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Time Representation and Temporal Reasoning from the Perspective of Non-Standard Analysis

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Abstract

This paper proceeds to develop models for representing time and reasoning about time from the perspective of non-standard analysis. It sets out a non-standard first-order theory and a non-standard qualitative approach for hyperreals. This first-order theory and this qualitative approach are based on the fact that any hyperreal is either infinitesimal, unlimited or appreciable. Within the first-order theory for hyperreal time presented in this paper, we establish a complete axiomatization and we prove that the associated membership problem is $PSPACE$-complete. Within the qualitative approach for hyperreal time presented in this paper, we establish qualitative constraint satisfaction problems and we prove that the associated consistency problem is in P.

Introduction

Many areas within computer science and artificial intelligence involve some sort of time representation and temporal reasoning through the intermediary of formal models. For defining these formal models, a system should consist of temporal individuals arranged in a temporal order. Various choices for temporal individuals have been investigated systematically and have given rise to multifarious temporal orders [Allen (1983), van Benthem (1991), Ladkin (1987), Ligozat (1991), Vilain and Kautz (1986)]. Traditional temporal structures consist of points in time ordered by a relation of precedence. The most obvious properties on precedence may be expressed in a first-order language with the non logical construct $<$ (... precedes ...).

The choices to be made concern the obvious conditions on precedence which may be formulated directly as axioms. One of the simplest axiomatization is the first-order theory of total dense orderings without endpoints, the structure of the reals being its most standard representative. This standard structure has rivals such as the rationals or the hyperreals. While reals, hence rationals, all belong to the same order of magnitude, it is the fact that hyperreals are either infinitesimal, unlimited or appreciable which sets them apart. The hyperreals form an ordered field that contains the real number system as a subfield, but also contains infinitely small (infinitesimal) numbers and infinitely large (unlimited) numbers.

In mathematics, these new entities offer new definitions of familiar concepts like convergence, continuity, etc [Robinson (1996), originally published in 1966 by North-Holland]. In other areas of science and technology, the hyperreal number system justifies the algebraic processing of small numbers and large numbers that researchers and engineers often do — witness its use in multifarious domains like market models [Cutland *et al.* (1991)] for modelling option pricing, electrical networks [Zemanian (2001)] for modelling infinite networks, etc. In computer science and artificial intelligence, hyperreals have been used for analysing texts [Becher *et al.* (2000)], reasoning about hybrid systems [Iwasaki *et al.* (1995)], etc.

This paper sets out a non-standard first-order theory and a non-standard qualitative approach for hyperreals where the construct $<$ mentioned above is split up into the following non logical constructs: $<_\epsilon$ (... precedes ... from an infinitesimal distance), $<_\omega$ (... precedes ... from an unlimited distance) and $<_1$ (... precedes ... from an appreciable distance). It departs from [Gagné and Plaice (1996), Nakamura and Fusaoka (2007), Weld (1990)] in that it proposes a complete examination of the mathematical foundations and the computational aspects of the precedence relations $<_\epsilon$, $<_\omega$ and $<_1$ between hyperreals. It also departs from [Davis (1999)] in that it does not consider infinitely many orders of magnitude between hyperreals.

The section-by-section breakdown of the paper is as follows. Motivating examples are given in the second section. Following the introduction to non-standard analysis proposed in [Goldblatt (1998)], the third section introduces a number of basic concepts such as the ordered field of hyperreals. The first-order theory for hyperreal time is presented in the fourth section. There, we establish a complete axiomatization and we prove that the associated membership problem is $PSPACE$-complete. The qualitative approach for hyperreal time is presented in the fifth section. There, we establish qualitative constraint satisfaction problems and we prove that the associated consistency problem is in P. All the proofs can be found in the appendix.

Motivating examples

Formal models for time representation and temporal reasoning from the non-standard perspective have been developed from a number of viewpoints [Becher *et al.* (2000), Iwasaki

et al. (1995)]. The majority of all this work has been concerned with the problem of granularity and calendars in time representation and temporal reasoning [Euzenat and Montanari (2005)].
In linguistic semantics, interpretation of sentences leads to the use of events and periods of time during which events occur. Consequently, suitable representations of time for analysing texts should admit temporal individuals at different levels of granularity. In [Becher *et al.* (2000)], the use of hyperreals provides a formal foundation for the theory of natural language analysis allowing to distinguish durative intervals from durationless intervals. The motivation for this work is to propose a definition of granularity levels of time on which consecutive events can be situated and compared. In this definition, for example, the distinction between "$\mathbf{a}$ precedes $\mathbf{b}$ and is infinitely close to it" and "$\mathbf{a}$ precedes $\mathbf{b}$ but is not in its immediate neighborhood" can be made explicit. In our setting, this distinction is definable by means of the constructs $<_\epsilon$ and $<_{\omega 1}$.
Electromechanical systems usually show an interplay between a level of discrete behaviour and a level of continuous behaviour. Consequently, suitable representations of time for reasoning about hybrid systems should admit temporal orders at different levels of behaviour. In [Iwasaki *et al.* (1995)], the use of hyperreals provides a formal model for the theory of hybrid systems allowing to characterize discrete actions occurring in the presence of continuous changes. The motivation for this work is to propose a logic for analysing the behaviour of hybrid systems. In this logic, for example, the distinction between "the instant at which $\mathbf{a}$ becomes true immediately precedes the beginning of $\mathbf{b}$'s execution" and "the instant at which $\mathbf{a}$ becomes true is equal to the beginning of $\mathbf{b}$'s execution" can be made explicit. In our setting, this distinction is definable by means of the constructs $<_\epsilon$ and $=$.

Ultrapower construction of the hyperreals

Following the introduction to non-standard analysis proposed in [Goldblatt (1998)], this section introduces a number of basic concepts.

Large sets

Let I be the set of all positive integers. We use $\mathbb{R}^I$ to denote the set of all real-valued sequences, $\mathcal{P}(I)$ to denote the power set of I and $\mathcal{P}(\mathcal{P}(I))$ to denote the power set of $\mathcal{P}(I)$. For a start, suppose that the notion of a large set of positive integers, in a sense that is to be determined, is at our disposal. Given $\mathbf{a}, \mathbf{b} \in \mathbb{R}^I$, we shall say that $\mathbf{a}$ agrees with $\mathbf{b}$ iff $\{n \in I: \mathbf{a}(n) = \mathbf{b}(n)\}$ is large. The set $\{n \in I: \mathbf{a}(n) = \mathbf{b}(n)\}$ may be thought of as a measure of the extent to which the statement "$\mathbf{a}$ agrees with $\mathbf{b}$" is true. In order to ensure that agreement between real-valued sequences is a non trivial equivalence relation, the following conditions must be satisfied:

- I is large,
- $\emptyset$ is not large,
- for all $X, Y \in \mathcal{P}(I)$, if X is large and Y is large then $X \cap Y$ is large.

Given $\mathbf{a}, \mathbf{b} \in \mathbb{R}^I$, we shall say that $\mathbf{a}$ precedes $\mathbf{b}$ iff $\{n \in I: \mathbf{a}(n) < \mathbf{b}(n)\}$ is large. The set $\{n \in I: \mathbf{a}(n) < \mathbf{b}(n)\}$ may be thought of as a measure of the extent to which the statement "$\mathbf{a}$ precedes $\mathbf{b}$" is true. In order to ensure that precedence between real-valued sequences is a total relation modulo agreement, the following condition must be satisfied:

- for all $X, Y \in \mathcal{P}(I)$, if $X \cup Y$ is large then X is large or Y is large.

The above conditions suggest to determine the notion of a large set of positive integers by means of ultrafilters on I.

