Improvement Operators

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Abstract

We introduce a new class of change operators. They are a generalization of usual iterated belief revision operators. The idea is to relax the success property, so the new information is not necessarily believed after the improvement. But its plausibility has increased in the epistemic state. So, iterating the process sufficiently many times, the new information will be finally believed. We give syntactical and semantical characterizations of these operators.

Introduction

Modelling belief change is a central topic in artificial intelligence, psychology and databases. One of the predominant approaches was proposed by Alchourrón, Gärdenfors and Makinson and is known as the AGM belief revision framework (Alchourrón, Gärdenfors, & Makinson 1985; Gärdenfors 1988; Katsuno & Mendelzon 1991). The main requirements imposed by AGM postulates are the principle of *coherence* asking to maintain consistency as far as possible, the so called principle of *minimal change* saying that we have to keep as much of the old information as possible, and the last important requirement is the principle of *primacy of update* (also called success property) that demands the new information to be true in the new belief base. The postulates proposed to characterize belief revision operators (Alchourrón, Gärdenfors, & Makinson 1985; Gärdenfors 1988; Katsuno & Mendelzon 1991; Hansson 1999) just aimed at capturing logically these principles.

A drawback of AGM definition of revision is that the conditions for the iteration of the process are very weak, and this is caused by the lack of expressive power of logical belief bases (Herzig, Konieczny, & Perussel 2003). In order to ensure good properties for the iteration of the revision process, one needs a more complex structure. So shifting from logical belief bases to epistemic states was proposed in (Darwiche & Pearl 1997). In this framework, one can define interesting iterated revision operators (Darwiche & Pearl 1997; Booth & Meyer 2006; Jin & Thielscher 2007; Konieczny & Pino Pérez 2000). Let us call these operators DP belief revision operators.

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Another framework that allows to define interesting iterated change operator is the one of Ordinal Conditional Functions (OCF), also named kappa-rankings, that was proposed by Spohn in (Spohn 1988), and further developed in (Williams 1994). An OCF can be represented by a function that associates an ordinal to every interpretation, with at least one interpretation taking the value 0. The ordinal associated to the interpretation represents the degree of disbelief of the interpretation. This notion can be used to define a *degree of acceptance* of a formula.

So a change operator in this framework, called a transmutation (Williams 1994), is a function that changes the degree of acceptance of a formula. This means that it requires more information than AGM/DP belief revision operators, since, in addition to the new information, one needs to give its new degree of acceptance. This has one important drawback, since one has to find this new degree somewhere! It it is not a problem if it is given by the application, but if the only received input is the new information, justifying an "arbitrary" degree of acceptance can be problematic. On the other hand, this more general framework allows to define interesting operators. It allows to define revision and contraction operators, by choosing the right degree of acceptance. In particular most of the works on DP iterared revision operators use OCF operators as examples (see e.g. (Darwiche & Pearl 1997; Jin & Thielscher 2007)). But it also allows to define restructuring operators (Williams 1994; Spohn 1988), that modify the OCF, without changing the believed formulae. Such operators do not exist in the classical DP belief revision framework, that obey the success property, that asks the new information to be believed after the change.

The aim of this paper is to define such restructuring-like operators in the standard AGM/DP framework. We want to define change operators on epistemic states that do not (necessarily) satisfy the success property, although still improving the plausibility of the new information. We call these operators improvement operators. This idea is quite intuitive since usual AGM/DP belief revision operators can be considered as too strong: after revising by a new information, this information will be believed. Most of the time this is the wanted behaviour for the revision operators. But in some cases it may be sensible to take into account the new information more cautiously. Maybe because we have some confidence in the source of the new information, but not enough for accepting unconditionally this new information. This can be seen as a kind of learning/reinforcement process: each time the agent receives a new information α, this formula will gain in plausibility in the epistemic state of the agent. And if the agent receives the same new information many times, then he will finally believe it.

Our operators are close in spirit to the bad day/good day approach of Booth and Meyer (also called abstract interval orders revision) (Booth & Meyer 2007; Booth, Meyer, & Wong 2006). Unlike their operators that need an extra information, our operators are defined in the usual DP framework. We give more details on this relationship at the end of the paper.

The rest of the paper is organized as follows: we give the preliminaries in the first section. The second section is devoted to the introduction of improvement operators. The third section is devoted to a discussion of the irrelevance of syntax property. In the fourth and fifth sections we state the main results concerning syntactical and semantical characterizations, namely two representation theorems. The fifth section shows an example and how to encode improvement using OCF. The sixth section contains some interesting properties of improvement operators. The last section is the conclusion. There is also an appendix containing the proofs of the main results.

Preliminaries

We consider a propositional language $\mathcal{L}$ defined from a finite set of propositional variables $\mathcal{P}$ and the standard connectives. Let $\mathcal{L}^*$ denote the set of consistent formulae of $\mathcal{L}$.

An interpretation ω is a total function from $\mathcal{P}$ to $\{0, 1\}$. The set of all interpretations is denoted $\mathcal{W}$. An interpretation ω is a model of a formula $\phi \in \mathcal{L}$ if and only if it makes it true in the usual truth functional way. $[[\alpha]]$ denotes the set of models of the formula α, i.e., $[[\alpha]] = \{\omega \in \mathcal{W} \mid \omega \models \alpha\}$. When $\{w_1, .., w_n\}$ is a set of models we denote by $\varphi_{w_1,..,w_n}$ a formula such that $[[\varphi_{w_1,..,w_n}]] = \{w_1, .., w_n\}$.

We will use epistemic states to represent the beliefs of the agent, as usual in iterated belief revision (Darwiche & Pearl 1997). An epistemic state Ψ represents the current beliefs of the agent, but also additional conditional information guiding the revision process (usually represented by a pre-order on interpretations, a set of conditionals, a sequence of formulae, etc). Let $\mathcal{E}$ denote the set of all epistemic states. A projection function $B : \mathcal{E} \longrightarrow \mathcal{L}^*$ associates to each epistemic state Ψ a consistent formula $B(\Psi)$, that represents the current beliefs of the agent in the epistemic state Ψ.

For simplicity purpose we will only consider in this paper consistent epistemic states and consistent new information. Thus, we consider change operators as functions $\circ$ mapping an epistemic state and a consistent formula into a new epistemic state, *i.e.* in symbols, $\circ : \mathcal{E} \times \mathcal{L}^* \longrightarrow \mathcal{E}$. The image of a pair (Ψ, α) under $\circ$ will be denoted by $\Psi \circ \alpha$.

We adopt the following notations:

- $\Psi \circ^n \alpha$ defined as: $\Psi \circ^1 \alpha = \Psi \circ \alpha$, $\Psi \circ^{n+1} \alpha = (\Psi \circ^n \alpha) \circ \alpha$
- $\Psi \star \alpha = \Psi \circ^n \alpha$, where n is the first integer such that $B(\Psi \circ^n \alpha) \vdash \alpha$.

Note that $\star$ is undefined if there is no n such that $B(\Psi \circ^n \alpha) \vdash \alpha$, but for all operators $\circ$ considered in this work, the associated operator $\star$ will be total, that is for any pair Ψ, α there will exist n such that $B(\Psi \circ^n \alpha) \vdash \alpha$ (see postulate **(I1)**) below.

Finally, let $\leq$ be a a total pre-order, i.e a reflexive ($x \leq x$), transitive ($(x \leq y \wedge y \leq z) \rightarrow x \leq z$) and total ($x \leq y \vee y \leq x$) relation over $\mathcal{W}$. Then the corresponding strict relation $<$ is defined as $x < y$ iff $x \leq y$ and $y \not\leq x$, and the corresponding equivalence relation $\simeq$ is defined as $x \simeq y$ iff $x \leq y$ and $y \leq x$. We denote $w \ll w'$ when $w < w'$ and there is no w'' such that $w < w'' < w'$. We also use the notation $min(A, \leq) = \{w \in A \mid \nexists w' \in A\, w' < w\}$.

When a set $\mathcal{W}$ is equipped with a total pre-order $\leq$, then this set can be splitted in different levels, that gives the ordered sequence of its equivalence classes $\mathcal{W} = \langle S_0, \ldots S_n \rangle$. So $\forall x, y \in S_i\ x \simeq y$. We say in that case that x and y are at the same level of the pre-order. And $\forall x \in S_i\ \forall y \in S_j$ $i < j$ implies $x < y$. We say in this case that x is in a lower level than y. We extend straightforwardly these definitions to compare subsets of equivalence classes, i.e if $A \subseteq S_i$ and $B \subseteq S_j$ then we say that A is in a lower level than B if $i < j$.

Improvement operators

First, let us state the basic logical properties that are asked for improvement operators.

