Temporalizing Cardinal Directions: From Constraint Satisfaction to Planning

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Abstract

Frank's cardinal direction calculus is one of the most prominent spatial constraint formalisms, which allows one to represent, and reason with, the relative position of objects in the Euclidean plane. Typical application fields of this calculus include geographical information systems (GIS), route finding and description systems, and navigation of robots that interact with humans. In this paper we investigate a constraint formalism which temporalizes the cardinal direction calculus with respect to Allen's interval algebra. In this constraint language it is possible to represent objects in the plane which change their absolute position in time. Since such changes entail changes of the relative positions of objects to other objects as well, we are interested in the question of how continuous change of objects is reflected in changes of the respective qualitative relations expressing these relative positions. We will show how continuous changes can be represented as operations to objects in grid-like structures. Based on this representation we finally propose a method for encoding temporalized spatial constraint satisfaction problems as deterministic planning problems.

Introduction

Qualitative Spatial Reasoning (QSR) abstracts from metrical details of the physical world and enables computers as well as artificial agents to make predictions about spatial relations, even when precise quantitative information is not available (Cohn 1997). From a practical point of view, QSR provides an abstraction layer that summarizes similar quantitative states into one qualitative description. A complementary view from the cognitive perspective is that the qualitative method *compares* features within the object domain rather than by *measuring* them in terms of some artificial external scale (Freksa 2004). This is the reason why qualitative descriptions are quite natural for humans.

Frank's cardinal direction calculus (Frank 1996) is one of the most prominent constraint formalisms in the domain of QSR. The cardinal direction calculus allows one to represent, and reason with the relative position of objects in the Euclidean plane. Typical application fields of this calculus include geographical information systems (GIS), route finding and description systems, and navigation of robots that interact with humans.

In this paper we investigate a constraint formalism, which temporalizes the cardinal direction calculus with respect to Allen's interval algebra. In this constraint language it is possible to represent objects in the plane, which change their absolute position in time. Since such changes usually entail changes of the relative positions of objects to other objects as well, we are interested in the question of how continuous change of objects is reflected in changes of the respective qualitative relations expressing these relative positions.

A temporalized cardinal direction calculus as outlined here may be applicable in scenarios known from logistics, planning, robot navigation, and multi-agent systems. For example, assume that a human in a control center has to navigate several unmanned aerial vehicles. To control the behavior of the vehicles, the operator may use constraints such as: As long as vehicle 1 is south of some fixed landmark, vehicle 2 should stay south or south-east of vehicle 1, but never north of a vehicle 3 which is to stay always south-east of vehicle 1, etc.

Spatio-temporal constraint languages have been previously discussed in the literature. For example, Bennett *et al.* (2002) investigated a temporalization of the topological $\mathcal{RCC}8$ calculus. Their approach was further developed by Gerevini & Nebel (2002) — many definitions and some techniques developed by them carry over to the new calculus discussed in this paper. From a more philosophical perspective, Galton (2000) discussed various facets of continuous change, in particular, how such changes can be consistently described at different levels of granularity and how their qualitative and quantitative descriptions are related to each other. In particular, his notion of dominance space will be implicitly used in this paper.

The aim of the paper is to work out the relationship between constraint satisfaction problems of temporalized calculi on the one hand side and deterministic planning problems on the other. For this we will first show how continuous changes can be represented as operations to objects in grid-like structures. Based on this representation we finally propose a method for encoding temporalized spatial constraint satisfaction problems as planning problems. The interesting point here is that the type of planning problem obtained by this encoding has not yet been discussed in the planning literature (at least to our knowledge).

The paper is organized as follows: In section 2 we introduce the language of the calculus $\mathcal{TCD}$, which temporalizes the cardinal direction calculus $\mathcal{CD}$ with respect to Allen's interval algebra. In section 3 we sketch how constraint satisfaction problems (formulated with respect to continuous time and continuous space) can be reformulated as discrete transitions in a finite grid. In particular, we will discuss some simple examples, which show that the constraint satisfaction problem of $\mathcal{TCD}$ cannot be solved by applying standard procedures for deciding the satisfiability problems of the component calculi from which $\mathcal{TCD}$ is built. Then in section 4, we investigate how the constraint satisfaction problem of $\mathcal{TCD}$ can be transformed into a planning problem. Finally, section 5 summarizes the results of the paper and gives a short overview of some questions that will be topics of future research.

Temporalizing Cardinal Directions

To begin with, let us recall the two constraint languages in which we will be interested in the following, namely Frank's cardinal direction calculus ($\mathcal{CD}$) and Allen's interval algebra ($\mathcal{IA}$).

In Allen's interval algebra (Allen 1983), intervals are represented as pairs of instants $\langle t_1, t_2 \rangle \in \mathbb{R}^2$ such that $t_1 < t_2$. By comparing start and endpoints of two intervals, one can identify thirteen (jointly exhaustive and pairwise disjoint) relations known in the literature as the Allen 13 relations (cf. Fig. 1). To represent imprecise knowledge, we consider disjunctions of base relations (usually written as sets of base relations). The satisfiability problem for $\mathcal{IA}$ is defined as follows: Given a finite (and maybe imprecise) description of the relations between intervals in terms of Allen relations, is this description consistent (satisfiable)? This problem is known to be NP-complete (Vilain, Kautz, & van Beek 1989). Tractable subclasses of the general problem were identified by Nebel & Bürckert (1995) and by Ligozat (1996).

Symbol	**Relation**	**Pictorial Representation**
$\prec$ (conv.: $\succ$)	*I* before *J*	
m (mi)	*I* meets *J*	
o (oi)	*I* overlaps *J*	
d (di)	*I* during *J*	
s (si)	*I* starts *J*	
f (fi)	*I* finishes *J*	
$=$	*I* equals *J*	

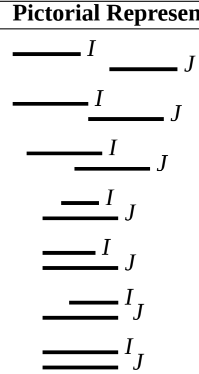


Figure 1: The thirteen basic relations of the Interval Algebra

Frank (1996) proposed two different approaches for representing cardinal directions: the first one uses cone-shaped directions such that each direction refers to a region of 45 degrees. The second one interprets the four main directions (*north*, *east*, etc.) as half-lines, while the intermediate regions (*north-east*, etc.) refer to quadrants (cf. Fig. 2). Ligozat (1998) worked out the advantages of the latter, *projection-based approach*, which we will follow here.