Ultrafilters

A set $U \in \mathcal{P}(\mathcal{P}(I))$ is said to have the finite intersection property iff the intersection of any finite number of elements of U is non empty. A set $U \in \mathcal{P}(\mathcal{P}(I))$ is said to be an ultrafilter on I iff

- $I \in U$,
- $\emptyset \notin U$,
- for all $X, Y \in \mathcal{P}(I)$, $X \cap Y \in U$ iff $X \in U$ and $Y \in U$,
- for all $X, Y \in \mathcal{P}(I)$, $X \cup Y \in U$ iff $X \in U$ or $Y \in U$.

These requirements imply that for all $X \in \mathcal{P}(I)$, $I \setminus X \in U$ iff $X \notin U$. A large supply of ultrafilters on I is provided by the ultrafilter theorem.

Proposition 1 *Let $U \in \mathcal{P}(\mathcal{P}(I))$. If U has the finite intersection property then there exists a set $U' \in \mathcal{P}(\mathcal{P}(I))$ such that $U \subseteq U'$ and U' is an ultrafilter on I.*

Let $n \in I$ be a positive integer. Consider the set $U_n = \{X \in \mathcal{P}(I)$: $n \in X\}$. Clearly, U_n is an ultrafilter on I. We call such ultrafilters the principal ultrafilters on I. Let $U_\omega = \{X \in \mathcal{P}(I)$: $I \setminus X$ is finite$\}$. As the reader can easily ascertain, U_ω has the finite intersection property. Hence, by the ultrafilter theorem, there exists a set $U'_\omega \in \mathcal{P}(\mathcal{P}(I))$ such that $U_\omega \subseteq U'_\omega$ and U'_ω is an ultrafilter on I. Such ultrafilters are called the non principal ultrafilters on I. It is a well-known fact that principal ultrafilters and non principal ultrafilters constitute a partition of the set of all ultrafilters on I.

Hyperreals

Let U be an ultrafilter on I. We define a binary relation $\equiv_U$ on $\mathbb{R}^I$ by putting

- $\mathbf{a} \equiv_U \mathbf{b}$ iff $\{n \in I: \mathbf{a}(n) = \mathbf{b}(n)\} \in U$

for each $\mathbf{a}, \mathbf{b} \in \mathbb{R}^I$. Note that $\equiv_U$ is an equivalence relation on $\mathbb{R}^I$. Given $\mathbf{a} \in \mathbb{R}^I$, we call the set of all $\mathbf{b} \in \mathbb{R}^I$ such that $\mathbf{a} \equiv_U \mathbf{b}$, denoted by $\mid \mathbf{a} \mid_{\equiv_U}$, the equivalence class with $\mathbf{a}$ as its representative modulo $\equiv_U$. The set of all equivalence classes modulo $\equiv_U$, denoted by $\mathbb{R}^I_{\mid \equiv_U}$, is called the quotient set of $\mathbb{R}^I$ modulo $\equiv_U$. We call the elements of $\mathbb{R}$ the real numbers while the elements of $\mathbb{R}^I_{\mid \equiv_U}$ are called the hyperreal numbers modulo $\equiv_U$. On $\mathbb{R}^I_{\mid \equiv_U}$, we define the binary relation $\prec_{\mid \equiv_U}$ and the binary operations $\oplus_{\mid \equiv_U}$ and $\otimes_{\mid \equiv_U}$ by putting

- $\mid \mathbf{a} \mid_{\equiv_U} \prec_{\mid \equiv_U} \mid \mathbf{b} \mid_{\equiv_U}$ iff $\{n \in I: \mathbf{a}(n) < \mathbf{b}(n)\} \in U$,

- $|\,\mathbf{a}\,|_{\equiv_U} \oplus_{|\equiv_U} |\,\mathbf{b}\,|_{\equiv_U}$ is $|\,\mathbf{a}+\mathbf{b}\,|_{\equiv_U}$,
- $|\,\mathbf{a}\,|_{\equiv_U} \otimes_{|\equiv_U} |\,\mathbf{b}\,|_{\equiv_U}$ is $|\,\mathbf{a}\times\mathbf{b}\,|_{\equiv_U}$

for each $\mathbf{a}, \mathbf{b} \in \mathbb{R}^I$. The binary relation $\prec_{|\equiv_U}$ and the binary operations $\oplus_{|\equiv_U}$ and $\otimes_{|\equiv_U}$ are well-defined seeing that for all $\mathbf{a}', \mathbf{b}' \in \mathbb{R}^I$ and for all $\mathbf{a}'', \mathbf{b}'' \in \mathbb{R}^I$, if $\mathbf{a}' \equiv_U \mathbf{a}''$ and $\mathbf{b}' \equiv_U \mathbf{b}''$ then $\{n \in I\colon \mathbf{a}'(n) < \mathbf{b}'(n)\} \in U$ iff $\{n \in I\colon \mathbf{a}''(n) < \mathbf{b}''(n)\} \in U$, $|\,\mathbf{a}'+\mathbf{b}'\,|_{\equiv_U} = |\,\mathbf{a}''+\mathbf{b}''\,|_{\equiv_U}$ and $|\,\mathbf{a}'\times\mathbf{b}'\,|_{\equiv_U} = |\,\mathbf{a}''\times\mathbf{b}''\,|_{\equiv_U}$.

Proposition 2 *The structure* $\langle \mathbb{R}^I_{|\equiv_U}, \prec_{|\equiv_U}, \oplus_{|\equiv_U}, \otimes_{|\equiv_U}\rangle$ *is an ordered field.*

For all reals $r \in \mathbb{R}$, we define the real-valued sequence $\mathbf{r} \in \mathbb{R}^I$ by putting

- $\mathbf{r}(n) = r$

for each $n \in I$.

Proposition 3 *The map* $r \in \mathbb{R} \mapsto |\,\mathbf{r}\,|_{|\equiv_U} \in \mathbb{R}^I_{|\equiv_U}$ *is an ordered-preserving field isomorphism from* $\mathbb{R}$ *into* $\mathbb{R}^I_{|\equiv_U}$.

The construction of $\mathbb{R}^I_{|\equiv_U}$ as the quotient set of $\mathbb{R}^I$ modulo $\equiv_U$ depends on the choice of the ultrafilter U on I. It has been shown that

- for all principal ultrafilters U on I, $\langle \mathbb{R}^I_{|\equiv_U}, \prec_{|\equiv_U}, \oplus_{|\equiv_U}, \otimes_{|\equiv_U}\rangle$ is isomorphic to $\langle \mathbb{R}, <, +, \times\rangle$,
- for all non principal ultrafilters U', U'' on I, $\langle \mathbb{R}^I_{|\equiv_{U'}}, \prec_{|\equiv_{U'}}, \oplus_{|\equiv_{U'}}, \otimes_{|\equiv_{U'}}\rangle$ is isomorphic to $\langle \mathbb{R}^I_{|\equiv_{U''}}, \prec_{|\equiv_{U''}}, \oplus_{|\equiv_{U''}}, \otimes_{|\equiv_{U''}}\rangle$.