Definition 1 *An operator $\circ$ is said to be a weak improvement operator if it satisfies (I1) to (I6):*
(I1) *There exists n such that $B(\Psi \circ^n \alpha) \vdash \alpha$*
(I2) *If $B(\Psi) \wedge \alpha \nvdash \bot$, then $B(\Psi \star \alpha) \equiv B(\Psi) \wedge \alpha$*
(I3) *If $\alpha \nvdash \bot$, then $B(\Psi \circ \alpha) \nvdash \bot$*
(I4) *For any positive integer n if $\alpha_i \equiv \beta_i$ for all $i \leq n$ then $B(\Psi \circ \alpha_1 \circ \cdots \circ \alpha_n) \equiv B(\Psi \circ \beta_1 \circ \cdots \circ \beta_n)$*
(I5) $B(\Psi \star \alpha) \wedge \beta \vdash B(\Psi \star (\alpha \wedge \beta))$
(I6) *If $B(\Psi \star \alpha) \wedge \beta \nvdash \bot$, then $B(\Psi \star (\alpha \wedge \beta)) \vdash B(\Psi \star \alpha) \wedge \beta$*

We have put together these properties because they allow to obtain a first basic representation theorem (see Theorem 1). These properties are very close to the usual ones for iterated belief revision (Darwiche & Pearl 1997). Note nevertheless that there is a real difference since in usual formulation $\star$ is a revision operator, whereas here it denotes a sequence of improvements.

Remark that (I3) is a straightforward consequence of the definition of the operator $\circ$, since we ask the new information and the epistemic states to be consistent. Although (I3) is redundant in our framework, we have chosen to put it explicitly to remain close to the usual DP postulates.

The main difference with usual belief revision operators is that we do not ask the fundamental success property $B(\Psi \circ \alpha) \vdash \alpha$. We ask instead the weaker (I1), that just requires that after a sequence of improvements, we will finally imply the new information. So this means that the (revision) operator $\star$ defined as a sequence of improvements $\circ$ is always defined.

Postulate (I4) is also stonger than the usual version of (Darwiche & Pearl 1997). We discuss it in the next Section.

Before establishing more specific postulates concerning the iteration by different formulas, we have to define new notions that help us to keep light notations.

Definition 2 *Let $\circ$ be a change operator satisfying (I1). Let α, β and Ψ be two formulae and an epistemic state respectively. We say that α is below β with respect to Ψ, given $\circ$, denoted $\alpha \prec^{\circ}_{\Psi} \beta$ (or simply $\alpha \prec_{\Psi} \beta$ if there is no ambiguity about $\circ$) if and only if $B(\Psi \star \alpha) \vdash B(\Psi \star (\alpha \vee \beta))$ and $B(\Psi \star \beta) \nvdash B(\Psi \star (\alpha \vee \beta))$.*

The pair (α, β) is Ψ-consecutive, denoted $\alpha \prec\!\!\prec^{\circ}_{\Psi} \beta$ (or simply $\alpha \prec\!\!\prec_{\Psi} \beta$ if there is no ambiguity about $\circ$) if and only if $\alpha \prec_{\Psi} \beta$ and there is no formula γ such that $\alpha \prec_{\Psi} \gamma \prec_{\Psi} \beta$.

The idea of these two definitions is that $\alpha \prec^{\circ}_{\Psi} \beta$ denotes that α is more entrenched (plausible) than β in the epistemic state Ψ. And $\alpha \prec\!\!\prec^{\circ}_{\Psi} \beta$ denotes the fact that α is a formula immediately more entrenched (plausible) than β.

Now we are ready to state the postulates concerning more specific properties of iteration:

Definition 3 *A weak improvement operator is said to be an improvement operator if it satisfies I7 to I11*
(I7) *If $\alpha \vdash \mu$ then $B((\Psi \circ \mu) \star \alpha) \equiv B(\Psi \star \alpha)$*
(I8) *If $\alpha \vdash \neg\mu$ then $B((\Psi \circ \mu) \star \alpha) \equiv B(\Psi \star \alpha)$*
(I9) *If $B(\Psi \star \alpha) \nvdash \neg\mu$ then $B((\Psi \circ \mu) \star \alpha) \vdash \mu$*
(I10) *If $B(\Psi \star \alpha) \vdash \neg\mu$ then $B((\Psi \circ \mu) \star \alpha) \nvdash \mu$*
(I11) *If $B(\Psi \star \alpha) \vdash \neg\mu$, $\alpha \wedge \mu \nvdash \bot$ and $\alpha \prec\!\!\prec_{\Psi} \alpha \wedge \mu$ then $B((\Psi \circ \mu) \star \alpha) \nvdash \neg\mu$*

A first observation about these postulates is that they are expressed in terms of both $\circ$ and $\star$. And that it is thanks to these several iterations until revision[1], modeled by $\star$, that we can define powerful properties on $\circ$. Postulates (I7), (I8) are close to the properties (C1) and (C2) of (Darwiche & Pearl 1997), but translated for weak improvement operators. Postulate (I9) is also close to the property of Independence in (Jin & Thielscher 2007) (called also property (P) in (Booth & Meyer 2006)), but also translated for weak improvement operators. Postulates (I9) and (I11) deals with the improvement of the new information, i.e. the increase of its plausibility in the epistemic state. Postulates (I10) and (I11) deals with the the cautiousness of the approach, i.e. they express the fact that the increase of the plausibility of the new information is limited. Postulate (I10) says that if after a sequence of improvements by α, the obtained epistemic state imply $\neg\mu$, then, if before the sequence of improvements by α we improve by μ, then it will not be enough to imply μ after the sequence of improvements. This means that it is not possible to go directly by an improvement from an epistemic state where a formula is believed to one where its negation is believed. And postulate (I11) captures some of the ideas behind improvement operators as "small change" operators. Basically it says that if $\neg\mu$ is believed when revising by α, but μ is quite plausible given α, then improving by μ before starting the sequence of improvements needed to revise by α will be enough to ensure the result to be consistent with μ.

[1]Note that the $\star$ operator satisfies the success property, so it can be called *revision* operator. We will use this term in the following. The fact that $\star$ is a true AGM/DP revision operator will be proved in Corollary 1.

Irrelevance of syntax

As it has been pointed out by many authors (see for instance (Darwiche & Pearl 1997; Booth & Meyer 2006)) the postulate of independence (or irrelevance) of syntax is a delicate matter for epistemic states. Actually a basic translation of Darwiche and Pearl (R*4) in our framework would lead to:

$$\text{If } \alpha \equiv \beta, \text{then } B(\Psi \circ \alpha) \equiv B(\Psi \circ \beta) \quad (1)$$

But even adding this postulate is not sufficient. Booth and Meyer have well illustrated this idea in (Booth & Meyer 2006). Actually, it is not enough that (1) holds in order to have a good iterative behavior with respect to revision by sequences of equivalent formulae. Consider:

$$\text{If } (\alpha \equiv \beta \ \& \ \gamma \equiv \theta), \text{then } B(\Psi \circ \alpha \circ \gamma) \equiv B(\Psi \circ \beta \circ \theta) \quad (2)$$

Postulate (1) doesn't entail postulate (2). So the good behaviour with respect to equivalent formulae is not guaranteed on two iterations. This is why Booth and Meyer have proposed in (Booth & Meyer 2006) to replace the usual postulate (1) by (2) in the usual DP framework.

We agree with Booth and Meyer that postulate (1) is not enough. But we think that (2) does not go far enough. In fact the example they give for showing that Postulate (1) does not avoid the problem at the second iteration can be easily extended to show that Postulate (2) does not avoid the problem at the third iteration. So one has to specify this for every number of iterations. That leads to the postulate **(I4)**.

The following example, which follows the same lines of the Example 1 in (Booth & Meyer 2006), shows that replacing the postulate (1) by the postulate (2) in the basic RAGM[2] framework is not enough to get the postulate **(I4)**.

Example 1 *Take a language with only two propositional letters p and q (in this order when we consider the interpretations). Let $\varphi_1 = p \vee \neg p$ and $\varphi_2 = q \vee \neg q$. Let Φ such that $\Phi \circ \varphi_1 \neq \Phi \circ \varphi_2$. Note that this is compatible with RAGM plus* (2)*, because the only constraint imposed by* (2) *is that $\leq_{\Phi\circ\varphi_1} = \leq_{\Phi\circ\varphi_2}$ but not that $\Phi \circ \varphi_1 = \Phi \circ \varphi_2$. Moreover we can take $B(\Phi \circ \varphi_1) = p \wedge q = B(\Phi \circ \varphi_2)$. Let Φ_1, Φ_2 be two epistemic states such that $B(\Phi_1) = p = B(\Phi_2)$, $00 <_{\Phi_1} 01$ and $01 <_{\Phi_2} 00$. Now, it is compatible with RAGM plus* (2) *to put $\leq_{\Phi\circ\varphi_1\circ p} = \leq_{\Phi_1}$ and $\leq_{\Phi\circ\varphi_2\circ\neg\neg p} = \leq_{\Phi_2}$. But then, by the representation, we have $B(\Phi \circ \varphi_1 \circ p \circ \neg p) = \neg p \wedge \neg q$ and $B(\Phi \circ \varphi_2 \circ \neg\neg p \circ \neg p) = \neg p \wedge q$, what clearly is a counter-example to* **(I4)**.

Nevertheless, it worths noticing that all the well-known iterated revision operators, as natural revision (Boutilier 1996), Darwiche and Pearl $\bullet$ operator (Darwiche & Pearl 1997), Nayak's lexicographic revision (Nayak 1994;

[2]RAGM is the name that Booth and Meyer give to AGM/DP belief revision operators satisfying (R*1)-(R*6) (Darwiche & Pearl 1997).