In what follows, we assume that *spatial* objects are points in the Euclidean plane $\mathbb{R}^2$. Given two point-like objects in the plane, the relative position between these objects can be described by one of the cardinal direction relations *north*, *north-east*, *east*, etc. (cf. Fig. 2). Note that these relative positions are defined with respect to a fixed reference frame. We will assume that this reference frame is fixed in time as well, i. e., a change of the relative spatial position always stems from an object changing its absolute position (not in a change of the underlying reference system). Ligozat (1998) examined the computational complexity of the general satisfiability problem (which is NP-complete) and he identified tractable subclasses.

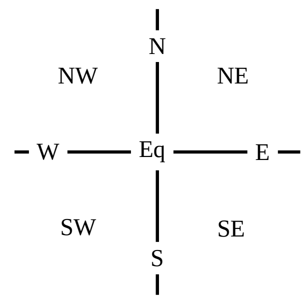


Figure 2: The nine base relations of the cardinal direction calculus

To put things a little bit more precise, let $V_{\mathcal{IA}}$ and $V_{\mathcal{CD}}$ be (disjoint) sets of variables. An $\mathcal{IA}$ *constraint* is a formula of the form $I\ R\ J$, where $I, J \in V_{\mathcal{IA}}$ and $R = \{r_1, \ldots, r_n\}$ is a (possibly empty) subset of the set of all $\mathcal{IA}$ base relations $\{\prec, \mathrm{m}, \mathrm{o}, \mathrm{s}, \mathrm{f}, \mathrm{d}, =, \succ, \mathrm{mi}, \mathrm{oi}, \mathrm{si}, \mathrm{fi}, \mathrm{di}\}$. Usually we write $I\ r\ J$ instead of $I\ \{r\}\ J$ when r is one of this base relations. A $\mathcal{CD}$ *constraint* is a formula of the form $x\ S\ y$, where $x, y \in V_{\mathcal{CD}}$ and $S = \{s_1, \ldots, s_m\}$ is a (possibly empty) subset of the set of all $\mathcal{CD}$ relations $\{\mathrm{N}, \mathrm{NE}, \mathrm{E}, \mathrm{SE}, \mathrm{S}, \mathrm{SW}, \mathrm{W}, \mathrm{NW}, \mathrm{Eq}\}$. An $\mathcal{IA}$ (resp. $\mathcal{CD}$) *constraint network* is a finite set of $\mathcal{IA}$ (resp. $\mathcal{CD}$) constraints. Let $V(C)$ be the set of variables occurring in a given constraint network C. An *assignment* for an $\mathcal{IA}$ constraint network C is a function $\tau : V(C) \to \mathbb{R}^2$ that assigns to each variable I that occurs in C a pair of real numbers $(\tau^1(I), \tau^2(I))$ such that $\tau^1(I) < \tau^2(I)$. The model relation w. r. t. an assignment τ for C is introduced as follows:

$$\begin{aligned}
\tau \models I \,\mathrm{m}\, J &\iff \tau^2(I) = \tau^1(J) \\
\tau \models I \,\mathrm{o}\, J &\iff \tau^1(I) < \tau^1(J) < \tau^2(I) < \tau^2(J) \\
\tau \models I \,\mathrm{d}\, J &\iff \tau^1(J) < \tau^1(I) < \tau^2(I) < \tau^2(J) \\
&\ldots \quad \text{(cf. Fig. 1)}
\end{aligned}$$

and

$$\tau \models I\ \{r_1, \ldots, r_n\}\ J \iff \tau \models I\ r_i\ J \text{ for some } 1 \leq i \leq n.$$

Analogously, an *assignment* for a $\mathcal{CD}$ constraint network C is a function $\gamma : V(C) \to \mathbb{R}^2$ that assigns to each variable x occurring in C, a pair of real numbers $(\gamma^1(x), \gamma^2(x))$. Here the model relation w. r. t. an assignment γ for C is defined as follows:

$$\begin{aligned}
\gamma \models x\,\mathrm{N}\,y &\iff \gamma^1(x) = \gamma^1(y) \text{ and } \gamma^2(x) > \gamma^2(y)\\
\gamma \models x\,\mathrm{NE}\,y &\iff \gamma^1(x) > \gamma^1(y) \text{ and } \gamma^2(x) > \gamma^2(y)\\
\gamma \models x\,\mathrm{E}\,y &\iff \gamma^1(x) > \gamma^1(y) \text{ and } \gamma^2(x) = \gamma^2(y)\\
&\ \ \dots \quad \text{(cf. Fig. 2)}
\end{aligned}$$

and

$$\gamma \models I\ \{s_1, \dots, s_m\}\ J \iff \gamma \models I\ s_j\ J \text{ for some } 1 \le j \le m.$$

Finally, an $\mathcal{IA}$ (resp. $\mathcal{CD}$) *constraint network*, C, is satisfiable if there exists an $\mathcal{IA}$ (resp. $\mathcal{CD}$) assignment that models all relations of C. In this case the assignment is said to be a *solution* of C

The temporalized cardinal direction calculus $\mathcal{TCD}$ combines constraint formulae of Allen's interval calculus with temporally annotated constraints of the cardinal direction calculus. More precisely, we define $\mathcal{TCD}$ constraint formulae as follows:

- Each interval constraint $I\ R\ J$ is a $\mathcal{TCD}$ constraint, i. e., $I\ R\ J$ is a $\mathcal{TCD}$ constraint for each pair of interval variables $I, J \in V_{\mathcal{IA}}$ and each set R of Allen relations.
- For each interval variable I, each pair of spatial variables x and y, and each set S of $\mathcal{CD}$ relations, $I : x\ S\ y$ is a $\mathcal{TCD}$ constraint.