Infinitesimal, unlimited and appreciable hyperreals

Let U be a fixed non principal ultrafilter on I. Given $\mathbf{a} \in \mathbb{R}^I$, we will denote more briefly as $|\,\mathbf{a}\,|$ the equivalence class $|\,\mathbf{a}\,|_{|\equiv_U}$ with $\mathbf{a}$ as its representative modulo $\equiv_U$. The quotient set $\mathbb{R}^I_{|\equiv_U}$ of $\mathbb{R}^I$ modulo $\equiv_U$ will be denoted more briefly as $^\star\mathbb{R}$. We will denote more briefly as $\prec^\star$ the binary relation $\prec_{|\equiv_U}$ on $\mathbb{R}^I_{|\equiv_U}$. The binary operations $\oplus_{|\equiv_U}$ and $\otimes_{|\equiv_U}$ on $\mathbb{R}^I_{|\equiv_U}$ will be denoted more briefly as $\oplus^\star$ and $\otimes^\star$. We shall say that the hyperreal $|\,\mathbf{a}\,| \in {}^\star\mathbb{R}$ is infinitesimal iff $|\,-r\,| \prec^\star |\,\mathbf{a}\,|$ and $|\,\mathbf{a}\,| \prec^\star |\,\mathbf{r}\,|$ for each real $r \in \mathbb{R}$ such that $r > 0$. For example, if $\epsilon \in \mathbb{R}^I$ is the real-valued sequence defined by putting

- $\epsilon(n) = 1/n$

for each $n \in I$ then $|\,\epsilon\,|$ is infinitesimal. The hyperreal $|\,\mathbf{a}\,| \in {}^\star\mathbb{R}$ is said to be unlimited iff $|\,\mathbf{a}\,| \prec^\star |\,-r\,|$ or $|\,\mathbf{r}\,| \prec^\star |\,\mathbf{a}\,|$ for each real $r \in \mathbb{R}$ such that $r > 0$. For example, if $\omega \in \mathbb{R}^I$ is the real-valued sequence defined by putting

- $\omega(n) = n$

for each $n \in I$ then $|\,\omega\,|$ is unlimited. We shall say that the hyperreal $|\,\mathbf{a}\,| \in {}^\star\mathbb{R}$ is appreciable iff $|\,\mathbf{a}\,|$ is neither infinitesimal nor unlimited. Hence, on $^\star\mathbb{R}$, we define the binary relations $\prec^\star_\epsilon$, $\prec^\star_\omega$ and $\prec^\star_1$ by putting

- $|\,\mathbf{a}\,| \prec^\star_\epsilon |\,\mathbf{b}\,|$ iff $|\,\mathbf{a}\,| \prec^\star |\,\mathbf{b}\,|$ and $|\,\mathbf{b}-\mathbf{a}\,|$ is infinitesimal,
- $|\,\mathbf{a}\,| \prec^\star_\omega |\,\mathbf{b}\,|$ iff $|\,\mathbf{a}\,| \prec^\star |\,\mathbf{b}\,|$ and $|\,\mathbf{b}-\mathbf{a}\,|$ is unlimited,
- $|\,\mathbf{a}\,| \prec^\star_1 |\,\mathbf{b}\,|$ iff $|\,\mathbf{a}\,| \prec^\star |\,\mathbf{b}\,|$ and $|\,\mathbf{b}-\mathbf{a}\,|$ is appreciable

for each $\mathbf{a}, \mathbf{b} \in \mathbb{R}^I$.

Non-standard first-order theory

The first-order theory for hyperreal time is presented in this section. We establish a complete axiomatization and prove decidability/complexity results.

Syntax

It is now time to meet the non-standard first-order language we will be working with. Let Var denote a countable set of individual variables (with typical members denoted x, y, etc). The set of all well-formed formulas (with typical members denoted ϕ, ψ, etc) of the non-standard first-order language is given by the rule

- $\phi ::= \bot \mid \neg\phi \mid (\phi' \vee \phi'') \mid \forall x\, \phi \mid x = y \mid x <_\epsilon y \mid x <_\omega y \mid x <_1 y$

where x and y range over Var. The intended meanings of the non logical constructs $<_\epsilon$, $<_\omega$ and $<_1$ are as follows:

- $x <_\epsilon y$: "x precedes y and $y - x$ is infinitesimal",
- $x <_\omega y$: "x precedes y and $y - x$ is unlimited",
- $x <_1 y$: "x precedes y and $y - x$ is appreciable".

We adopt the standard definitions for the remaining Boolean operations and for the existential quantifier. The notion of a subformula is standard. We adopt the standard rules for omission of the parentheses. A number of other constructs can be defined in terms of the primitive ones as follows:

- $x <_{\epsilon\omega} y ::= x <_\epsilon y \vee x <_\omega y$,
- $x <_{\epsilon 1} y ::= x <_\epsilon y \vee x <_1 y$,
- $x <_{\omega 1} y ::= x <_\omega y \vee x <_1 y$.

The length for a formula ϕ, denoted by $|\,\phi\,|$, is defined to be the number of symbols in ϕ. Formulas in which every individual variable in an atomic subformula is in the scope of a corresponding quantifier are called sentences.

Semantics

Models for the non-standard first-order language are 4-tuples $M = \langle \mathcal{R}, \prec_\epsilon, \prec_\omega, \prec_1\rangle$ where $\mathcal{R}$ is a non empty set of instants and $\prec_\epsilon$, $\prec_\omega$ and $\prec_1$ are binary relations on $\mathcal{R}$. An assignment on M is a function f that assigns an element $f(x)$ of $\mathcal{R}$ to each $x \in Var$. Satisfaction is a 3-place relation $\models$ between a model $M = \langle \mathcal{R}, \prec_\epsilon, \prec_\omega, \prec_1\rangle$, an assignment f on M and a formula ϕ. It is inductively defined as usual. In particular,

- $M \models_f x <_\epsilon y$ iff $f(x) \prec_\epsilon f(y)$,
- $M \models_f x <_\omega y$ iff $f(x) \prec_\omega f(y)$,
- $M \models_f x <_1 y$ iff $f(x) \prec_1 f(y)$.

Notice that the satisfaction of $M \models_f \phi$ depends only on the values of f for those individual variables which are free in ϕ. In particular, if ϕ is a sentence then the satisfaction of $M \models_f \phi$ is completely independent of f.

A complete axiomatization

A model $M = \langle \mathcal{R}, \prec_\epsilon, \prec_\omega, \prec_1\rangle$ is said to be standard iff it satisfies the following sentences:

$IRRE$ • $\forall x\, x \not<_\epsilon x$,

- $\forall x\, x \not<_\omega x$,
- $\forall x\, x \wedge x \not<_1 x$,

$DISJ$ • $\forall x\, \forall y\, (x <_\epsilon y \rightarrow x \not<_\omega y \wedge x \not<_1 y)$,
- $\forall x\, \forall y\, (x <_\omega y \rightarrow x \not<_\epsilon y \wedge x \not<_1 y)$,
- $\forall x\, \forall y\, (x <_1 y \rightarrow x \not<_\epsilon y \wedge x \not<_\omega y)$,

$TRAN$ • $\forall x\, \forall y\, (\exists z\, (x <_\epsilon z \wedge z <_\epsilon y) \rightarrow x <_\epsilon y)$,
- $\forall x\, \forall y\, (\exists z\, (x <_\epsilon z \wedge z <_\omega y) \rightarrow x <_\omega y)$,
- $\forall x\, \forall y\, (\exists z\, (x <_\epsilon z \wedge z <_1 y) \rightarrow x <_1 y)$,
- $\forall x\, \forall y\, (\exists z\, (x <_\omega z \wedge z <_\epsilon y) \rightarrow x <_\omega y)$,
- $\forall x\, \forall y\, (\exists z\, (x <_\omega z \wedge z <_\omega y) \rightarrow x <_\omega y)$,
- $\forall x\, \forall y\, (\exists z\, (x <_\omega z \wedge z <_1 y) \rightarrow x <_\omega y)$,
- $\forall x\, \forall y\, (\exists z\, (x <_1 z \wedge z <_\epsilon y) \rightarrow x <_1 y)$,
- $\forall x\, \forall y\, (\exists z\, (x <_1 z \wedge z <_\omega y) \rightarrow x <_\omega y)$,
- $\forall x\, \forall y\, (\exists z\, (x <_1 z \wedge z <_1 y) \rightarrow x <_1 y)$,