Konieczny & Pino Pérez 2000) for instance, satisfy **(I4)**. The reason behind this phenomenon is that all these operators satisfy the following property

$$\text{If } \alpha \equiv \beta \text{ and } \leq_\Psi = \leq_\Phi \text{ then } \leq_{\Psi\circ\alpha} = \leq_{\Phi\circ\beta} \quad (3)$$

In the presence of the other RAGM properties, the previous property entails the property (2). But the converse is not true as we can see via the example 1.

Remark also that usual belief revision operators satisfy all other weak improvement properties (with $n = 1$ in (I1)), so weak improvements operators are a generalization of usual DP belief revision operators (Darwiche & Pearl 1997).

Representation theorem

Let us first define strong faithful assignements.

Definition 4 *A function $\Psi \mapsto \leq_\Psi$ that maps each epistemic state Ψ to a total pre-order on interpretations $\leq_\Psi$ is said to be a* strong faithful assignment *if and only if:*

1. *If $w \models B(\Psi)$ and $w' \models B(\Psi)$, then $w \simeq_\Psi w'$*
2. *If $w \models B(\Psi)$ and $w' \not\models B(\Psi)$, then $w <_\Psi w'$*
3. *For any positive integer n if $\alpha_i \equiv \beta_i$ for any $i \leq n$ then $\leq_{\Psi\circ\alpha_1\circ\cdots\circ\alpha_n} = \leq_{\Psi\circ\beta_1\circ\cdots\circ\beta_n}$*

Note that conditions 1 and 2 are equivalent to $[[B(\Psi)]] = \min(\mathcal{W}, \leq_\Psi)$, and are the usual ones for faithful assignment (Darwiche & Pearl 1997). Condition 3 is a very natural condition that links pre-orders associated to iteration of improvements: two sequences of improvements of the same pre-order by equivalent formulae lead to the same pre-order.

Let us first show a first representation theorem on weak improvement operators, before turning on the more interesting iteration properties.

Theorem 1 *A change operator $\circ$ is a weak improvement operator if and only if there exists a strong faithful assignment that maps each epistemic state Ψ to a total pre-order on interpretations $\leq_\Psi$ such that*

$$[[B(\Psi \star \alpha)]] = \min([[\alpha]], \leq_\Psi) \quad (4)$$

It is easy to check that the faithful assignment representing $\circ$ in the previous theorem is unique.

An obvious corrolary of the previous Theorem and its proof is the following one:

Corollary 1 *If $\circ$ is a weak improvement operator, then $\star$ is an AGM/DP revision operator, i.e. it satisfies (R*1)-(R*6) of (Darwiche & Pearl 1997).*

As a consequence of the previous theorem we have also the following trichotomy property:

Proposition 1 *Let $\circ$ be a weak improvement operator. Then*

$$B(\Psi \star (\alpha \vee \beta)) = \begin{cases} B(\Psi \star \alpha) \text{ or} \\ B(\Psi \star \beta) \text{ or} \\ B(\Psi \star \alpha) \vee B(\Psi \star \beta) \end{cases}$$

Let us now give two corollaries of these results, that are useful to understand the definitions of $\prec_\Psi$ and $\prec\!\!\prec_\Psi$, and that will be useful in the proof of the main Theorem (Theorem 2).

Corollary 2 *Let $\circ$ be a weak improvement operator. Then $\alpha \prec_\Psi \beta$ if and only if there exist w, w' such that $w \in [[B(\Psi \star \alpha)]]$, $w' \in [[B(\Psi \star \beta)]]$, $w <_\Psi w'$.*

Corollary 3 *Let $\circ$ be a weak improvement operator. Then $\alpha \prec\!\!\prec_\Psi \beta$ if and only if there exist w, w' such that $w \in [[B(\Psi \star \alpha)]]$, $w' \in [[B(\Psi \star \beta)]]$, $w <_\Psi w'$ and there is no w'' such that $w <_\Psi w'' <_\Psi w'$.*

Main result

Let us turn now to the main representation result about improvement operators.

Definition 5 *Let $\circ$ be a weak improvement operator and $\Psi \mapsto \leq_\Psi$ its corresponding strong faithful assignment. The assignment will be called a gradual assignment if the properties S1, S2, S3, S4 and S5 are satisfied*

(S1) *If $w, w' \in [[\alpha]]$ then $w \leq_\Psi w' \Leftrightarrow w \leq_{\Psi\circ\alpha} w'$*
(S2) *If $w, w' \in [[\neg\alpha]]$ then $w \leq_\Psi w' \Leftrightarrow w \leq_{\Psi\circ\alpha} w'$*
(S3) *If $w \in [[\alpha]], w' \in [[\neg\alpha]]$ then $w \leq_\Psi w' \Rightarrow w <_{\Psi\circ\alpha} w'$*
(S4) *If $w \in [[\alpha]], w' \in [[\neg\alpha]]$ then $w' <_\Psi w \Rightarrow w' \leq_{\Psi\circ\alpha} w$*
(S5) *If $w \in [[\alpha]], w' \in [[\neg\alpha]]$ then $w' \ll_\Psi w \Rightarrow w \leq_{\Psi\circ\alpha} w'$*

Properties (S1) and (S2) correspond to usual properties (CR1) and (CR2) for DP iterated revision operators (Darwiche & Pearl 1997). Property (S3) is the new property proposed in (Jin & Thielscher 2007; Booth & Meyer 2006), and that forces to increase the plausibility of the models of the new information. Property (S4) shows how the increase of plausibility of the models of the new information is limited by improvement operators. This is an important difference with usual DP iterated revision operators (Darwiche & Pearl 1997; Jin & Thielscher 2007; Booth & Meyer 2006). Property (S5) asks (together with (S4)) that if a model of $\neg\alpha$ is just a little more plausible than a model of α, then after improvement the two models will have the same plausibility.

Theorem 2 *A change operator $\circ$ is an improvement operator if and only if there exists a gradual assignment such that*

$$[[B(\Psi \star \alpha)]] = \min([[\alpha]], \leq_\Psi)$$

This theorem has important consequences. In particular the relationship between $\leq_\Psi$ and $\leq_{\Psi\circ\alpha}$ imposed by Definition 5 is very tight. Actually, the total pre-order $\leq_{\Psi\circ\alpha}$ is completely determined by $\leq_\Psi$ and α as it will be stated in Proposition 2. We first need the following lemma:

Lemma 1 *Let $\circ$ be an improvement operator and $\Psi \mapsto \leq_\Psi$ its gradual assignment. If $w <_\Psi w'$, $w \in [[\neg\alpha]]$, $w' \in [[\alpha]]$ and $w \not\ll_\Psi w'$ then $w <_{\Psi\circ\alpha} w'$.*

This Lemma is interesting since it gives the missing relation between $\leq_{\Psi\circ\alpha}$ and $\leq_\Psi$, since all other relations are given by the properties of Definition 5.

Proposition 2 *Let $\circ$ be an improvement operator and $\Psi \mapsto \leq_\Psi$ its gradual assignment. Then for every formula α, the pre-order $\leq_{\Psi\circ\alpha}$ is completely determined by $\leq_\Psi$ and $[[\alpha]]$.*

$w \in [[\alpha]]$	$w' \in [[\alpha]]$	$w \leq_\Psi w' \Leftrightarrow w \leq_{\Psi\circ\alpha} w'$	**(S1)**
$w \in [[\neg\alpha]]$	$w' \in [[\neg\alpha]]$	$w \leq_\Psi w' \Leftrightarrow w \leq_{\Psi\circ\alpha} w'$	**(S2)**
$w \in [[\alpha]]$	$w' \in [[\neg\alpha]]$	$w <_\Psi w' \Leftrightarrow w <_{\Psi\circ\alpha} w'$ $w \simeq_\Psi w' \Rightarrow w <_{\Psi\circ\alpha} w'$ $w' \ll_\Psi w \Rightarrow w \simeq_{\Psi\circ\alpha} w'$ $w' <_\Psi w \wedge w' \not\ll_\Psi w \Rightarrow w <_{\Psi\circ\alpha} w'$	**(S3)** **(S3)** **(S4) & (S5)** **(Lemma 1)**

Table 1: From $\leq_\Psi$ to $\leq_{\Psi\circ\alpha}$

This proposition is a very important one since it says, in a sense, that there is a unique improvement operator. In fact one can define different improvement operators by assigning to the initial epistemic state of the sequence different pre-orders. Once this pre-order is known, Proposition 2 tells us that there is no more freedom on the choice of subsequent pre-orders.

So clearly if ones considers pre-orders on interpretations as epistemic states (recall that the representation theorem just says that we can associate a pre-order on interpretation to each epistemic state, it does not presume anything on the exact nature of these epistemic states), then there is a unique improvement operator.

The exact construction of $\leq_{\Psi\circ\alpha}$ from $\leq_\Psi$ is given in Table 1. Roughly speaking, $\leq_{\Psi\circ\alpha}$ is obtained by shifting down one level the models of α in the total pre-order $\leq_\Psi$. This will be stated more formally in the next section.

Concrete example: Improvement via OCF

Let us now show how to implement an improvement operator via OCF. Let us denote Ord the class of ordinals.

Definition 6 *An Ordinal Conditional Function (OCF) κ is a function from the set of interpretations $\mathcal{W}$ to the set of ordinals such that at least one interpretation is assigned* 0.
A function from the set of interpretations $\mathcal{W}$ to the set of ordinals will be called a free OCF.
The set of OCF will be denoted $\mathcal{K}$.