A *$\mathcal{TCD}$ constraint network* is a finite set of $\mathcal{TCD}$ constraints. A standard interpretation of this constraint formalism is based on the following ingredients: we use the linear ordering of the reals for interpreting interval variables and points of the Euclidean plane $\mathbb{R}^2$ for interpreting spatial variables. More precisely, we assign to each spatial variable a continuous path in the Euclidean plane.

A typical example of a $\mathcal{TCD}$ constraint network is the following:

$$I\,\mathrm{m}\,J,\ \ I : x\ \{\mathrm{NE}, \mathrm{NW}\}\ y,\ I : y\,\mathrm{E}\,z,\ J : x\,\mathrm{SW}\,y,\ J : y\,\mathrm{S}\,z.$$

This network expresses that during interval I, point x is north-east or north-west of y and y is east of z. Then, in interval J, which immediately succeeds I, x is south-west of y, etc.

Definition 1 An *interpretation* for a $\mathcal{TCD}$ constraint network C is an ordered pair $\langle \tau, \gamma \rangle$, where

- τ is an $\mathcal{IA}$ assignment for $V(C) \cap V_{\mathcal{IA}}$.
- γ assigns to each instant $t \in \mathbb{R}$ a $\mathcal{CD}$ assignment $\gamma_t : V(C) \cap V_{\mathcal{CD}} \to \mathbb{R}^2$ such that for each variable x, the function $\hat{\gamma}_x : \mathbb{R} \to \mathbb{R}^2, t \mapsto \gamma_t(x)$, is continuous (with respect to the usual topologies).

The point $\hat{\gamma}_x(t)$ is referred to as *the position* of x at time point t, and $\hat{\gamma}_x^1(t)$ and $\hat{\gamma}_x^2(t)$ refer to the x- and the y-coordinate of this position, respectively.

We then define the model relation as follows:

$$\begin{aligned}
\langle \tau, \gamma \rangle \models I\ R\ J &\iff \tau \models I\ R\ J\\
\langle \tau, \gamma \rangle \models I : x\ S\ y &\iff \gamma_t \models x\ S\ y \text{ for each } \tau^1(I) < t < \tau^2(I).
\end{aligned}$$

Note that we only require that the spatial constraints hold in the interior of the interval. This is necessary since if these spatial constraints need to hold at the starting and endpoint of the interval as well, then it would not be possible that a base relation holding between two objects in some interval I changes to a different base relation between these objects in any interval met by I.

Definition 2 (a) An interpretation $\langle \tau, \gamma \rangle$ for a $\mathcal{TCD}$ constraint network C is said to be a *model* of C if $\langle \tau, \gamma \rangle \models \phi$ for each $\phi \in C$.

(b) An interpretation $\langle \tau, \gamma \rangle$ is *collision-free* if for each pair of spatial variables x and y in C, $\hat{\gamma}_x(t) \neq \hat{\gamma}_y(t)$ for all $t \in \mathbb{R}$.

(c) An interpretation $\langle \tau, \gamma \rangle$ obeys the *"same place, same thing"-principle* if for each pair of spatial variables x and y in C, it holds: if $\hat{\gamma}_x(t) = \hat{\gamma}_y(t)$ for some $t \in \mathbb{R}$, then $\hat{\gamma}_x = \hat{\gamma}_y$.

In the following, we will focus on collision-free interpretations. The reason for this lies in the idea that no two distinct physical objects can occupy the same place ("same place, same thing"). If we apply this principle to $\mathcal{TCD}$ constraint networks, we can eliminate spatial constraints in which Eq occurs by propagating the identity relation between two variables to all other (temporally annotated) spatial constraints. In fact, if there is a $\mathcal{TCD}$ constraint network that is true in a "same place, same thing"-interpretation, then the corresponding $\mathcal{TCD}$ constraint network in which Eq does not occur, is true in a collision free interpretation, and vice versa.

Our definition of a model requires object paths to be continuous. Continuity constraints, however, cannot be expressed by $\mathcal{TCD}$ formulae, but by *deduction rules* only. For example, consider network containing the constraint $I : x\ \{\mathrm{N}, \mathrm{NE}, \mathrm{SE}, \mathrm{S}\}\ y$. This constraint is satisfied by a continuous $\mathcal{TCD}$ interpretation if and only if either $I : x\ \{\mathrm{N}, \mathrm{NE}\}\ y$ or $I : x\ \{\mathrm{SE}, \mathrm{S}\}\ y$ is satisfied by that interpretation. To show this, let t be an instant in I such that $x\ \{\mathrm{N}, \mathrm{NE}\}\ y$ is false at t. For reductio ad absurdum assume that there is an instant t' such that $x\ \{\mathrm{SE}, \mathrm{S}\}\ y$ is true at t'. If $t' < t$ (analogously for $t < t'$), there must be an instant $t < t'' < t'$ in which $x\ \{\mathrm{Eq}, \mathrm{E}\}\ y$ is true — in contradiction to the fact that $x\ \{\mathrm{N}, \mathrm{NE}, \mathrm{SE}, \mathrm{S}\}\ y$ must be true at t''.