$UNIV$ • $\forall x\, \forall y\, (x = y \vee x <_\epsilon y \vee x <_\omega y \vee x <_1 y \vee y <_\epsilon x \vee y <_\omega x \vee y <_1 x)$,

$DENS$ • $\forall x\, \forall y\, (x <_\epsilon y \rightarrow \exists z\, (x <_\epsilon z \wedge z <_\epsilon y))$,
- $\forall x\, \forall y\, (x <_\omega y \rightarrow \exists z\, (x <_\epsilon z \wedge z <_\omega y))$,
- $\forall x\, \forall y\, (x <_1 y \rightarrow \exists z\, (x <_\epsilon z \wedge z <_1 y))$,
- $\forall x\, \forall y\, (x <_\omega y \rightarrow \exists z\, (x <_\omega z \wedge z <_\epsilon y))$,
- $\forall x\, \forall y\, (x <_\omega y \rightarrow \exists z\, (x <_\omega z \wedge z <_\omega y))$,
- $\forall x\, \forall y\, (x <_\omega y \rightarrow \exists z\, (x <_\omega z \wedge z <_1 y))$,
- $\forall x\, \forall y\, (x <_1 y \rightarrow \exists z\, (x <_1 z \wedge z <_\epsilon y))$,
- $\forall x\, \forall y\, (x <_\omega y \rightarrow \exists z\, (x <_1 z \wedge z <_\omega y))$,
- $\forall x\, \forall y\, (x <_1 y \rightarrow \exists z\, (x <_1 z \wedge z <_1 y))$,

$PRED$ • $\forall x\, \exists y\, y <_\epsilon x$,
- $\forall x\, \exists y\, y <_\omega x$,
- $\forall x\, \exists y\, y <_1 x$,

$SUCC$ • $\forall x\, \exists y\, x <_\epsilon y$,
- $\forall x\, \exists y\, x <_\omega y$,
- $\forall x\, \exists y\, x <_1 y$.

Note that the model $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star \rangle$ is standard. Obviously, the set of all the above sentences has not the finite model property. Nevertheless, seeing that it has infinite models, it has models in each infinite power. To illustrate the truth of this, let $\langle S, <_S \rangle$, $\langle T, <_T \rangle$ and $\langle U, <_U \rangle$ be total dense orderings without endpoints. If $\mathcal{R} = S \times T \times U$ then the model $M = \langle \mathcal{R}, \prec_\epsilon, \prec_\omega, \prec_1 \rangle$ defined by putting

- $(s, t, u) \prec_\epsilon (s', t', u')$ iff $s = s'$, $t = t'$ and $u <_U u'$,
- $(s, t, u) \prec_\omega (s', t', u')$ iff $s <_S s'$,
- $(s, t, u) \prec_1 (s', t', u')$ iff $s = s'$ and $t <_T t'$

is standard. We call it the $3D$ model defined over $\mathcal{R}$. Let $M = \langle \mathcal{R}, \prec_\epsilon, \prec_\omega, \prec_1 \rangle$ be a standard model. We define the binary relations $\sim_\epsilon$ and $\sim_1$ on $\mathcal{R}$ by putting

- $a \sim_\epsilon b$ iff $a = b$ or $a \prec_\epsilon b$ or $b \prec_\epsilon a$,
- $a \sim_1 b$ iff $a \sim_\epsilon b$ or $a \prec_1 b$ or $b \prec_1 a$

for each $a, b \in \mathcal{R}$. It is a simple matter to check that $\sim_\epsilon$ and $\sim_1$ are equivalence relations on $\mathcal{R}$ such that every equivalence class in $\mathcal{R}$ modulo $\sim_\epsilon$ consists of infinitely many instants, every equivalence class in $\mathcal{R}$ modulo $\sim_1$ is made up of infinitely many equivalence classes modulo $\sim_\epsilon$ and $\mathcal{R}$ consists of infinitely many equivalence classes modulo $\sim_1$. In the sequel, the following notations will be used for all $a \in \mathcal{R}$:

- $[a]_\epsilon = \{b \in \mathcal{R}\colon a \sim_\epsilon b\}$,
- $[a]_1 = \{b \in \mathcal{R}\colon a \sim_1 b\}$.

Let $M = \langle \mathcal{R}, \prec_\epsilon, \prec_\omega, \prec_1 \rangle$ and $M' = \langle \mathcal{R}', \prec_\epsilon', \prec_\omega', \prec_1' \rangle$ be models. M is said to be elementary embeddable in M' iff there exists a mapping σ on $\mathcal{R}$ into $\mathcal{R}'$ such that for all formulas ϕ and for all assignments f on M, $M \models_f \phi$ iff $M' \models_{\sigma \circ f} \phi$. To illustrate the value of countable standard models, we shall prove the following proposition:

Proposition 4 *Let $M = \langle \mathcal{R}, \prec_\epsilon, \prec_\omega, \prec_1 \rangle$ and $M' = \langle \mathcal{R}', \prec_\epsilon', \prec_\omega', \prec_1' \rangle$ be standard models. If M is countable then M is elementary embeddable in M'.*

Let $M = \langle \mathcal{R}, \prec_\epsilon, \prec_\omega, \prec_1 \rangle$ and $M' = \langle \mathcal{R}', \prec_\epsilon', \prec_\omega', \prec_1' \rangle$ be models. We shall say that M and M' are elementary equivalent iff M and M' satisfy the same sentences. Let us remark that for all models $M = \langle \mathcal{R}, \prec_\epsilon, \prec_\omega, \prec_1 \rangle$ and $M' = \langle \mathcal{R}', \prec_\epsilon', \prec_\omega', \prec_1' \rangle$, if M is elementary embeddable in M' then M and M' are elementary equivalent. As a corollary of proposition 4 we obtain that

Proposition 5 *Any two standard models are elementary equivalent.*

The first-order theory HY of standard models has the following list of proper axioms: $IRRE$, $DISJ$, $TRAN$, $UNIV$, $DENS$, $PRED$ and $SUCC$. There are several results about HY:

Proposition 6 *1. HY is countably categorical, i.e. HY has only, up to isomorphism, one countable model.*

2. HY is not categorical in any uncountable power, i.e. HY has non isomorphic models in any uncountable power.

3. HY is maximal consistent, i.e. HY is a maximal first-order theory with respect to the inclusion relation among consistent first-order theories.

4. HY is complete with respect to $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star \rangle$, i.e. every sentence satisfied in $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star \rangle$ is in HY.

Decidability/complexity results

In this section, we address the problem of the computational complexity of HY. The results are summarized in the following proposition:

Proposition 7 *1. HY is decidable, i.e. there is a mechanical procedure to determine whether or not a given sentence is in HY.*

2. The membership problem in HY is $PSPACE$-complete, i.e. the membership problem in HY is solvable by a deterministic algorithm using a polynomial amount of space whereas any problem solvable by a deterministic algorithm using a polynomial amount of space can be reduced to the membership problem in HY.

Let $M = \langle \mathcal{R}, \prec_\epsilon, \prec_\omega, \prec_1 \rangle$ be a model, R be a binary relation on $\mathcal{R}$ and $\mathcal{L}$ be a restriction of the non-standard first-order language. We shall say that R is definable with $\mathcal{L}$ in M iff there exists a formula $\phi(x, y)$ in $\mathcal{L}$ such that for all assignments f on M, $f(x)\ R\ f(y)$ iff $M \models_f \phi(x, y)$. Notice that

$\cdot$	$\cdot^{-1}$
$<_\omega$	$>_\omega$
$<_1$	$>_1$
$<_\epsilon$	$>_\epsilon$
$>_\epsilon$	$<_\epsilon$
$>_1$	$<_1$
$>_\omega$	$<_\omega$

Table 1: The operation of converse on the atoms.