Let us now state how to implement improvement using the framework of OCF. More precisely we will give two results: first we will show how to simply compute the resulting pre-order after an improvement, using a translation through free OCFs. Then, we will see how to define an improvement operator in the OCF framework.

So, what we do first is the following: giving $\leq_\Psi$ and α we describe $\leq_{\Psi\circ\alpha}$ using the machinery of OCF. At this point let us recall the equivalent view of a total pre-order $\leq$ introduced in the preliminaries: a total pre-order over $\mathcal{W}$ can be seen as the splitting of the set $\mathcal{W}$ in different levels (the equivalent classes), $\langle S_0, \ldots S_n \rangle$m the ordered sequence of its equivalence classes. Thus, $\forall x, y \in S_i\ x \simeq y$ and $\forall x \in S_i$ $\forall y \in S_j\ i < j$ implies $x < y$.

Let κ be the canonical representative of $\leq_\Psi$, *i.e.* if $\leq_\Psi$ has n levels and w is in the level i, $\kappa(w) = i$.

Consider now the following free OCF:

$$\kappa_\alpha(w) = \begin{cases} \kappa(w) & \text{if } w \models \alpha \\ \kappa(w) + 1 & \text{if } w \models \neg\alpha \end{cases}$$

It is not hard to see that this free OCF represents $\leq_{\Psi\circ\alpha}$, that is, the total pre-order associated to this function in the natural way ($w \leq_{\kappa_\alpha} w'$ iff $\kappa_\alpha(w) \leq \kappa_\alpha(w')$) satisfies all the properties of Table 1. So this result just aims at illustrating simply the behaviour of improvement operators in terms of total pre-orders via free OCF.

Note however that the free OCF used above is very particular, since it is build from the pre-order $\leq_\Psi$. In order to be able to define an improvement operator on any given OCF, this requires much more difficult definitions in order to modelize the smooth increase of plausibility of improvement operators.

So now we turn to a plain representation of an improvement operator $\circ$ in the full OCF framework. Thus, we assume that epistemic states are indeed OCF's and $\circ : \mathcal{K} \times \mathcal{L}^* \longrightarrow \mathcal{K}$. In this framework, we define the function B by putting $[[B(\kappa)]] = \{w : \kappa(w) = 0\}$.

Remember that $\kappa(\alpha) = min\{\kappa(w) : w \in [[\alpha]]\}$. Given κ and α, an OCF and a consistent formula respectively, we are going to define the new OCF $\kappa \circ \alpha$ by cases according to $\kappa(\alpha) > 0$ or $\kappa(\alpha) = 0$. In the first case ($\kappa(\alpha) > 0$) we perform the sliding down the models of α via an auxiliary function called $f^\kappa_{\alpha\downarrow}$ defined below. In the second case ($\kappa(\alpha) = 0$) we simulate the sliding down the models of α via an auxiliary function called $f^\kappa_{\alpha\uparrow}$ defined below that performs the sliding up the models of $\neg\alpha$. The functions $f^\kappa_{\alpha\downarrow} : [[\alpha]] \longrightarrow Ord$ and $f^\kappa_{\alpha\uparrow} : [[\neg\alpha]] \longrightarrow Ord$ are defined by putting

$$f^\kappa_{\alpha\downarrow}(w) = \begin{cases} max\{\rho : \exists w' \in [[\neg\alpha]]\ \kappa(w') = \rho\ \&\ \rho < \kappa(w) \\ \qquad \&\ \nexists w'' \in [[\alpha]]\ \rho < \kappa(w'') < \kappa(w)\} \\ \qquad\qquad \text{if this set is nonempty} \\ \kappa(w) - 1 \quad \text{otherwise} \end{cases}$$

$f^\kappa_{\alpha\downarrow}$ maps w, a model of α, into the first rank below $\kappa(w)$ where there is a model of $\neg\alpha$ in the case that there is no models of α strictly in between this two levels. Otherwise $f^\kappa_{\alpha\downarrow}$ maps w into $\kappa(w) - 1$.

$$f^\kappa_{\alpha\uparrow}(w) = \begin{cases} min\{\rho : \exists w' \in [[\alpha]]\ \kappa(w') = \rho\ \&\ \kappa(w) < \rho \\ \qquad \&\ \nexists w'' \in [[\neg\alpha]]\ \kappa(w) < \kappa(w'') < \rho\} \\ \qquad\qquad \text{if this set is nonempty} \\ \kappa(w) + 1 \quad \text{otherwise} \end{cases}$$

$f^{\kappa}_{\alpha\uparrow}$ maps w, a model of $\neg\alpha$, into the first rank above $\kappa(w)$ where there is a model of α in the case that there is no models of $\neg\alpha$ strictly in between this two ranks. Otherwise $f^{\kappa}_{\alpha\uparrow}$ maps w into $\kappa(w)+1$.

Again, we define two functions mapping worlds into ordinals according to whether or not $\kappa(\alpha)=0$. When $\kappa(\alpha)=0$ we put

$$\kappa\uparrow\neg\alpha(w)=\begin{cases}\kappa(w) & \text{if } w\models\alpha\\ f^{\kappa}_{\alpha\uparrow}(w) & \text{if } w\models\neg\alpha\end{cases}$$

and when $\kappa(\alpha)>0$ we put

$$\kappa\downarrow\alpha(w)=\begin{cases}\kappa(w) & \text{if } w\models\neg\alpha\\ f^{\kappa}_{\alpha\downarrow}(w) & \text{if } w\models\alpha\end{cases}$$

Finally we define $\kappa\circ\alpha$ by putting

$$\kappa\circ\alpha=\begin{cases}\kappa\uparrow\neg\alpha & \text{if } \kappa(\alpha)=0\\ \kappa\downarrow\alpha & \text{if } \kappa(\alpha)>0\end{cases}$$

Let us now take an example in order to see how it works.

Example 2 *Consider a language with propositional variables p,q and r in this order. Let κ be the OCF with image $\{0,1,2,4,5\}$ described in the diagram below and α a formula such that $\alpha=\neg p$. The following diagrams shows κ, $\kappa\circ\alpha$, $\kappa\circ\alpha\circ\alpha$ and $\kappa\circ\alpha\circ\alpha\circ\alpha$ (the models of α are in boldface):*

5	110	110	5
4	**011**	− − − − − −	4
3	− − − − − −	**011**	3
2	**010 000 001**	− − − − − −	2
1	101 100	101 100 **010 000 001**	1
0	111	111	0
	κ	$\kappa\circ\alpha$	

6		110	6
5	110	− − − − − −	5
4	− − − − − −	− − − − − −	4
3	− − − − − −	− − − − − −	3
2	− − − − − −	101 100	2
1	101 100 **011**	111 **011**	1
0	111 **010 000 001**	**010 000 001**	0
	$\kappa\circ\alpha\circ\alpha$	$\kappa\circ\alpha\circ\alpha\circ\alpha$	

Properties of improvement operators

Let us give now some additional properties on improvement operators, that illustrate how it relates with existing operators.

Proposition 3 *Improvement operators can not be represented as Spohn's Conditionalisation nor Williams' Adjustment.*

This is quite an intuitive result since Conditionalisation and Adjustement operate the same "global" change on the interpretation ranks, whereas, as sum up in Table 1, improvement requires a more adaptative behaviour (that depends more on the ranks of the other interpretations).

Proposition 4
- *There exists n such that $\Psi\circ^n\alpha$ is Nayak's lexicographic revision (Nayak 1994; Konieczny & Pino Pérez 2000). Let us note $\Psi\star_{lex}\alpha=\Psi\circ^n\alpha$.*
- *Actually, the first n such that $\Psi\star_{lex}\alpha=\Psi\circ^n\alpha$ is a fixed point for improvement by α, in the sense that $\leq_{\Psi\star_{lex}\alpha}=\leq_{\Psi\star_{lex}\alpha\circ\alpha}$.*

This can be shown easily with the help of Proposition 2 or with the representation of $\leq_{\Psi\circ\alpha}$ via the free OCF. In fact it requires at most k iterations where k is the level in $\leq_\Psi$ of the worst world of α (i.e. the model of α at the highest level) to reach this fixed point.

This last proposition is interesting since it illustrate the fact that the process of improvement does not stop as soon as the new information is believed. So in particular:

Proposition 5 *$B(\Psi)\vdash\alpha$ does not imply that $\leq_{\Psi\circ\alpha}=\leq_\Psi$ and therefore does not imply $\Psi\circ\alpha=\Psi$.*

It is worth noticing that even if we have a fixed point in the sense of Proposition 4, *i.e.* $\leq_\Psi=\leq_{\Psi\circ\alpha}$ we can have $\Psi\neq\Psi\circ\alpha$. The operator defined via the OCF is an example of such a situation.

Remark 1 *Improvements operators can be used to define contractions operators. Actualy, define $\Psi\odot\alpha=\Psi\circ^n\neg\alpha$ where n is the smallest integer such that $\Psi\circ^n\neg\alpha\nvdash\alpha$. Then, $\odot$ is a contraction operator.*

Let us now elaborate on the links between improvement operators and the bad day/good day approach of Booth et al. (Booth & Meyer 2007; Booth, Meyer, & Wong 2006) (also called abstract interval orders revision). In both cases the change is small, in the sense that the increase of plausibility of the models of the new information is limited. A first difference is that their operators are defined as revision of total pre-orders, whereas improvements are defined on general DP epistemic states. A second, more important difference between Booth et al. approach and ours is that they need an extra-logical information in order to guide the process, whereas our operators are completely defined in the usual DP framework. This is an important improvement, that allows for instance to easily iterate the process.