Using some simple facts from linear algebra, one can easily verify the following lemma — note that linear functions (as considered in the lemma) are continuous, since $\mathbb{R}^2$ is a topological $\mathbb{R}$-vector space:

Lemma 3 *Let $p_1, p_2, v_1, v_2 \in \mathbb{R}^2$, let $q_1 := p_1 + v_1$ and $q_2 := p_2 + v_2$, and let $f_1, f_2 : [0, 1] \to R^2$ be linear functions defined by $f_1(t) := p_1 + tv_1$ and $f_2(t) := p_2 + tv_2$.*

(a) *If $p_1\,\mathrm{N}\,p_2$ and $q_1\,\mathrm{N}\,q_2$, then $f_1(t)\,\mathrm{N}\,f_2(t)$, for each $0 \le t \le 1$.*

(b) *If $p_1\,\mathrm{N}\,p_2$ and $q_1\,\mathrm{NE}\,q_2$, then $f_1(t)\,\mathrm{NE}\,f_2(t)$ for each $0 < t \le 1$ (analogously for* NW *instead of* NE*).*

(c) *If $p_1\,\mathrm{N}\,p_2$ and $q_1\,\mathrm{E}\,q_2$, then $f_1(t)\,\mathrm{NE}\,f_2(t)$ for each $0 < t < 1$ (analogously for* W *instead of* E*).*

(d) *Analogous claims hold for the other $\mathcal{CD}$ base relations.* □

Lemma 4 *Let $f,g : I \to \mathbb{R}^2$ be continuous functions, where $I := [t_0,t_1]$ is a fixed closed interval in $\mathbb{R}$.*

(a) *If f and g do not collide in I and if the set $\{ t \in I : f(t)\,\mathrm{N}\,g(t)\}$ is not empty, then it has a maximum (minimum). Analogous claims hold for the relations* E, S, *and* W.

(b) *The set $\{ t \in I : f(t)\,\mathrm{NE}\,g(t)\}$ has no maximum (minimum) distinct from t_1 (t_0), even if it is not empty. Analogous claims hold for the relations* SE, SW, *and* NW.

(c) *The set $\{ t \in I : f(t)\,\mathrm{Eq}\,g(t)\}$ has a maximum if it is not empty.*

Proof. We only sketch the proof for claim (a) ((b) and (c) can be proven in a similar manner). Consider the set $\hat{N} := \{ t \in I : f(t)\,\mathrm{N}\,g(t)\}$. By assumption this set is upper-bounded and not empty, hence $\sup \hat{N}$ exists. We show that $t^* := \sup \hat{N}$ is in $\hat{N}$. For reductio ad absurdum suppose that $t^* \notin \hat{N}$. Since f and g are continuous, so is the function $f - g$, $t \mapsto f(t) - g(t)$, as well as its projections $f^1 - g^1$ and $f^2 - g^2$. Now if $f^1(t^*) = g^1(t^*)$, it follows $f^2(t^*) < g^2(t^*)$ (since $t^* \notin \hat{N}$ and f and g do not collide) and thus $(f^2 - g^2)(t^*) < 0$. On the other hand, for each $t \in \hat{N}$, $(f^2 - g^2)(t) > 0$. Since $(f^2 - g^2)$ is continuous, there must be a t' with $\hat{N} \le t' < t^*$ such that $(f^2 - g^2)(t') = 0$—in contradiction to the assumption that t^* is the supremum of $\hat{N}$. In the cases that $f^1(t^*) < g^1(t^*)$ or that $f^1(t^*) > g^1(t^*)$, consider the continuous function $f^1 - g^1$ in order to obtain a contradiction. □

By applying this lemma we can immediately prove the following propositions:

Corollary 5 *Let $I = (t_1,t_2)$ be an open interval. Let $f,g : (t_1,t_2] \to \mathbb{R}^2$ be continuous functions that do not collide in $(t_1,t_2]$. Let s and s' be $\mathcal{CD}$ base relations such that $f(t)\,s\,g(t)$, for each $t \in I$, and $f(t_2)\,s'\,g(t_2)$.*

(a) *If $s = \mathrm{N}$, then $s' \in \{\mathrm{N}\}$. Conversely, if $s' = \mathrm{N}$, then $s \in \{\mathrm{NW},\mathrm{N},\mathrm{NE}\}$.*

(b) *If $s = \mathrm{NE}$, then $s' \in \{\mathrm{N},\mathrm{NE},\mathrm{E}\}$. Conversely, if $s' = \mathrm{NE}$, then $s \in \{\mathrm{NE}\}$.*

(c) *If $s = \mathrm{E}$, then $s' \in \{\mathrm{E}\}$. Conversely, if $s' = \mathrm{E}$, then $s \in \{\mathrm{NE},\mathrm{E},\mathrm{SE}\}$.*

(d) *If $s = \mathrm{SE}$, then $s' \in \{\mathrm{E},\mathrm{SE},\mathrm{S}\}$. Conversely, if $s' = \mathrm{SE}$, then $s \in \{\mathrm{SE}\}$.*

(e) *Analogous claims hold for the other $\mathcal{CD}$ base relations.*

Proof. (a) Choose an instant t with $t_1 < t < t_2$ and consider the closed interval $[t,t_2]$. Then the set $\hat{N}$ defined in the proof of Lemma 4 (a) has a maximum. This maximum cannot be contained in the open interval (t,t_2). Thus $\max \hat{N}$ must be t_2, and hence $s' = \mathrm{N}$. The second claim follows from general continuity considerations.

(b) The crucial claim is the second one: Suppose that $s' = \mathrm{NE}$. Fix an arbitrary t with $t_1 < t < t_2$ and consider the interval $[t,t_2]$. Then the set $\widehat{\mathrm{NE}}$ does not take a minimum distinct from t. If $s \neq \mathrm{NE}$, t_2 would be such a minimum.

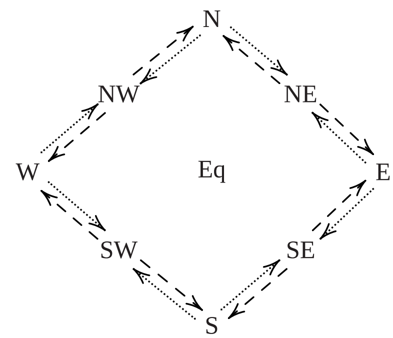


Figure 3: The neighborhood graph of the cardinal direction calculus in the case that the "same-place, same thing"-principle is enforced. Dotted arrows represent continuous transitions from a point scenario into an interval scenario. Dashed lines represent transitions from intervals to points.

