact that parts are sometimes enclosed (spatially or otherwise) within their whole.

The relation \diamond , and the sets of relations same- P ($\xi(P)$), related- P ($\rho(P, f)$), and within- S , $\diamond(S, \leq)$, are the three building blocks that we will use to characterize part-whole relations. A part-whole relation between class (set) A and class (set) B will be denoted $pW(A, B)$ and characterized by at least the following axiom:

$$pW(A, B) \subseteq D(A, B) \cap \diamond.$$

If, in addition, A and B have same- P values for properties P_1 and P_2 and A is within B with respect to property S , we have the following axiom:

$$pW(A, B) \subseteq$$

$$D(A, B) \cap \diamond \cap \xi(P_1) \cap \xi(P_2) \cap \diamond(S).$$

The right-hand side of the above inequality represents the set of necessary conditions that need to be met by the association between A and B . We name the right-hand-side the Necessary Relation for the A-part-of-B relation. The necessary relation of $pW(A, B)$ will be denoted by $\mathcal{N}(pW(A, B))$. In the example above, we have:

$$\mathcal{N}(pW(A, B)) = D(A, B) \cap \diamond \cap \xi(P_1) \cap \xi(P_2) \cap \diamond(S).$$

We extend the definition of necessary relation to any relation R . Relation Q is said to be a necessary relation for R if and only if $R \subseteq Q$.

Note that because \subseteq is monotonous with respect to intersection and composition, we have the following propositions:

Proposition:

Given two relations R and R' , given two relations $\mathcal{N}(R)$, and $\mathcal{N}(R')$, the following statements are true:

- $\mathcal{N}(R) \cap \mathcal{N}(R')$ is a necessary relation for $R \cap R'$.
- $\mathcal{N}(R) \circ \mathcal{N}(R')$ is a necessary relation for $R \circ R'$.

Reasoning about Part-Whole relations

The main impetus of this work has been to capture part-whole relations in a way that accounts for variability (different relations are constructed from different combinations of the building blocs) and commonality (common set of building blocks), and allow us to reason about them.

Reasoning about part-whole relations consists of:

- Computing the composition of part-whole relations, i.e. given that $pW(A, B)$ and $pW(B, C)$, what can we say about the relation between A and C ?
- Computing and compositions of part-whole relations with classification relations, i.e. given that $pW(A, B)$ and $B \subseteq C$, what is the relationship between A and C ?

Our reasoning approach is based on two keys premises:

We characterize relations rather than define them. In other words, we reason about necessary relations rather than about the relations directly.

We define an semi-algebra based on the set of primitive relations and the operations of composition and intersection. Each necessary relation is an intersection of relations from the set $\mathcal{B} = \{D, \diamond, \xi, \rho, \diamond\}$. Given two relations $R1$ and $R2$, with their necessary relations $\mathcal{N}1$ and $\mathcal{N}2$. The composition $\mathcal{N}1 \circ \mathcal{N}2$ is a necessary relation to $R1 \circ R2$. But also, every superset of $\mathcal{N}1 \circ \mathcal{N}2$ is a necessary relation for $R1 \circ R2$. The composition $\mathcal{N}1 \circ \mathcal{N}2$ is the intersection of compositions of pairs of relations from the set \mathcal{B} . Therefore, it suffices to compute the 5 by 5 compositions of primitive relations. The 5 by 5 table is available in the extended version of this paper [2] where composition of part-whole with other associations is also examined. The framework can be expanded to add other primitives as needed. The reasoning supported by this framework is supported by the relational algebra, whereby the necessary relation of an expression can be computed directly and automatically through table look up.

We are currently extending this framework by investigating the representation of horizontal relationships (relationships between different parts of the same whole), and a wider variety of vertical relationships, notably vertical relationships that involve the whole and more than one of its parts.

References

- 1 Baader, F. *et. Al. The Description Logics Handbook*, Cambridge, 2002.
- 2 Mili, F. W. Li Algebraic approach to specifying part-whole associations, Technical report, 2004.