Actually, given $\leq_\Psi$, it is possible to define $\preceq$, a $\leq_\Psi$-*faithful tpo* (see (Booth, Meyer, & Wong 2006)), such that the revision of $\preceq$ by α, in the sense of Booth-Meyer-Wong, is exactly $\leq_{\Psi\circ\alpha}$. So, according to this link, improvement operators could be considered in a sense as a special case of Booth and Meyer operators.

Finally there is a very interesting behaviour of improvement operator with respect to long term behaviour. When working in a finite framework, no existing iterated belief revision operator escapes one of the following limit cases after a long course of revisions: maxichoice revision, or full meet revision.

Full meet revision means that the beliefs of the new epistemic state is either the conjunction of the new information with the beliefs of the old epistemic state it if is consistent, or just the new information otherwise. This is problematic, since it means that after a long course of revision the agent has lost all his beliefs. But for instance Lehmann's operators (Lehmann 1995) lead to this limit case when working on finite frameworks.

Maxichoice revision means that the revision leads to an epistemic states whose beliefs are a complete formula. This is also problematic, since it means that when this situation is reached, any revision by any formula will allow to be completely determined about each issue. It can be argued that this is due to the long revision history, that allows the agent to have a very precise view of the world. But still it seems sensible to be able to be uncertain about some issues, and to lose some certainties sometimes. Note that most of DP-like operators lead to this limit case (Darwiche & Pearl 1997; Nayak 1994; Boutilier 1996; Konieczny & Pino Pérez 2000; Booth & Meyer 2006; Jin & Thielscher 2007).

Note that it is not the case in the framework of OCFs, after any sequence of conditionalization, or adjustment, it is possible to reach any other OCF by some sequence. This is one of the advantage of using a more quantitative framework (using a degree of acceptance for each input formula), compared to the fully qualitative one that is the DP iterated belief revision framework.

It is interesting to note that improvement operators have no limit case. Actually, after any sequence of improvements, it is possible to reach any formula (as beliefs of an epistemic state) by an adequate sequence of improvements (the same result holds for associated pre-orders: any pre-order can be reached after an adequate sequence of improvements starting from any other pre-order). Whereas for DP iterated belief revision operators it is not the case: after some sequences of revisions, some formulae (or pre-orders) are not reachable anymore.

The following proposition summarizes this property of improvement operators:

Proposition 6 *Let $\leq$ be any pre-order on interpretations and $\leq_\Psi$ the pre-order associated to Ψ then there exists a sequence of formulae $\alpha_1, \ldots, \alpha_n$ such that $\leq_{\Psi \circ \alpha_1 \circ \ldots \circ \alpha_n} = \leq$.*

Improvement operators are, as far as we know, the first change operators defined in the DP framework that allows to avoid these limit cases.

Conclusion

We have introduced a new family of change operators called improvement operators. These operators have a more cautious behaviour than usual DP iterated revision operators. The main iterated revision operators of the literature satisfy all the properties of weak improvement operators. In that respect weak improvement operators can be considered as a generalization of iterated revision operators.

An essential point for being able to state logical properties and theorems on improvement operators is the interesting relationship between $\circ$ and its corresponding revision operator $\star$.

(I11), the last postulate required for improvement operators is very strong, in the sense that it determines in a unique way the pre-orders associated to the improvement. Thus, there are room to explore some variants of **(I11)** leading to other interesting weak improvements operators. We keep this as future work.

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Appendix

Proof of Theorem 1: (*only if*) Let $\circ$ be a weak improvement operator. We define an assignment $\Psi \mapsto \leq_\Psi$ by putting

$$w \leq_\Psi w' \text{ if and only if } w \models B(\Psi \star \varphi_{w,w'})$$

By I4, the relation $\leq_\Psi$ is well defined, *i.e.* it does not depend on the choice of the formula $\varphi_{w,w'}$. We prove now that $\leq_\Psi$ is a total pre-order.

Totality: Let w, w' be any interpretations (eventually $w = w'$). By I1, $[[\Psi \star \varphi_{w,w'}]] \subseteq \{w, w'\}$ and by definition $[[\Psi \star \varphi_{w,w'}]] \neq \emptyset$, so $w \in [[\Psi \star \varphi_{w,w'}]]$ or $w' \in [[\Psi \star \varphi_{w,w'}]]$ (or both), *i.e.* $w \leq_\Psi w'$ or $w' \leq_\Psi w$.

Transitivity: Suppose $w \leq_\Psi w'$ and $w' \leq_\Psi w''$. We want to show that $w \leq_\Psi w''$, that is to say $w \in [[B(\Psi \star \varphi_{w,w''})]]$. Suppose towards a contradiction that it is not the case, so $w \notin [[B(\Psi \star \varphi_{w,w''})]]$. As by definition $[[B(\Psi \star \varphi_{w,w''})]] \neq \emptyset$, this means that $[[B(\Psi \star \varphi_{w,w''})]] = \{w''\}$. Let us consider two cases:

1) If $B(\Psi \star \varphi_{w,w',w''}) \wedge \varphi_{w,w''}$ is not consistent. Then, as by definition $B(\Psi \star \varphi_{w,w',w''}) \nvdash \perp$, we have $[[B(\Psi \star \varphi_{w,w',w''})]] = \{w'\}$. In this case as $B(\Psi \star \varphi_{w,w',w''}) \wedge \varphi_{w,w'}$ is consistent, by I5 and I6 and I4 we get that $[[B(\Psi \star \varphi_{w,w'})]] = [[B(\Psi \star \varphi_{w,w',w''}) \wedge \varphi_{w,w'}]] = \{w'\}$. This means by definition that $w' <_\Psi w$. Contradiction.

2) If $B(\Psi \star \varphi_{w,w',w''}) \wedge \varphi_{w,w''}$ is consistent. Then by I5 and I6 and I4 we get that $[[B(\Psi \star \varphi_{w,w''})]] = [[B(\Psi \star \varphi_{w,w',w''}) \wedge \varphi_{w,w''}]] = \{w''\}$. This means by definition that $w'' <_\Psi w$. Contradiction.

Let us prove now equation (4). First we will prove that $[[B(\Psi \star \alpha)]] \subseteq \min([[\alpha]], \leq_\Psi)$. Take w in $[[B(\Psi \star \alpha)]]$. Thus, $B(\Psi \star \alpha) \wedge \varphi_{w,w'} \not\vdash \perp$ for any $w' \in [[\alpha]]$. Then, by I5 and I6 and I4, $B(\Psi \star \alpha) \wedge \varphi_{w,w'} \equiv B(\Psi \star \varphi_{w,w'})$. Therefore $w \in [[B(\Psi \star \varphi_{w,w'})]]$, that is $w \leq_\Psi w'$ for any $w' \in [[\alpha]]$ what exactly means $w \in \min([[\alpha]], \leq_\Psi)$.

Now we will prove the converse inclusion that is $\min([[\alpha]], \leq_\Psi) \subseteq [[B(\Psi \star \alpha)]]$. Suppose that $w \in \min([[\alpha]], \leq_\Psi)$. We want to show that $w \in [[B(\Psi \star \alpha)]]$. Towards a contradiction suppose that $w \notin [[B(\Psi \star \alpha)]]$. Let w' be a model of $B(\Psi \star \alpha)$.Then, by I5 and I6 and I4, $B(\Psi \star \alpha) \wedge \varphi_{w,w'} \equiv B(\Psi \star \varphi_{w,w'})$. By assumption $w \notin [[B(\Psi \star \alpha)]]$ so $[[B(\Psi \star \varphi_{w,w'})]] = \{w'\}$. Therefore $w' <_\Psi w$, contradicting the minimality of w in $[[\alpha]]$ with respect to $\leq_\Psi$.

Now we prove the conditions of the strong faithful assignment. First to show conditions 1 and 2 it is equivallent to show that $[[B(\Psi)]] = \min(\mathcal{W}, \leq_\Psi)$. Suppose that $w \models$

$B(\Psi)$. We want to see that $w \leq_\Psi w'$ for any interpretation w'. In order to do that, let w' be a interpretation. Note that $w \models B(\Psi) \wedge \varphi_{w,w'}$, so $B(\Psi) \wedge \varphi_{w,w'} \not\vdash \bot$. Then, by I2, $B(\Psi \star \varphi_{w,w'}) \equiv B(\Psi) \wedge \varphi_{w,w'}$. Therefore, $w \models B(\Psi \star \varphi_{w,w'})$, *i.e.* $w \leq_\Psi w'$. This proves that $[[B(\Psi)]] \subseteq \min(\mathcal{W}, \leq_\Psi)$. For the converse inclusion take $w \in \min(\mathcal{W}, \leq_\Psi)$. Towards a contradiction, suppose that $w \notin [[B(\Psi)]]$. Let w' be a model of $B(\Psi)$. Then $[[B(\Psi) \wedge \varphi_{w,w'}]] = \{w'\}$. Thus, by I2, $B(\Psi \star \varphi_{w,w'}) \equiv B(\Psi) \wedge \varphi_{w,w'}$ and therefore $[[B(\Psi \star \varphi_{w,w'})]] = \{w'\}$, *i.e.* $w' <_\Psi w$, contradicting the minimality of w with respect to $\leq_\Psi$.