The other claims can be proven in a similar manner. □

It is clear that claims analogous to those in the corollary hold for transitions from points to intervals (i. e., if we consider possible qualitative transitions from t_1 into the interval (t_1,t_2)). Furthermore, if we iterate the possible transitions described in this corollary, we obtain the following claim:

Corollary 6 *Let $I = (t_1,t_2)$ and $I' = (t_2,t_3)$ be open intervals. Let $f,g : (t_1,t_3) \to \mathbb{R}^2$ be continuous functions that do not meet in (t_1,t_3). Let s and s' be $\mathcal{CD}$ base relations such that $f(t)\,s\,g(t)$, for each $t \in I$, and $f(t)\,s'\,g(t)$, for each $t \in I'$.*

(a) *If $s = \mathrm{N}$, then $s' \in \{\mathrm{NW},\mathrm{N},\mathrm{NE}\}$.*

(b) *If $s = \mathrm{E}$, then $s' \in \{\mathrm{NE},\mathrm{E},\mathrm{SE}\}$.*

(c) *If $s = \mathrm{S}$, then $s' \in \{\mathrm{SE},\mathrm{S},\mathrm{SW}\}$.*

(d) *If $s = \mathrm{W}$, then $s' \in \{\mathrm{SW},\mathrm{W},\mathrm{NW}\}$.*

(e) *If $s = \mathrm{NE}$, then $s' \in \{\mathrm{NW},\mathrm{N},\mathrm{NE},\mathrm{E},\mathrm{SE}\}$.*

(f) *If $s = \mathrm{SE}$, then $s' \in \{\mathrm{NE},\mathrm{E},\mathrm{SE},\mathrm{S},\mathrm{SW}\}$.*

(g) *If $s = \mathrm{SW}$, then $s' \in \{\mathrm{SE},\mathrm{S},\mathrm{SW},\mathrm{W},\mathrm{NW}\}$.*

(h) *If $s = \mathrm{NW}$, then $s' \in \{\mathrm{SW},\mathrm{W},\mathrm{NW},\mathrm{N},\mathrm{NE}\}$.* □

The neighborhood graph of the cardinal direction calculus (depicted in Fig. 3) is essential for solving $\mathcal{TCD}$ constraint networks. In fact, we will try to construct a solution of such a network by defining a transition function between spatial scenarios that obeys the constraints expressed in the neighborhood graph. The correctness of this graph, which is crucial for this method, follows from Cor. 5.

Definition 7 Let C be a $\mathcal{TCD}$ constraint network.

- C is said to be *reduced* if for each pair of interval variables I and J, there is an Allen base relation r such that $I\,r\,J \in C$.
- C is said to *strongly reduced* if C is reduced and contains exactly one constraint of the form $I : x\,S\,y$ for each interval variable I and each pair of point variables x and y.

Lemma 8 *Each reduced $\mathcal{TCD}$ constraint network C can be transformed in polynomial time into a logically equivalent* *constraint network C' such that C' is strongly reduced and contains only the Allen relations* m *and* $\prec$.

Proof. Adapt a proof in (Gerevini & Nebel 2002). ☐

In the following, let V be a finite subset of $V_{\mathcal{CD}}$. A *scenario* for V is a $\mathcal{CD}$ constraint network C such that for each pair of variables $x, y \in V$, there is a base relation s with $x\,s\,y \in C$.

Definition 9 Let $\mathcal{I} = (I_i)_{1 \le i \le n}$ be a sequence of interval variables. A *chronicle* for V w. r. t. $\mathcal{I}$ is a sequence of $\mathcal{CD}$ constraint networks $(C_j)_{0 \le j \le 2n}$ such that

- (i) Each C_j is a consistent scenario for V (note that w. r. t. $\mathcal{CD}$ scenarios path-consistency decides satisfiability).
- (ii) For each $1 \le i \le n$ and each pair of variables $x, y \in V$, if $x\,r\,y \in C_{2i-2}$ and $x\,r'\,y \in C_{2i-1}$, then $r = r'$ or r' is a relation that is a dotted arrow neighbor of r in the neighborhood graph depicted in Fig. 3 (C_{2j} represents a "point scenario").
- (iii) For each $1 \le i \le n$ and each pair of variables $x, y \in V$, if $x\,r\,y \in C_{2i-1}$ and $x\,r'\,y \in C_{2i}$, then $r = r'$ or r' is a relation that is a dashed arrow neighbor of r in the neighborhood graph depicted in Fig. 3 (in this case C_{2i-1} represents an "interval scenario").

Proposition 10 *Let $V = \{v_1, \ldots, v_m\}$ be a set of $\mathcal{CD}$ variables, $\mathcal{I} = (I_i)_{1 \le i \le n}$ be a sequence of interval variables, and let $(C_j)_{0 \le j \le 2n}$ be a chronicle for V and $\mathcal{I}$. Then there exists a collision-free and continuous $\mathcal{TCD}$ interpretation $\langle \tau, \gamma \rangle$ such that*

(a) $\langle \tau, \gamma \rangle \models I_i \,\mathrm{m}\, I_{i+1}$*, for each* $1 \le i \le n-1$.
(b) $\langle \tau, \gamma \rangle \models I_i : C_{2i-1}$*, for each* $1 \le i \le n$.

Proof. The idea of the proof is as follows: Since each C_j is satisfiable, choose for each $0 \le j \le 2n$ a $\mathcal{CD}$ assignment γ_j satisfying C_j. Then construct for each variable step-by-step a continuous path by a piece-wise linear function, where each piece describes the transition from the scenario C_j to the scenario C_{j+1}. More precisely, we first set $\tau(i) := (2i-2, 2i)$. Then define

$$\hat{\gamma}_x(t) := \begin{cases} \gamma_0(x) & \text{if } t < 0 \\ \gamma_{2n}(x) & \text{if } t > 2n \\ \gamma_{2i-2}(x) + (t - 2i - 2) \cdot \\ \quad (\gamma_{2i-1}(x) - \gamma_{2i-2}(x)) & \text{if } 2i-2 \le t \le 2i-1 \\ \gamma_{2i-1}(x) + (t - 2i - 1) \cdot \\ \quad (\gamma_{2i}(x) - \gamma_{2i-1}(x)) & \text{if } 2i-1 \le t \le 2i \end{cases}$$

It is clear that these settings define a continuous interpretation. By applying Lemma 3 it is easy to verify that this interpretation is collision-free as well and that claim (b) is satisfied. ☐

TCD Constraint Satisfaction Problems

The general constraint satisfaction problem TCDSAT is defined as the problem to decide whether a $\mathcal{TCD}$ constraint network is satisfiable. However, usually we are not only interested in finding an interpretation that models the constraint network, but in the probably harder problem of finding a continuous interpretation of the network. For application scenarios, interpretations that avoid collisions of objects may be even more interesting. Hence in this paper we are interested in the following version of the general satisfiability problem (referred to as TCDSAT-CCF):

Instance: A $\mathcal{TCD}$ constraint network C.