$\cdot$	$\cdot\circ <_\omega$	$\cdot\circ <_1$	$\cdot\circ <_\epsilon$	$\cdot\circ >_\epsilon$	$\cdot\circ >_1$	$\cdot\circ >_\omega$
$<_\omega$	$\{<_\omega\}$	$\{<_\omega\}$	$\{<_\omega\}$	$\{<_\omega\}$	$\{<_\omega\}$	all_7
$<_1$	$\{<_\omega\}$	$\{<_1\}$	$\{<_1\}$	$\{<_1\}$	all_5	$\{>_\omega\}$
$<_\epsilon$	$\{<_\omega\}$	$\{<_1\}$	$\{<_\epsilon\}$	all_3	$\{>_1\}$	$\{>_\omega\}$
$>_\epsilon$	$\{<_\omega\}$	$\{<_1\}$	all_3	$\{>_\epsilon\}$	$\{>_1\}$	$\{>_\omega\}$
$>_1$	$\{<_\omega\}$	all_5	$\{>_1\}$	$\{>_1\}$	$\{>_1\}$	$\{>_\omega\}$
$>_\omega$	all_7	$\{>_\omega\}$	$\{>_\omega\}$	$\{>_\omega\}$	$\{>_\omega\}$	$\{>_\omega\}$

Table 2: Composition of primitive relations in hyperpoint algebra.

Proposition 8 *1. $=$ is definable with $<_\epsilon$ in any standard model.*

2. $=$ is not definable with $<_\omega$ and $<_1$ in any standard model.

3. $\prec_\epsilon$ is not definable with $=$, $<_\omega$ and $<_1$ in any standard model.

4. $\prec_\omega$ is not definable with $=$, $<_\epsilon$ and $<_1$ in any standard model.

5. $\prec_1$ is not definable with $=$, $<_\epsilon$ and $<_\omega$ in any standard model.

Moreover,

Proposition 9 *1. $=$ is definable with $<_{\epsilon\omega}$ in any standard model,*

2. $=$ is definable with $<_{\epsilon 1}$ in any standard model,

3. $=$ is not definable with $<_{\omega 1}$ in any standard model.

4. $\prec_\epsilon$ is definable with $<_{\epsilon\omega}$ in any standard model,

5. $\prec_\epsilon$ is not definable with $=$ and $<_{\epsilon 1}$ in any standard model,

6. $\prec_\epsilon$ is not definable with $=$ and $<_{\omega 1}$ in any standard model.

7. $\prec_\omega$ is definable with $<_{\epsilon\omega}$ in any standard model,

8. $\prec_\omega$ is not definable with $=$ and $<_{\epsilon 1}$ in any standard model,

9. $\prec_\omega$ is not definable with $=$ and $<_{\omega 1}$ in any standard model.

10. $\prec_1$ is not definable with $=$ and $<_{\epsilon\omega}$ in any standard model,

11. $\prec_1$ is not definable with $=$ and $<_{\epsilon 1}$ in any standard model,

12. $\prec_1$ is not definable with $=$ and $<_{\omega 1}$ in any standard model.

Non-standard qualitative approach

The qualitative approach for hyperreal time is presented in this section. We establish qualitative constraint satisfaction problems and prove tractability results.

Hyperpoint algebra

The hyperpoint algebra, say $\mathcal{H}$, has 7 atoms: $<_\omega, <_1, <_\epsilon, =$, $>_\epsilon$, $>_1$ and $>_\omega$. $\mathcal{H}$'s identity, also denoted as id, is $=$. The operation $^{-1}$ of converse on the atoms but id is given in table 1. Of course, the converse of id is id itself. Composition $\circ$ of primitive relations but id in hyperpoint algebra is defined by table 2, where all_3 is the relation $\{<_\epsilon, id, >_\epsilon\}$, all_5 is the relation $\{<_1, <_\epsilon, id, >_\epsilon, >_1\}$ and all_7 is the relation $\{<_\omega, <_1, <_\epsilon, id, >_\epsilon, >_1, >_\omega\}$. Evidently, the composition of a primitive relation with id is the primitive relation itself.

A natural representation of $\mathcal{H}$ is obtained by taking as base all hyperreal numbers. Each of the 7 atoms is then interpreted in the obvious way. We will denote more briefly as $\mathcal{B}_{HY}$ the set of all $\mathcal{H}$'s atoms. The set of all $\mathcal{H}$'s relations is defined as $\mathcal{P}(\mathcal{B}_{HY})$, the power set of $\mathcal{B}_{HY}$. It contains 2^7 relations. Each relation of $\mathcal{P}(\mathcal{B}_{HY})$ can be seen as the union of its atoms. The operation of converse on the relations of $\mathcal{P}(\mathcal{B}_{HY})$ and the composition of relations in $\mathcal{P}(\mathcal{B}_{HY})$ are defined in the following way:

- $R^{-1} = \{A^{-1} \colon A \in R\}$,
- $R \circ S = \bigcup\{A \circ B \colon A \in R \text{ and } B \in S\}$.

Notice that for all $a, b \in {}^\star\mathbb{R}$, if there exists $c \in {}^\star\mathbb{R}$ such that the pair $\langle a, c\rangle$ satisfies the relation $R \in \mathcal{P}(\mathcal{B}_{HY})$ and the pair $\langle c, b\rangle$ satisfies the relation $S \in \mathcal{P}(\mathcal{B}_{HY})$ then $\langle a, b\rangle$ satisfies $R \circ S$. As a result of these definitions, we can prove that

Proposition 10 *The structure $\langle \mathcal{P}(\mathcal{B}_{HY}), \emptyset, -, \cup, id, ^{-1}, \circ\rangle$ is a relational algebra.*

We will say that a subset of $\mathcal{P}(\mathcal{B}_{HY})$ is a subclass iff it is closed under the operations of intersection, converse and composition. Given a subset E of $\mathcal{P}(\mathcal{B}_{HY})$, we use $sc(E)$ to denote the least subclass which contains E. By items 3, 4 and 5 of proposition 8,

- $<_\epsilon \notin sc(\{id, <_\omega, <_1\})$,
- $<_\omega \notin sc(\{id, <_\epsilon, <_1\})$,
- $<_1 \notin sc(\{id, <_\epsilon, <_\omega\})$.

Convex closure

Let us arrange $\mathcal{H}$'s atoms in a partial order illustrated in figure 1 which defines a lattice called the hyperpoint lattice.

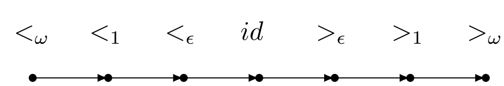

Figure 1.