Now for condition 3 suppose $\alpha_i \equiv \beta_i$ for any $i \leq n$ we want to show $\leq_{\Psi\circ\alpha_1\circ\cdots\circ\alpha_n} = \leq_{\Psi\circ\beta_1\circ\cdots\circ\beta_n}$. We proceed by induction on $k = 0, \ldots, n$. For $k = 0$ is trivial because $\leq_\Psi = \leq_\Psi$. For shortening the notation put $\Theta_k = \Psi \circ \alpha_1 \circ \cdots \circ \alpha_k$ and $\Gamma_k = \Psi \circ \beta_1 \circ \cdots \circ \beta_k$. Thus our induction hypothesis is $\leq_{\Theta_k} = \leq_{\Gamma_k}$. We want to show that $\leq_{\Theta_k\circ\alpha_{k+1}} = \leq_{\Gamma_k\circ\beta_{k+1}}$. In order to do that, we prove that each level of $\leq_{\Theta_k\circ\alpha_{k+1}}$ is equal to the corresponding level of $\leq_{\Gamma_k\circ\beta_{k+1}}$. This is done by induction on the number of levels of $\leq_{\Theta_k\circ\alpha_{k+1}}$. We sketch the proof. For the level 0: we want to see that $min(\mathcal{W}, \leq_{\Theta_k\circ\alpha_{k+1}}) = min(\mathcal{W}, \leq_{\Gamma_k\circ\beta_{k+1}})$. By equation (4), we have $min(\mathcal{W}, \leq_{\Theta_k\circ\alpha_{k+1}}) = [[B(\Gamma_k \circ \alpha_{k+1})]]$ and $min(\mathcal{W}, \leq_{\Gamma_k\circ\beta_{k+1}}) = [[B(\Gamma_k \circ \beta_{k+1})]]$. By I4$^+$, $[[B(\Theta_k \circ \alpha_{k+1})]] = [[B(\Gamma_k \circ \beta_{k+1})]]$. Therefore, $min(\mathcal{W}, \leq_{\Theta_k\circ\alpha_{k+1}}) = min(\mathcal{W}, \leq_{\Gamma_k\circ\beta_{k+1}})$. Now suppose that the first i levels of $\leq_{\Theta_k\circ\alpha_{k+1}}$ correspond exactly to the first i levels of $\leq_{\Gamma_k\circ\beta_{k+1}}$. We will prove that the level $i+1$ of $\leq_{\Theta_k\circ\alpha_{k+1}}$ is contained in the level $i+1$ of $\leq_{\Gamma_k\circ\beta_{k+1}}$ (and with a symmetrical argument we will prove the inverse inclusion). Towards a contradiction suppose that w is in the level $i+1$ of $\leq_{\Theta_k\circ\alpha_{k+1}}$ and w is not in the level $i+1$ of $\leq_{\Gamma_k\circ\beta_{k+1}}$. Take w' in the level $i+1$ of $\leq_{\Gamma_k\circ\beta_{k+1}}$. As the first i levels of $\leq_{\Theta_k\circ\alpha_{k+1}}$ and $\leq_{\Gamma_k\circ\beta_{k+1}}$ are equal, w' is in a level j, with $j > i$ for the pre-order $\leq_{\Theta_k\circ\alpha_{k+1}}$. Consider now the formula $\varphi_{w,w'}$. Then it is clear that $w \in min([[\varphi_{w,w'}]], \leq_{\Theta_k\circ\alpha_{k+1}})$ and $w \notin min([[\varphi_{w,w'}]], \leq_{\Gamma_k\circ\beta_{k+1}})$. From this, by equation (4), follows $[[B(\Theta_k \circ \alpha_{k+1} \circ \varphi_{w,w'})]] \neq [[B(\Gamma_k \circ \beta_{k+1} \circ \varphi_{w,w'})]]$, contradicting I4$^+$.

(*if*) Suppose that we have a strong faithful assignment $\Psi \mapsto \leq_\Psi$ such that equation (4) holds. We want to check that I1-I6 hold.

(I1) Follows from equation 4.

(I2) Let us first show that $B(\Psi) \wedge \alpha \vdash B(\Psi \star \alpha)$. If $w \models B(\Psi) \wedge \alpha$ this means that $w \in \min(\mathcal{W}, \leq_\Psi)$. So for any $w' \in \mathcal{W}$, we have $w \leq_\Psi w'$. This is in particular true for all the models of α, so $w \in \min(\alpha, \leq_\Psi)$, that is, by definition, $w \models B(\Psi\star\alpha)$. Let us now show that $B(\Psi\star\alpha) \vdash B(\Psi)\wedge\alpha$. By definition $w \models B(\Psi\star\alpha)$ means $w \in \min([[\alpha]], \leq_\Psi)$. So $w \models \alpha$. Let us show that $w \models B(\Psi)$. Suppose that it is not the case. In this case, and since by hypothesis $B(\Psi)\wedge\alpha \nvdash \bot$ we can choose a $w' \in [[B(\Psi) \wedge \alpha]]$. So, as $w' \in [[B(\Psi)]]$ and $w \notin [[B(\Psi)]]$, we have $w' <_\Psi w$. But as $w', w \models \alpha$, this implies that $w \notin \min([[\alpha]], \leq_\Psi)$. Contradiction.

(I4) Suppose $\alpha_i \equiv \beta_i$ for any $i \leq n$ we want to show $B(\Psi \circ \alpha_1 \circ \cdots \circ \alpha_n) \equiv B(\Psi \circ \beta_1 \circ \cdots \circ \beta_n)$. By equation (4), this is equivalent to prove $min([[\alpha_n]], \leq_{\Psi\circ\alpha_1\circ\cdots\circ\alpha_{n-1}}) = min([[\beta_n]], \leq_{\Psi\circ\beta_1\circ\cdots\circ\beta_{n-1}})$. But this is clear because, by S6, we have $\leq_{\Psi\circ\alpha_1\circ\cdots\circ\alpha_{n-1}} = \leq_{\Psi\circ\beta_1\circ\cdots\circ\beta_{n-1}}$ and by hypothesis $[[\alpha_n]] = [[\beta_n]]$.

(I5 and I6) By equation (4) we have $[[B(\Psi \star (\alpha \wedge \beta))]] = \min([[\alpha \wedge \beta]], \leq_\Psi)$ and $[[B(\Psi \star \alpha) \wedge \beta]] = \min([[\alpha]], \leq_\Psi) \cap [[\beta]]$. Thus, it is enough to see that

$$\min([[\alpha]], \leq_\Psi) \cap [[\beta]] = \min([[\alpha \wedge \beta]], \leq_\Psi)$$

under the hypothesis $\min([[\alpha]], \leq_\Psi) \cap [[\beta]] \neq \emptyset$. It is quite clear that $\min([[\alpha]], \leq_\Psi) \cap [[\beta]] \subseteq \min([[\alpha \wedge \beta]], \leq_\Psi)$. For the other inclusion take $w \in \min([[\alpha \wedge \beta]], \leq_\Psi)$. As w is in $[[\beta]]$ it remains to see that $w \in \min([[\alpha]], \leq_\Psi)$. We know that $w \in [[\alpha]]$. We claim that it is minimal in $[[\alpha]]$ with respect to $\leq_\Psi$. Towards a contradiction, suppose that ir is not the case. As $\leq_\Psi$ is a total pre-order there exists $w' \in \min([[\alpha]], \leq_\Psi)$ such that $w' <_\Psi w$. By hypothesis, there exists $w'' \in \min([[\alpha]], \leq_\Psi) \cap [[\beta]]$. Again as$\leq_\Psi$ is a total pre-order, $w' \sim_\Psi w''$, therefore $w'' <_\Psi w$ contradicting the minimality of w in $[[\alpha \wedge \beta]]$ with respect to $\leq_\Psi$. ∎

Proof of Proposition 1: If $\min([[\alpha]], \leq_\Psi)$ and $\min([[\beta]], \leq_\Psi)$ are in the same level with respect to $\leq_\Psi$ then $\min([[(]]\alpha \vee \beta), \leq_\Psi) = \min([[\alpha]], \leq_\Psi) \cup \min([[\beta]], \leq_\Psi)$. Thus, by Theorem 1, $[[B(\Psi \star (\alpha \vee \beta))]] = [[B(\Psi \star \alpha)]] \cup [[B(\Psi \star \beta)]]$. Otherwise, $\min([[\alpha]], \leq_\Psi)$ is in a lower level than $\min([[\beta]], \leq_\Psi)$ or $\min([[\beta]], \leq_\Psi)$ is in a lower level than $\min([[\alpha]], \leq_\Psi)$. In the first case, $\min([[(]]\alpha \vee \beta), \leq_\Psi) = \min([[\alpha]], \leq_\Psi)$. Thus, by the Theorem 1, $[[B(\Psi \star (\alpha \vee \beta))]] = [[B(\Psi \star \alpha)]]$. In the second case, $\min([[(]]\alpha \vee \beta), \leq_\Psi) = \min([[\beta]], \leq_\Psi)$. Thus, by the Theorem 1, $[[B(\Psi \star (\alpha \vee \beta))]] = [[B(\Psi \star \beta)]]$. ∎