Question: Is there a *continuous* and *collision-free* interpretation that models C?

It is an easy exercise to verify that TCDSAT and hence TCDSAT-CCF as well are NP-hard:

Theorem 11 *TCDSAT and TCDSAT-CCF are NP-hard.*

Proof. We can reformulate a proof given by Gerevini & Nebel (2002). For this we "model" the properties of the $\mathcal{RCC}8$ relations DC and EC used there by the $\mathcal{CD}$ relations E and NE. Hence consider the $\mathcal{TCD}$ CSP $\{I : x\,\mathrm{E}\,y, J : x\,\mathrm{NE}\,y\}$. This implies that $I\,\{<, \mathrm{m}, \mathrm{mi}, >\}\,J$. But the smallest set of interval relations that contains this relation as well as each of the base relations of $\mathcal{IA}$ and that is closed under intersections, converse formations, and compositions is the set of all interval relations (Nebel & Bürckert 1995). The NP-hardness of the satisfiability problem for $\mathcal{IA}$ thus implies that TCDSAT is NP-hard as well. ☐

We now know the lower bound complexity, but the interesting part is to determine upper bounds. At least for very simple $\mathcal{TCD}$ constraint networks, namely such networks in which only base relations occur, we can decide satisfiability as follows: First transform such a network into a strongly reduced one. In a second step consider the sub-networks defined by maximal m-chains of interval variables $(I_1\,\mathrm{m}\ldots\mathrm{m}\,I_n)$ in the strongly reduced network. If each such sub-network can be extended to a chronicle, then each sub-network is satisfiable (cf. Proposition 10) and hence the original network is satisfiable as well.

In the general case, however, we only have m-chains of intervals that annotate $\mathcal{CD}$ formulae containing disjunctions of $\mathcal{CD}$ relations. For this reason it is crucial to find a suitable sequence of $\mathcal{CD}$ scenarios that are consistent with the disjunctive descriptions in the constraint network. This problem can be restated as follows: Assume that we have a $\mathcal{TCD}$ constraint $I : C$, a spatial scenario C_s, which holds at the beginning of I, and a spatial scenario C_f, which holds at the end of I. How can we transform (continuously and in a collision free manner) the start scenario into the final scenario such that the constraints holding in I, i. e., those of C, are never violated? Hence the task is to find a sequence $\mathcal{I} = (I_i)_{1 \le i \le n}$ of interval variables and then a chronicle $(C_j)_{0 \le j \le 2n}$ such that $C_0 = C_\mathrm{s}$, $C_{2n} = C_\mathrm{f}$, and each of the constraints in C_j $(1 \le j \le 2n-1)$ is consistent with C.

To sum up, the problem we will be concerned with in the rest of the paper is defined as the following *transformation problem*:

Instance: An initial $\mathcal{CD}$ scenario C_s, a final $\mathcal{CD}$ scenario C_f, and a $\mathcal{CD}$ constraint network $C_{\mathrm{s/f}}$.

Question: Is there a chronicle from C_s to C_f that does not violate the constraints in $C_{s/f}$?

The idea for analyzing this problem is to consider possible transformations in grid structures, which in turn can be encoded as planning problems. A *grid* is a set of the form $G = \{ (i,j) \in \mathbb{N}^2 : 1 \leq i,j \leq n \}$ for some natural number $n \geq 1$. n is referred to as the *size* of G.

Obviously, the problem of testing satisfiability of $\mathcal{CD}$ constraint networks with respect to the plane is equivalent to the corresponding problem of finding a satisfying assignment in a (sufficiently large) grid. More precisely, if a $\mathcal{CD}$ constraint network with n variables is satisfiable, then it is satisfiable in a grid of size n — this can be proven by induction in the number of variables. For this reason, the transformation problem presented above can be equivalently restated as a transformation problem of *grid scenes*, that is, instantiations of scenarios in a grid. A solution of such a transformation problem in grids is a *grid chronicle*, i. e., a sequence of grid scenes $(\sigma_i)_{0 \leq j \leq 2n}$ such that each transition from σ_i to σ_{i+1} obeys the continuity constraints expressed in the neighborhood graph (cf. Fig. 3).

For a given assignment to the variables occurring in a given $\mathcal{CD}$ constraint network, one can compute in polynomial time (in the number of the variables) whether the assignment models the constraint network. Moreover, an easy combinatorial argument shows that only a finite number of transformations between two scenarios is possible:

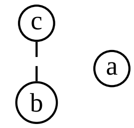
Lemma 12 *(a) There are only* 8^n *distinct* $\mathcal{CD}$ *scenarios in* n *variables.*

(b) Let C *and* C' *be* $\mathcal{CD}$ *scenarios for the same set of variables. If there is a chronicle transforming* C *into* C'*, then this chronicle passes through at most* 8^n *distinct scenarios.* □

This gives us only an imprecise upper bound of the complexity, i. e., so far the problem is only in EXPTIME. But we aim at showing that the upper bound complexity of the transformation problem is in PSPACE.

When does there exist a solution for an instance of the transformation problem $\langle C_s, C_f, C_{s/f} \rangle$? To answer this question, let us consider a typical example (presented in Fig. 4), in which such a problem has no solution. We say, that an object is a *hiker* if it changes its qualitative position w. r. t. another object during its movement. A *guard* for a hiker y is an object x such that there is a relation R distinct from the universal relation with $x\,R\,y \in C_{s/f}$. A *gate* for a hiker x is a pair of guards (y,z), which x cannot pass according to the constraints in $C_{s/f}$ (cf. Fig. 4).