The convex relations of $\mathcal{H}$ correspond to the intervals in the lattice. For each relation R of $\mathcal{H}$, there exists a least convex relation of $\mathcal{H}$ which contains R. According to Ligozat's notation (Ligozat 1998), we denote this relation by $\mathbf{I}(R)$. Notice that for all $a, b \in {}^\star\mathbb{R}$, if there exists $c \in {}^\star\mathbb{R}$ such that the pair $\langle a, c\rangle$ satisfies the relation $R \in \mathcal{P}(\mathcal{B}_{HY})$ and

$\cdot$	$\dim(\cdot)$
$<_\omega$	3
$<_1$	2
$<_\epsilon$	1
id	0
$>_\epsilon$	1
$>_1$	2
$>_\omega$	3

Table 3: The dimension of $\mathcal{H}$'s atoms.

the pair $\langle c, b\rangle$ satisfies the relation $S \in \mathcal{P}(\mathcal{B}_{HY})$ then $\langle a, b\rangle$ satisfies $\mathbf{I}(R \circ S)$. Obviously, $\mathbf{I}(\mathbf{I}(R)) = \mathbf{I}(R)$ for each $R \in \mathcal{P}(\mathcal{B}_{HY})$. Moreover, we can prove that

Proposition 11 *Let $R, S \in \mathcal{P}(\mathcal{B}_{HY})$.*

1. *If $R \subseteq S$ then $\mathbf{I}(R) \subseteq \mathbf{I}(S)$.*
2. $\mathbf{I}(R^{-1}) = \mathbf{I}(R)^{-1}$.
3. $\mathbf{I}(R \circ S) = \mathbf{I}(R) \circ \mathbf{I}(S)$.

It follows that

Proposition 12 *The set $Conv(\mathcal{H})$ of all convex relations of $\mathcal{H}$ is a subclass.*

For illustration, the convex relations of $\mathcal{H}$ are

- $\emptyset$,
- $\{<_\omega\}, \{<_1\}, \{<_\epsilon\}, \{id\}, \{>_\epsilon\}, \{>_1\}, \{>_\omega\}$,
- $\{<_\omega, <_1\}, \{<_1, <_\epsilon\}, \{<_\epsilon, id\}, \{id, >_\epsilon\}, \{>_\epsilon, >_1\}, \{>_1, >_\omega\}$,
- $\{<_\omega, <_1, <_\epsilon\}, \{<_1, <_\epsilon, id\}, \{<_\epsilon, id, >_\epsilon\}, \{id, >_\epsilon, >_1\}, \{>_\epsilon, >_1, >_\omega\}$,
- $\{<_\omega, <_1, <_\epsilon, id\}, \{<_1, <_\epsilon, id, >_\epsilon\}, \{<_\epsilon, id, >_\epsilon, >_1\}, \{id, >_\epsilon, >_1, >_\omega\}$,
- $\{<_\omega, <_1, <_\epsilon, id, >_\epsilon\}, \{<_1, <_\epsilon, id, >_\epsilon, >_1\}, \{<_\epsilon, id, >_\epsilon, >_1, >_\omega\}$,
- $\{<_\omega, <_1, <_\epsilon, id, >_\epsilon, >_1\}, \{<_1, <_\epsilon, id, >_\epsilon, >_1, >_\omega\}$,
- $\{<_\omega, <_1, <_\epsilon, id, >_\epsilon, >_1, >_\omega\}$.

The dimension of $\mathcal{H}$'s atoms is defined by table 3. We extend the dimension to any relation R of $\mathcal{H}$ in the following way:

- $\dim(R) = \max\{\dim(A)\colon A \in R\}$.

We can prove that

Proposition 13 *Let $R, S \in \mathcal{P}(\mathcal{B}_{HY})$.*

1. *If $R \subseteq S$ then $\dim(R) \leq \dim(S)$.*
2. $\dim(R^{-1}) = \dim(R)$.
3. $\dim(R \circ S) = \max\{\dim(R), \dim(S)\}$.

Let $R \in \mathcal{P}(\mathcal{B}_{HY})$. Obviously, $\dim(\mathbf{I}(R) \setminus R) < \dim(\mathbf{I}(R))$.

Qualitative constraint satisfaction problems

We represent the information about the relative positions between hyperpoints by a particular constraint satisfaction problem: a hyperpoint network. The variables of such a constraint network represent some hyperpoints and the constraints are defined by relations of $\mathcal{H}$. More formally, a hyperpoint network is a structure of the form $\mathcal{N} = \langle V, C\rangle$ where V is a finite set of variables and C is a function assigning to each pair $\langle x, y\rangle$ of V's variable a relation $C(x, y)$ of $\mathcal{H}$. $\mathcal{N}$ is said to be non empty iff

- for all $x, y \in V$, $C(x, y) \neq \emptyset$.

Given a hyperpoint network, the main problem is to know whether it is consistent, i.e. whether we can associate with each one of its variables a hyperreal so that its constraints are satisfied. We will call this problem the consistency problem. Let $\mathcal{N} = \langle V, C\rangle$ be a hyperpoint network. We shall say that $\mathcal{N}$ is path-consistent (respectively: weakly path-consistent) iff

- for all $x, y, z \in V$, $C(x, z) \circ C(z, y) \supseteq C(x, y)$ (respectively: $\mathbf{I}(C(x, z) \circ C(z, y)) \supseteq C(x, y)$).

Given a hyperpoint network $\mathcal{N} = \langle V, C\rangle$, the path-consistency (respectively: weak path-consistency) method consists in iterating the triangulation operation

- for all $x, y, z \in V$, $C(x, y) := (C(x, z) \circ C(z, y)) \cap C(x, y)$ (respectively: $C(x, y) := \mathbf{I}(C(x, z) \circ C(z, y)) \cap C(x, y)$)

until a fixed point is reached. This method is polynomial, seeing that it can be implemented in $\mathcal{O}(Card(V))^3$. Notice that

Proposition 14 *The path-consistency (respectively: weak path-consistency) method is sound, i.e. it does not remove an atom which participates to a consistent instantiation of the given hyperpoint network.*

To close this section, we establish the following propositions:

Proposition 15 *Let $\mathcal{N} = \langle V, C\rangle$ be a hyperpoint network. If $\mathcal{N}$ is weakly path-consistent then $\mathbf{I}(\mathcal{N})$ is path-consistent.*

Proposition 16 *Let $\mathcal{N} = \langle V, C\rangle$ be a hyperpoint network. If $\mathcal{N}$ is convex and path-consistent then either it is empty or it admits a consistent instantiation of maximal dimension.*

Tractability results

In this section, we address the problem of the tractability of the consistency problem. The desired result is given in the following proposition:

Proposition 17 *The consistency problem is in P, i.e. the consistency problem is solvable by a deterministic algorithm using a polynomial amount of time.*

Conclusion

There has been a considerable amount of work that has looked at the issues of using non-standard analysis in computer science and artificial intelligence: Gagné and Plaice [Gagné and Plaice (1996)] present a non-standard temporal database system, Nakamura and Fusaoka [Nakamura and Fusaoka (2007)] present a non-standard interpretation of hybrid automata, Weld [Weld (1990)] presents a non-standard technique for analysing real-world systems, etc. In these papers, the possibility of ordering events within a single instant is of the utmost importance.

Formal languages in which one can assign a proper meaning to the association of statements about different grained temporal domains have been considered whereas several papers use granularities to address the problem of representing calendars and reasoning about calendars. See [Euzenat and Montanari (2005)] for a survey. Nevertheless, it seems that the non-standard first-order theory of representing time and the non-standard qualitative approach to reasoning about time presented in this paper constitute the first step towards a logic of time based on the hyperreals.

Much remains to be done. For example, adding the symbols $+$ and $\times$ for the binary operations $\oplus^\star$ and $\otimes^\star$ on ${}^\star\mathbb{R}$, we may want to add some arithmetic to our non-standard first-order language. By proposition 2, we know that the structure $\langle {}^\star\mathbb{R}, \prec^\star, \oplus^\star, \otimes^\star \rangle$ is an ordered field. Nevertheless, we do not know anything about the complete axiomatization and the decidability/complexity of the set of all sentences in this extended non-standard first-order language that are satisfied in the structure $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star \rangle$.