Proof of Corollary 2: (*only if*) Assume that $\alpha \prec_\Psi \beta$, that is $B(\Psi\star\alpha) \vdash B(\Psi\star(\alpha\vee\beta))$ and $B(\Psi\star\beta) \not\vdash B(\Psi\star(\alpha\vee\beta))$. By Proposition 1 and its proof, necessarily $\min([[\alpha]], \leq_\Psi)$ is in a lower level than $\min([[\beta]], \leq_\Psi)$. Thus, by Theorem 1, it is enough to take $w \in \min([[\alpha]], \leq_\Psi)$ and $w' \in \min([[\beta]], \leq_\Psi)$ to get $w \in [[B(\Psi \star \alpha)]]$, $w' \in [[B(\Psi \star \beta)]]$, $w <_\Psi w'$.
(*if*) Assume that there exist w, w' such that $w \in [[B(\Psi \star \alpha)]]$, $w' \in [[B(\Psi \star \beta)]]$, $w <_\Psi w'$. Then, by Theorem 1, $\min([[\alpha]], \leq_\Psi)$ is in a lower level than $\min([[\beta]], \leq_\Psi)$. Then, by Proposition 1 and its proof, $B(\Psi \star (\alpha \vee \beta)) \equiv B(\Psi \star \alpha)$. On the other hand $B(\Psi \star \beta) \not\vdash B(\Psi \star (\alpha \vee \beta))$ because $\min([[\alpha]], \leq_\Psi)$ and $\min([[\beta]], \leq_\Psi)$ are not in the same level. Therefore $\alpha \prec_\Psi \beta$

∎

Proof of Corollary 3: (*only if*) Assume $\alpha \prec\!\!\!\cdot_\Psi \beta$. By Corollary 2, we get w, w' such that $w \in [[B(\Psi \star \alpha)]]$, $w' \in [[B(\Psi \star \beta)]]$, $w <_\Psi w'$. Towards a contradiction, suppose that there exists w'' such that $w <_\Psi w'' <_\Psi w'$. But it is clear, using Corollary 2, that $\alpha \prec_\Psi \varphi_{w''} \prec_\Psi \beta$ contradicting the fact $\alpha \prec\!\!\!\cdot_\Psi \beta$.
(*if*) Assume there exist w, w' such that $w \in [[B(\Psi \star \alpha)]]$,

$w' \in [\![B(\Psi \star \beta)]\!]$, $w <_\Psi w'$ and there is no w'' such that $w <_\Psi w'' <_\Psi w'$. By Corollary 2, $\alpha \prec_\Psi \beta$. Thus, the only possibility for $\alpha \not\prec\!\!\!\prec_\Psi \beta$, is the existence of γ such that $\alpha \prec_\Psi \gamma \prec_\Psi \beta$. Again, by Corollary 2, taking $w'' \in [\![B(\Psi \star \gamma)]\!]$ we have $w <_\Psi w'' <_\Psi w'$, a contradiction. ∎

Proof of Theorem 2: (*only if*) By Theorem 1 we know that there exists an epistemic assignment $\Psi \mapsto \leq_\Psi$ such that the equation (4) holds. Thus, it remains to prove that the assignment is indeed a gradual assignment, *i.e.* it satisfies S1, S2, S3, S4 and S5.

(S1) Suppose $w, w' \in [\![\alpha]\!]$. Thus, $\varphi_{w,w'} \vdash \alpha$. By I7, $B((\Psi \circ \alpha) \star \varphi_{w,w'}) \equiv B(\Psi \star \varphi_{w,w'})$. Then by equation (4) we have

$$\begin{aligned} w \leq_{\Psi\circ\alpha} w' \quad &\Leftrightarrow \quad w \in \min(\{w, w'\}, \leq_{\Psi\circ\alpha}) \\ &\Leftrightarrow \quad w \in [\![B((\Psi \circ \alpha) \star \varphi_{w,w'})]\!] \\ &\Leftrightarrow \quad w \in [\![B(\Psi \star \varphi_{w,w'})]\!] \\ &\Leftrightarrow \quad w \in \min(\{w, w'\}, \leq_\Psi) \\ &\Leftrightarrow \quad w \leq_\Psi w' \end{aligned}$$

(S2) The proof is analogous to the one of S1 but using I8 instead of I7.

(S3) Suppose that $w \in [\![\alpha]\!]$, $w' \in [\![\neg\alpha]\!]$ and $w \leq_\Psi w'$. We want to show that $w <_{\Psi\circ\alpha} w'$. As $w \leq_\Psi w'$, necessarily $w \in \min(\{w, w'\}, \leq_\Psi)$ what, by Equation (4), means $w \in [\![B(\Psi \star \varphi_{w,w'})]\!]$. Then $B(\Psi \star \varphi_{w,w'}) \nvdash \neg\alpha$, so by I9, $B((\Psi \circ \alpha) \star \varphi_{w,w'}) \vdash \alpha$. Then, by I1 and I3, $[\![B((\Psi \circ \alpha) \star \varphi_{w,w'})]\!] = \{w\}$. From this, using Equation (4), we get $w <_{\Psi\circ\alpha} w'$.

(S4) Suppose that $w \in [\![\alpha]\!]$, $w' \in [\![\neg\alpha]\!]$ and $w' <_\Psi w$. We want to show that $w' \leq_{\Psi\circ\alpha} w$. From the hypothesis $w' <_\Psi w$ we get $\min(\{w, w'\}, \leq_\Psi) = \{w'\}$. Then, by Equation (4), $[\![B(\Psi \star \varphi_{w,w'})]\!] = \{w'\}$. Therefore $B(\Psi \star \varphi_{w,w'}) \vdash \neg\alpha$. Thus, by I10, $B((\Psi \circ \alpha) \star \varphi_{w,w'}) \nvdash \alpha$. Then $w' \in [\![B((\Psi \circ \alpha) \star \varphi_{w,w'})]\!]$, and by Equation (4), $w' \in \min(\{w, w'\}, \leq_{\Psi\circ\alpha})$, *i.e.* $w' \leq_{\Psi\circ\alpha} w$.

(S5) Suppose that $w \in [\![\alpha]\!]$, $w' \in [\![\neg\alpha]\!]$, $w' <_\Psi w$ and that there is no w'' such that $w' <_\Psi w'' <_\Psi w$. We want to show that $w \leq_{\Psi\circ\alpha} w'$. From $w' <_\Psi w$ we have $\min(\{w, w'\}, \leq_\Psi) = \{w'\}$, so from Theorem 1 we have $[\![B(\Psi \star \varphi_{w,w'})]\!] = \{w'\}$. So $B(\Psi \star \varphi_{w,w'}) \vdash \neg\alpha$. On the other hand, the assumptions with the Corollary 3 gives us $\varphi_{w,w'} \prec\!\!\!\prec_\Psi \varphi_{w,w'} \wedge \alpha$. Then, by I11, $B((\Psi \circ \alpha) \star \varphi_{w,w'}) \nvdash \neg\alpha$, that means by Theorem 1, $w \leq_{\Psi\circ\alpha} w'$.

(*if*) By Theorem 1 we know that $\circ$ is a weak improvement operator. Thus, it remains to check that I7-I11 hold.

(I7) Suppose that $\alpha \vdash \mu$, *i.e.* $[\![\alpha]\!] \subseteq [\![\mu]\!]$. We want to show $B((\Psi \circ \mu) \star \alpha) \equiv B(\Psi \star \alpha)$. By Equation (4) this is equivalent to prove that $\min([\![\alpha]\!], \leq_{\Psi\circ\mu}) = \min([\![\alpha]\!], \leq_\Psi)$. That is a straightforward consequence of S1 that gives (since $[\![\alpha]\!] \subseteq [\![\mu]\!]$) $\forall w, w' \models \alpha, w \leq_\Psi w'$ iff $w \leq_{\Psi\circ\mu} w'$.

(I8) Suppose that $\alpha \vdash \neg\mu$, *i.e.* $[\![\alpha]\!] \subseteq [\![\neg\mu]\!]$. We want to show $B((\Psi \circ \mu) \star \alpha) \equiv B(\Psi \star \alpha)$. By Equation (4) this is equivalent to prove that $\min([\![\alpha]\!], \leq_{\Psi\circ\mu}) = \min([\![\alpha]\!], \leq_\Psi)$. Like for (I7), this is a straightforward consequence of S2 that gives (since $[\![\alpha]\!] \subseteq [\![\neg\mu]\!]$) $\forall w, w' \models \alpha, w \leq_\Psi w'$ iff $w \leq_{\Psi\circ\mu} w'$.