Hence, a $\mathcal{TCD}$ constraint network is inconsistent, whenever its corresponding transformation problem contains (at least) one impassable gate. This result can be used to check whether some $\mathcal{TCD}$ problem instances are unsatisfiable, but the existence of impassable gates is not a sufficient criterion, because there can exist cycles in the solution path. For instance, suppose that a hiker a cannot pass the gate (g_1, g_2), before g_1 passes a gate (g'_1, g'_2) and so on. For this reason,



$$C_s = \{c\,\mathrm{N}\,b, a\,\mathrm{NE}\,b, a\,\mathrm{SE}\,c\}$$
$$C_f = \{c\,\mathrm{N}\,b, a\,\mathrm{SW}\,c, a\,\mathrm{NW}\,b\}$$
$$C_{s/f} = \{a\,\{\mathrm{SE},\mathrm{E},\mathrm{NE},\mathrm{N},\mathrm{NW},\mathrm{W},\mathrm{SW}\}\,c,\; a\,\{\mathrm{NE},\mathrm{E},\mathrm{SE},\mathrm{S},\mathrm{SW},\mathrm{W},\mathrm{NW}\}\,b,\; b\,\{\mathrm{N},\mathrm{NW},\mathrm{W},\mathrm{SW},\mathrm{S}\}\,c\}.$$

Figure 4: A gate. Object a cannot pass the guards b and c, as stated in the constraint network $C_{s/f}$. It has to move south of b and north of c. By changing the positions of b and c (sometimes this is not possible) a can pass first c and then b.

we can not prima facie exclude the possibility that a transition chain from some initial to an end scenario consists of 8^n different scenarios.

A Planning Encoding

In the following we present a method that can be used to encode transformation problems (described in the previous section) as (non-standard) deterministic planning problems. A side-effect of this encoding will be that the upper bound of the complexity of TCDSAT-CCF is PSPACE.[1]

First recall the basic notion of a deterministic planning problem:

Definition 13 An instance of a *deterministic planning problem* is a 4-tuple $\langle P, I, O, G \rangle$ consisting of a set of state variables P, an (initial) state I, a set O of operators over P, and a propositional formula G over P (describing the set of *goal states*).

Let $\langle C_s, C_f, C_{s/f} \rangle$ be a transformation problem for n objects. Consider a fixed grid of size n. Our planning encoding uses n^3 Boolean (position) variables $p_{x,i,j}$, which are true if and only if object x is at position (i,j) in the given grid. In Cor. 5, we saw that it is necessary to make it explicit whether a state represents a point scenario or an interval scenario. For this reason, we introduce a further state variable *int* which is true in states associated to interval scenarios, and else false.

The initial state I is an assignment of these position variables that is consistent with the scenario C_s. The goal state is a propositional formula expressing the final scenario C_f. For example, $x\,\mathrm{N}\,y$ can be encoded as the formula $\bigvee_{1 \leq i,j \leq n} (p_{x,i,j} \wedge \bigvee_{k>j} p_{y,i,k})$. To define the operators as one-step movements in the grid, we first define a "neighborhood

[1] A straight-forward encoding, namely by using $\mathcal{CD}$ constraints as state variables, leads to a non-deterministic planning problem, which in general is EXPTIME-hard.

relation" N:

$$N(p_{x,i,j}, p_{x,k,l}) = \begin{cases} \top & \text{if } k \in \{i-1, i, i+1\}, \\ & l \in \{j-1, j, j+1\}, \\ & \text{and } (k,l) \neq (i,j) \\ \bot & \text{else.} \end{cases}$$

Then we define an operator, $Move_O(X,(i,j),(k,l))$ that moves an object x from one position to a neighboring position:

$$\begin{array}{ll} \textit{Prec:} & p_{x,i,j} \wedge N(p_{x,i,j}, p_{x,k,l}) \\ \textit{Effec:} & \neg p_{x,i,j} \wedge p_{x,k,l} \end{array}$$

Following, we describe under which condition a transition from one state (of a plan) to another state is possible. For this let $\langle c_1, e_1\rangle \ldots \langle c_l, e_l\rangle$ be a set of operators over P. Let s be an arbitrary state. An operator $\langle c, e\rangle$ is *applicable* in s if $s \models c$. A (possibly empty) set O' of operators $o_1, \ldots, o_n$ is *applicable* in s if

(a) O' contains at most one operator for each object x;

(b) each operator in O' is applicable in s;

(c) after the application of the operators, no two objects share the same position;

(d) the resulting scenario is consistent with the conditions of Cor. 5 (depending on whether the variable *int* is true or not).

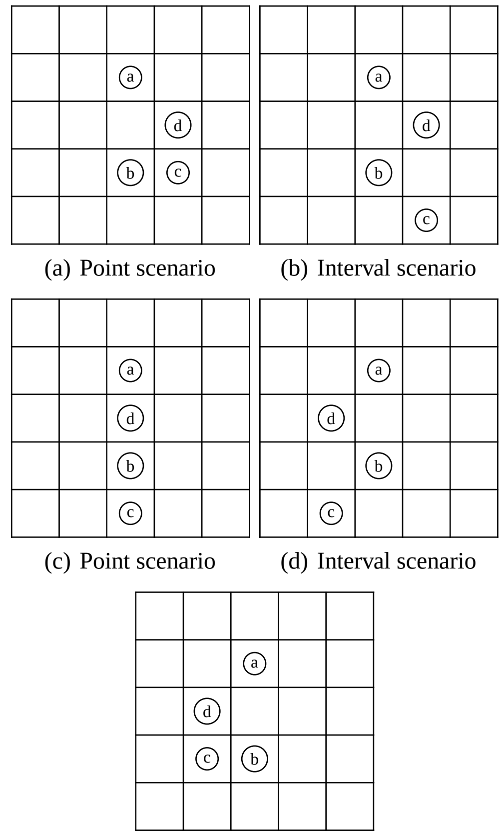
After any application of a set of operators, the value interval variable *int* is changed. A *successor state* s' of s is a state obtained by applying an (applicable) set of operators (*multi-operator application*). A *(multi-operator) plan* is a sequence of multi-operator applications on the initial state to a state satisfying the goal condition. Finally, a plan *respects* a $\mathcal{CD}$ constraint network C if each of the states in the plan (distinct from the initial and goal state) satisfies the formula obtained by the encoding of C.