An even further extension of our non-standard first-order language would be the following: add non-standard formulas of the form $x \ll y$ with intended meaning "x's order of magnitude is less than y's order of magnitude". Davis [Davis (1999)] presents algorithms using a polynomial amount of time that can solve sets of constraints of the form "the order of magnitude of the Euclidean distance between points x and y is less than the order of magnitude of the Euclidean distance between points z and t". Nevertheless, we do not know anything about the complete axiomatization and the decidability/complexity of the set of all sentences in this extended non-standard first-order language that are satisfied in the structure $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star \rangle$.

In other respects, there is the question of the development of a non-standard qualitative approach à la Allen [Allen (1983)] where one considers, in the structure $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star \rangle$, the intervals whose endpoints are within an appreciable distance. We do not know anything about the qualitative constraint satisfaction problems and the tractability results that one should consider within the framework of such appreciable intervals.

There is also the question of the development of a non-standard temporal logic based on the idea of associating with $<_\epsilon$, $<_\omega$ and $<_1$ the temporal connectives G_ϵ, G_ω and G_1 being read, in the structure $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star \rangle$, "at every instant within the infinitesimal future of the present instant, it will always going to be the case that ...", "at every instant within the unlimited future of the present instant, it will always going to be the case that ..." and "at every instant within the appreciable future of the present instant, it will always going to be the case that ...". We do not know anything about the complete axiomatization and the decidability/complexity of the set of all formulas in this non-standard temporal language that are valid in the structure $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star \rangle$.

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Appendix

Proof of proposition 1: See [Goldblatt (1998)]. ⊣
Proof of proposition 2: See [Goldblatt (1998)]. ⊣
Proof of proposition 3: See [Goldblatt (1998)]. ⊣
Proof of proposition 4: Let $M = \langle \mathcal{R}, \prec_\epsilon, \prec_\omega, \prec_1 \rangle$ and $M' = \langle \mathcal{R}', \prec'_\epsilon, \prec'_\omega, \prec'_1 \rangle$ be standard models. Suppose that M is countable, we demonstrate that M is elementary embeddable in M'. We need to consider an injective mapping σ on the $\sim_1$-partition of $\mathcal{R}$ into the $\sim'_1$-partition of $\mathcal{R}'$ such that for all $a, b \in \mathcal{R}$, if $a \prec_\omega b$ then for all $a' \in \sigma([a]_1)$ and for all $b' \in \sigma([b]_1)$, $a' \prec'_\omega b'$. For each equivalence class $[a]_1$ in the $\sim_1$-partition of $\mathcal{R}$, we also need an injective mapping $\tau_{[a]_1}$ on the $\sim_\epsilon$-partition of $[a]_1$ into the $\sim'_\epsilon$-partition of $\sigma([a]_1)$ such that for all $b, c \in [a]_1$, if $b \prec_1 c$ then for all $b' \in \tau_{[a]_1}([b]_\epsilon)$ and for all $c' \in \tau_{[a]_1}([c]_\epsilon)$, $b' \prec'_1 c'$. Finally, for each equivalence class $[a]_\epsilon$ in the $\sim_\epsilon$-partition of $\mathcal{R}$, we need to consider an injective mapping $\mu_{[a]_\epsilon}$ on $[a]_\epsilon$ into $\tau_{[a]_1}([a]_\epsilon)$ such that for all $b, c \in [a]_\epsilon$, if $b \prec_\epsilon c$ then $\mu_{[a]_\epsilon}(b) \prec'_\epsilon \mu_{[a]_\epsilon}(c)$. Now, let ν be the injective mapping on $\mathcal{R}$ into $\mathcal{R}'$ defined with

- $\nu(a) = \mu_{[a]_\epsilon}(a)$.

To see that ν is an elementary embedding of M into M', we invite the reader to show by induction on the complexity of formulas ϕ that for all assignments f on M, $M \models_f \phi$ iff $M' \models_{\nu \circ f} \phi$. ⊣
Proof of proposition 5: Let $M = \langle \mathcal{R}, \prec_\epsilon, \prec_\omega, \prec_1 \rangle$ and $M' = \langle \mathcal{R}', \prec'_\epsilon, \prec'_\omega, \prec'_1 \rangle$ be standard models. We demonstrate that M and M' are elementary equivalent. Let $\langle S, <_S \rangle$, $\langle T, <_T \rangle$ and $\langle U, <_U \rangle$ be countable total dense orderings without endpoints. If $\mathcal{R}'' = S \times T \times U$ then the $3D$ model $M'' = \langle \mathcal{R}'', \prec''_\epsilon, \prec''_\omega, \prec''_1 \rangle$ defined over $\mathcal{R}''$ is countable. By proposition 4, M'' is elementary embeddable in M and M'' is elementary embeddable in M'. Hence, M and M' are elementary equivalent. ⊣
Proof of proposition 6: **1**. Let $M = \langle \mathcal{R}, \prec_\epsilon, \prec_\omega, \prec_1 \rangle$ and $M' = \langle \mathcal{R}', \prec'_\epsilon, \prec'_\omega, \prec'_1 \rangle$ be standard models. Suppose that M and M' are countable, we demonstrate that M is isomorphic to M'. We need to consider a bijective mapping σ on the $\sim_1$-partition of $\mathcal{R}$ onto the $\sim'_1$-partition of $\mathcal{R}'$ such that for all $a, b \in \mathcal{R}$, if $a \prec_\omega b$ then for all $a' \in \sigma([a]_1)$ and for all $b' \in \sigma([b]_1)$, $a' \prec'_\omega b'$. For each equivalence class $[a]_1$ in the $\sim_1$-partition of $\mathcal{R}$, we also need a bijective mapping $\tau_{[a]_1}$ on the $\sim_\epsilon$-partition of $[a]_1$ onto the $\sim'_\epsilon$-partition of $\sigma([a]_1)$ such that for all $b, c \in [a]_1$, if $b \prec_1 c$ then for all $b' \in \tau_{[a]_1}([b]_\epsilon)$ and for all $c' \in \tau_{[a]_1}([c]_\epsilon)$, $b' \prec'_1 c'$. Finally, for each equivalence class $[a]_\epsilon$ in the $\sim_\epsilon$-partition of $\mathcal{R}$, we need to consider a bijective mapping $\mu_{[a]_\epsilon}$ on $[a]_\epsilon$ onto $\tau_{[a]_1}([a]_\epsilon)$ such that for all $b, c \in [a]_\epsilon$, if $b \prec_\epsilon c$ then $\mu_{[a]_\epsilon}(b) \prec'_\epsilon \mu_{[a]_\epsilon}(c)$. Now, let ν be the bijective mapping on $\mathcal{R}$ onto $\mathcal{R}'$ defined with

- $\nu(a) = \mu_{[a]_\epsilon}(a)$.