(I9) Let us remark from the fact that $\leq_\Psi$ and $\leq_{\Psi\circ\mu}$ are total pre-orders, that postulate S3 is equivalent to the following one:

(S3') If $w \in [\![\mu]\!], w' \in [\![\neg\mu]\!]$ then $w' \leq_{\Psi\circ\mu} w \quad \Rightarrow \quad w' <_\Psi w$

Now suppose that $B(\Psi \star \alpha) \nvdash \neg\mu$. We want to show $B((\Psi \circ \mu) \star \alpha) \vdash \mu$. Towards a contradiction, suppose that $B((\Psi \circ \mu) \star \alpha) \nvdash \mu$, *i.e.* there exists $w \in [\![B((\Psi \circ \mu) \star \alpha)]\!]$ such that $w \notin [\![\mu]\!]$. By Equation (4) $w \in \min([\![\alpha]\!], \leq_{\Psi\circ\mu})$, so for any $w' \in [\![\alpha]\!]$, $w \leq_{\Psi\circ\mu} w'$. By the assumption, there exists $w'' \in [\![B(\Psi \star \alpha)]\!] \cap [\![\mu]\!]$. In particular, by I1, $w'' \in [\![\alpha]\!]$. Thus, $w \leq_{\Psi\circ\mu} w''$. On the other hand, by Equation (4) $w'' \in \min([\![\alpha]\!], \leq_\Psi)$. As $w \in [\![\neg\mu]\!]$ and $w'' \in [\![\mu]\!]$, and $w \leq_{\Psi\circ\mu} w''$, by S3', $w <_\Psi w''$. But, since $w \in [\![\alpha]\!]$, this contradicts the minimality of w'' in $[\![\alpha]\!]$ with respect to $\leq_\Psi$.

(I10) Let us remark from the fact that $\leq_\Psi$ and $\leq_{\Psi\circ\mu}$ are total pre-orders, the postulate S4 is equivalent to the following one:

(S4') If $w \in [\![\mu]\!], w' \in [\![\neg\mu]\!]$ then $w <_{\Psi\circ\mu} w' \quad \Rightarrow \quad w \leq_\Psi w'$

Suppose that $B(\Psi \star \alpha) \vdash \neg\mu$. We want to show $B((\Psi \circ \mu) \star \alpha) \nvdash \mu$. Towards a contradiction suppose that $B((\Psi \circ \mu) \star \alpha) \vdash \mu$. Let w, w' be such that $w \models B((\Psi \circ \mu) \star \alpha)$ and $w' \models B(\Psi \star \alpha)$. By the assumptions $w \in [\![\mu]\!]$ and $w' \in [\![\neg\mu]\!]$. By Equation (4), $w \in \min([\![\alpha]\!], \leq_{\Psi\circ\mu})$ and $w' \in \min([\![\alpha]\!], \leq_\Psi)$. By the assumptions, $w \not\simeq_{\Psi\circ\mu} w'$ and $w \not\simeq_\Psi w'$, because if not $w' \in \min([\![\alpha]\!], \leq_{\Psi\circ\mu})$ or $w \in \min([\![\alpha]\!], \leq_\Psi)$. But this is impossible because in the first case $w' \in [\![\mu]\!]$, a contradiction and in he second case $w \in [\![\neg\mu]\!]$, a contradiction. Thus, necessarily $w <_{\Psi\circ\mu} w'$ and $w' <_\Psi w$. As we have $w \in [\![\mu]\!]$, $w' \in [\![\neg\mu]\!]$ and $w <_{\Psi\circ\mu} w'$, by S4', $w \leq_\Psi w'$, a contradiction.

(I11) Assume $B(\Psi \star \alpha) \vdash \neg\mu$, $\alpha \wedge \mu \nvdash \bot$ and $\alpha \prec\!\!\!\prec_\Psi \alpha \wedge \mu$. We want to show that $B((\Psi \circ \mu) \star \alpha) \nvdash \neg\mu$. Towards a contradiction, suppose that $B((\Psi \circ \mu) \star \alpha) \vdash \neg\mu$. Let w, w' such that $w' \in [\![B(\Psi \star \alpha)]\!]$ and $w \in [\![B(\Psi \star (\alpha \wedge \mu))]\!]$. By the assumptions we have $w' \in [\![\neg\mu]\!]$ and $w \in [\![\mu]\!]$. By Corollary 3, $w' <_\Psi w$ and there is no w'' such that $w' <_\Psi w'' <_\Psi w$. By S5, $w \leq_{\Psi\circ\mu} w'$. By S4, $w' \leq_{\Psi\circ\mu} w$. Therefore, $w \simeq_{\Psi\circ\mu} w'$. That means that $[\![B(\Psi \star \alpha)]\!]$ and $[\![B(\Psi \star \alpha \wedge \mu)]\!]$ are in the same level with respect to $\leq_{\Psi\circ\mu}$. We claim that this level is the level of $\min([\![\alpha]\!], \leq_{\Psi\circ\mu})$. But this is a contradiction because we have $w \in \min(\min([\![\alpha]\!], \leq_{\Psi\circ\mu}))$ and therefore $w \models \neg\mu$ which contradicts the fact that $w \models \mu$. Now we turn to the proof of our claim. Towards a contradiction, suppose the claim is not true. Then, necessarily there is $w'' \in \min([\![\alpha]\!], \leq_{\Psi\circ\mu})$ such that $w'' <_{\Psi\circ\mu} w$. We consider two cases: $w'' \in [\![\mu]\!]$ and $w'' \in [\![\neg\mu]\!]$. In the case $w'' \in [\![\mu]\!]$, we don't have $w'' <_\Psi w$ because $w \in \min([\![\alpha \wedge \mu]\!], \leq_\Psi)$. Therefore $w \leq_\Psi w''$. Then, by S1, $w \leq_{\Psi\circ\mu} w''$, a contradiction. In the case $w'' \in [\![\neg\mu]\!]$, we don't have $w'' <_\Psi w'$ because $w' \in \min([\![\alpha]\!], \leq_\Psi)$. Therefore $w' \leq_\Psi w''$. Then, by S2, $w' \leq_{\Psi\circ\mu} w''$, that is $w \leq_{\Psi\circ\mu} w''$, a contradiction. ∎

Proof of Lemma 1: Define $A = \{w'' \in \mathcal{W} : w <_\Psi w'' <_\Psi w'\}$. By the assumptions $w <_\Psi w'$ and $w \not\ll_\Psi w'$, the set A is nonempty. Thus $A \cap [[\neg\alpha]] \neq \emptyset$ or $A \cap [[\alpha]] \neq \emptyset$. We consider first the case $A \cap [[\neg\alpha]] \neq \emptyset$. Take $w'' \in max(A \cap [[\neg\alpha]], \leq_\Psi)$. By definition of A, $w <_\Psi w''$ and $w'' <_\Psi w'$. We consider two subcases:

- $w'' \ll_\Psi w'$. In this situation, we conclude by S4 and S5 $w'' \sim_{\Psi\circ\alpha} w'$. By S2, $w <_{\Psi\circ\alpha} w''$. Therefore by transitivity $w <_{\Psi\circ\alpha} w'$.
- $w'' \not\ll_\Psi w'$. In this situation we take w''' such that $w'' \ll_\Psi w'''$. Its clear that $w''' <_\Psi w'$ and by definition of w'', $w''' \in [[\alpha]]$. By S4 and S5, $w'' \simeq_{\Psi\circ\alpha} w'''$. Thus $w <_{\Psi\circ\alpha} w'''$. By S1, $w''' <_{\Psi\circ\alpha} w'$. Then, by transitivity, $w <_{\Psi\circ\alpha} w'$.

For the second case, $A \cap [[\alpha]] \neq \emptyset$, we proceed with an analogous reasoning, but this time taking $w'' \in min(A \cap [[\alpha]], \leq_\Psi)$. ∎

Proof of Proposition 2: Towards a contradiction, suppose that we have $\leq^1_{\Psi\circ\alpha} \neq \leq^2_{\Psi\circ\alpha}$ and both pre-orders obey to (S1-S5). Let w, w' be witness of this inequality. Thus, without lost of generality, we can suppose $w <^1_{\Psi\circ\alpha} w'$ and $w' \leq^2_{\Psi\circ\alpha} w$. By S1, it is not the case $w, w' \in [[\alpha]]$, since otherwise by $w <^1_{\Psi\circ\alpha} w'$ we obtain $w <_\Psi w'$ and by $w' \leq^2_{\Psi\circ\alpha} w$ we obtain $w' \leq_\Psi w$ and a contradiction. Similarly by S2, it is not the case $w, w' \in [[\neg\alpha]]$. Thus, the only possibilities are $w \in [[\alpha]]$ and $w' \in [[\neg\alpha]]$ or $w \in [[\neg\alpha]]$ and $w' \in [[\alpha]]$.
We consider the first case, *i.e.* $w \in [[\alpha]]$ and $w' \in [[\neg\alpha]]$. As $w' \leq^2_{\Psi\circ\alpha} w$, by S3, $w \not\leq_\Psi w'$, *i.e.* $w' <_\Psi w$. If $w' \not\ll_\Psi w$ then, by the Lemma 1 $w' <^1_{\Psi\circ\alpha} w$, a contradiction. If $w' \ll_\Psi w$, by S4 and S5, $w' \simeq^1_{\Psi\circ\alpha} w$, again a contradiction. Now, we consider the second case, *i.e.* $w \in [[\neg\alpha]]$ and $w' \in [[\alpha]]$. As $w <^1_{\Psi\circ\alpha} w'$, by S3, $w' \not\leq_\Psi w$, *i.e.* $w <_\Psi w'$. Suppose $w \ll_\Psi w'$. Then, by S5, $w' \leq^1_{\Psi\circ\alpha} w$ a contradiction. So $w \not\ll_\Psi w'$, and by Lemma 1, $w <^2_{\Psi\circ\alpha} w'$, a contradiction. ∎

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