This definition of a plan extends the classical definition of a plan as used in AI planning domain (Rintanen 2005). Obviously, a multi-operator plan exists if a plan in the usual sense exists, but not vice versa.

Let us round out this section by discussion some examples, which illustrate the concepts introduced here. First we present a transformation problem that is only solvable by a multi-operator plan, but not by a classical plan.

$$\begin{aligned} C_{\mathrm{s}} &= \{d\,\mathrm{SE}\,a, d\,\mathrm{NE}\,b, d\,\mathrm{N}\,c, c\,\mathrm{E}\,b, a\,\mathrm{N}\,b, a\,\mathrm{NW}\,c\} \\ C_{\mathrm{f}} &= \{d\,\mathrm{SW}\,a, d\,\mathrm{NW}\,b, d\,\mathrm{N}\,c, c\,\mathrm{W}\,b, a\,\mathrm{N}\,b, a\,\mathrm{NE}\,c\} \\ C_{\mathrm{s/f}} &= \{d\ \{\mathrm{SE},\mathrm{S},\mathrm{SW}\}\ a,\ b\ \{\mathrm{NE},\mathrm{N},\mathrm{NW}\}\ a,\ a\,\mathrm{N}\,b, \\ &\qquad d\,\mathrm{N}\,c,\ c\ \{\mathrm{E},\mathrm{SE},\mathrm{S},\mathrm{SW},\mathrm{W}\}\ b\} \end{aligned}$$

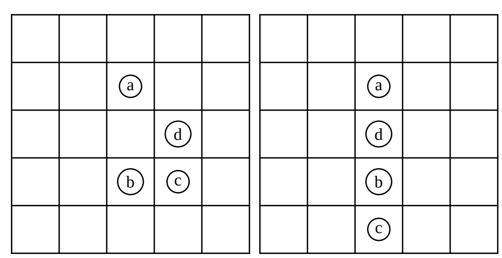
The multi-operator plan solving this CSP can be outlined as follows (cp. Fig. 5): Object d, which always has to stay north of object c, has to transform its position from $d\,\mathrm{SE}\,a$ to $d\,\mathrm{SW}\,a$, while respecting the transformation constraint $d\ \{\mathrm{SE},\mathrm{S},\mathrm{SW}\}\ a$. Since c cannot pass north of object b it has to move south of b. So object c changes its position from $b\,\mathrm{E}\,c$ to $b\,\mathrm{SE}\,c$. Then a multi-operator application is necessary (since otherwise the constraint $c\,\mathrm{N}\,b$ is violated), that is, we move concurrently both objects c and d to the left. In the next step, this simultaneous movement is repeated. Finally object c moves one step north, and we have reached the goal position.



(a) Point scenario (b) Interval scenario (c) Point scenario (d) Interval scenario (e) Point scenario

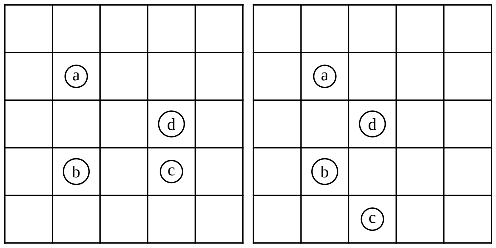
Figure 5: A multi-operator plan defining a chronicle

In the situation of the example, an immediate transition from the first to the third scenario (cf. Fig. 6) would not be possible, since this transition contradicts the continuity constraints expressed in the neighborhood graph (cf. Fig. 3): the relation $b\,\mathrm{W}\,c$ changes (in one step) into $b\,\mathrm{N}\,c$. However, the same movements of the objects c and d, respectively, are possible if the initial scenario is slightly changed (cf. Fig. 7)



(a) Point scenario (b) Interval scenario

Figure 6: An example of an impossible transition if continuity constraints are enforced



(a) Point scenario (b) Interval scenario

Figure 7: A possible transition if the initial scenario in Fig. 6 is slightly changed

Using these definitions, one can prove the following claim:

Lemma 14 *An instance of the transformation problem* $\langle C_s, C_f, C_{s/f} \rangle$ *has a solution if and only if there exists a multi-operator plan from the initial scenario* C_s *to the goal scenario* C_f *that respects* $C_{s/f}$. □

To sum up, we may outline the idea how the satisfiability problem of $\mathcal{TCD}$ can be solved by an encoding as a deterministic planning problem:

1. Each instance of the TCDSAT-CCF problem can be transformed into a satisfiability equivalent instance of the transformation problem.
2. Each instance of the transformation problem can be equivalently restated as a transformation problem of grid scenes.
3. Each transformation problem in a grid is satisfiable iff there exists a deterministic plan.

Since the encoding of transformation problems presented here only uses deterministic effects and since the number of possible multi-operator applications is limited, it can be shown that we have a deterministic planning problem. From this it follows (Bylander 1994) that the satisfiability problem TCDSAT-CCF is in PSPACE.

Summary and Outlook

In this paper we have outlined semantical concepts of the temporalized spatial constraint formalism $\mathcal{TCD}$. We showed that simple instances of constraint networks can be solved by constructing chronicles of spatial scenarios, which reflect semantically well-defined continuous movements of objects. For harder instances, we proposed to encode constraint networks as deterministic planning problems. A side-effect of this encoding is that the satisfiability problem for $\mathcal{TCD}$ constraint networks can be shown to be in PSPACE. But the more interesting point is that we provide a constructive method that connects constraint satisfaction problems and planning problems with multi-operator applications.

Acknowledgments

This work was partially supported by the Deutsche Forschungsgemeinschaft (DFG) as part of the Transregional Collaborative Research Center SFB/TR 8 Spatial Cognition. We would like to thank Bernhard Nebel for helpful discussions. We also gratefully acknowledge the suggestions of three anonymous reviewers, who helped improving the paper.

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