Obviously, ν is an isomorphism of M onto M'.
2. Let α be an uncountable power. We demonstrate that we can find standard models $M = \langle \mathcal{R}, \prec_\epsilon, \prec_\omega, \prec_1 \rangle$ and $M' = \langle \mathcal{R}', \prec'_\epsilon, \prec'_\omega, \prec'_1 \rangle$ of power α such that M is not isomorphic to M'. Let $\langle T, <_T \rangle$ be a countable total dense ordering without endpoints, $\langle U, <_U \rangle$ be a total dense ordering without endpoints of power α, $\langle T', <_{T'} \rangle$ be a total dense ordering without endpoints of power α and $\langle U', <_{U'} \rangle$ be a countable total dense ordering without endpoints. If $\mathcal{R} = \mathbb{R} \times T \times U$ and $\mathcal{R}' = \mathbb{R} \times T' \times U'$ then the $3D$ models $M = \langle \mathcal{R}, \prec_\epsilon, \prec_\omega, \prec_1 \rangle$ and $M' = \langle \mathcal{R}', \prec'_\epsilon, \prec'_\omega, \prec'_1 \rangle$ defined over $\mathcal{R}$ and $\mathcal{R}'$ are of power α. Seeing that images of equivalence classes in $\mathcal{R}$ modulo $\sim_\epsilon$ under an isomorphism are equivalence classes in $\mathcal{R}'$ modulo $\sim'_\epsilon$, a cardinality argument immediately gives that M is not isomorphic to M'.
3. By proposition 5.
4. Immediately follows from item 3, since the model $\langle {}^\star\mathbb{R}, \prec^\star_\epsilon, \prec^\star_\omega, \prec^\star_1 \rangle$ is standard. ⊣
Proof of proposition 7: **1**. Simple application of item 3 of proposition 6.
2. To show that the membership problem in HY is $PSPACE$-complete, we have to show that it is in the class $PSPACE$ and it is $PSPACE$-hard. Firstly, we demonstrate that the membership problem in HY is in the class $PSPACE$. It suffices to observe that the membership problem in HY is in the class AP, i.e. the membership problem in HY is solvable by an alternating algorithm using a polynomial amount of time. An alternating algorithm to solve the membership problem in HY consists, given a sentence $Q_1 x_1 \ \ldots Q_n x_n \ \phi(x_1, \ldots, x_n)$ in prenex normal form, in arranging all possible assignments of the variables $x_1, \ldots, x_n$ as the leaves of a tree of depth n: the root of the tree contains the assignment of x_1, then we branch on the assignment of x_2, then the assignment of x_3, and so on. We can turn this tree into a Boolean circuit, where all gates at the i-th level are $\wedge$ gates if Q_i is $\forall$ and $\vee$ gates otherwise. A leaf of the tree is "true" if the corresponding assignment satisfies ϕ and "false" otherwise. It is immediate from this construction that $Q_1 x_1 \ \ldots Q_n x_n \ \phi(x_1, \ldots, x_n)$ is in HY iff the value of this Boolean circuit is "true". Since $AP = PSPACE$ [Papadimitriou (1994)], then the membership problem in HY is in the class $PSPACE$. Secondly, we demonstrate that the membership problem in HY is $PSPACE$-hard. It suffices to observe that HY is a conservative extension of the first-order theory EQ^∞ of identity in all infinite models. Since the membership problem in EQ^∞ is $PSPACE$-hard [Balbiani and Tinchev (2007), Stockmeyer (1977)], then the membership problem in HY is $PSPACE$-hard. ⊣
Proof of proposition 8: By proposition 5, it suffices to consider definability in $\langle {}^\star\mathbb{R}, \prec^\star_\epsilon, \prec^\star_\omega, \prec^\star_1 \rangle$.
1. It suffices to observe that for all assignments f on $\langle {}^\star\mathbb{R}, \prec^\star_\epsilon, \prec^\star_\omega, \prec^\star_1 \rangle$, $f(x) = f(y)$ iff $\langle {}^\star\mathbb{R}, \prec^\star_\epsilon, \prec^\star_\omega, \prec^\star_1 \rangle \models_f \forall z \ (x <_\epsilon z \leftrightarrow y <_\epsilon z)$.

2. We demonstrate that $=$ is not definable with $<_\omega$ and $<_1$ in $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$. Assume that there exists a formula $\phi(x,y)$ in $<_\omega$ and $<_1$ such that for all assignments f on $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$, $f(x) = f(y)$ iff $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle \models_f \phi(x,y)$. Let $a \in {}^\star\mathbb{R}$ and $b', b'' \in [a]_\epsilon$ be such that $b' \neq b''$. We need to consider a surjective mapping σ on $[a]_\epsilon$ onto itself such that $\sigma(b') = \sigma(b'')$. Now, let τ be the surjective mappping on ${}^\star\mathbb{R}$ onto itself such that for all $c \in {}^\star\mathbb{R}$,

- if $c \in [a]_\epsilon$ then $\tau(c) = \sigma(c)$,
- if $b \notin [a]_\epsilon$ then $\tau(c) = c$.

Notice that $\tau(b') = \tau(b'')$. As a simple exercise, we invite the reader to show by induction on the complexity of formulas ψ in $<_\omega$ and $<_1$ that for all assignments f on $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$, $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle \models_f \psi$ iff $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle \models_{\tau\circ f} \psi$. Hence, for all assignments f on $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$, $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle \models_f \phi(x,y)$ iff $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle \models_{\tau\circ f} \phi(x,y)$. Thus, for all assignments f on $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$, $f(x) = f(y)$ iff $\tau(f(x)) = \tau(f(y))$. If f is an assignment on $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$ such that $f(x) = b'$ and $f(y) = b''$ then $b' = b''$ iff $\tau(b') = \tau(b'')$: a contradiction.
3. We demonstrate that $\prec_\epsilon^\star$ is not definable with $=$, $<_\omega$ and $<_1$ in $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$. Assume that there exists a formula $\phi(x,y)$ in $=$, $<_\omega$ and $<_1$ such that for all assignments f on $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$, $f(x) \prec_\epsilon^\star f(y)$ iff $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle \models_f \phi(x,y)$. Let $a, b \in {}^\star\mathbb{R}$ be such that $a \prec_\epsilon^\star b$. We need to consider a bijective mapping σ on $\{a\}$ onto $\{b\}$ such that $\sigma(a) = b$. Now, let τ be the bijective mappping on ${}^\star\mathbb{R}$ onto itself such that for all $c \in {}^\star\mathbb{R}$,

- if $c \in \{a\}$ then $\tau(c) = \sigma(c)$,
- if $c \in \{b\}$ then $\tau(c) = \sigma^{-1}(c)$,
- if $c \notin \{a\}$ and $c \notin \{b\}$ then $\tau(c) = c$.

Notice that $\tau(a) \not\prec_\epsilon^\star \tau(b)$. As a simple exercise, we invite the reader to show by induction on the complexity of formulas ψ in $=$, $<_\omega$ and $<_1$ that for all assignments f on $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$, $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle \models_f \psi$ iff $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle \models_{\tau\circ f} \psi$. Hence, for all assignments f on $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$, $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle \models_f \phi(x,y)$ iff $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle \models_{\tau\circ f} \phi(x,y)$. Thus, for all assignments f on $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$, $f(x) \prec_\epsilon^\star f(y)$ iff $\tau(f(x)) \prec_\epsilon^\star \tau(f(y))$. If f is an assignment on $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$ such that $f(x) = a$ and $f(y) = b$ then $a \prec_\epsilon^\star b$ iff $\tau(a) \prec_\epsilon^\star \tau(b)$: a contradiction.
4. We demonstrate that $\prec_\omega^\star$ is not definable with $=$, $<_\epsilon$ and $<_1$ in $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$. Assume that there exists a formula $\phi(x,y)$ in $=$, $<_\epsilon$ and $<_1$ such that for all assignments f on $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$, $f(x) \prec_\omega^\star f(y)$ iff $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle \models_f \phi(x,y)$. Let $a, b \in {}^\star\mathbb{R}$ be such that $a \prec_\omega^\star b$. We need to consider a bijective mapping σ on $[a]_1$ onto $[b]_1$ preserving $\prec_\epsilon^\star$ and $\prec_1^\star$. Now, let τ be the bijective mappping on ${}^\star\mathbb{R}$ onto itself such that for all $c \in {}^\star\mathbb{R}$,

- if $c \in [a]_1$ then $\tau(c) = \sigma(c)$,
- if $c \in [b]_1$ then $\tau(c) = \sigma^{-1}(c)$,
- if $c \notin [a]_1$ and $c \notin [b]_1$ then $\tau(c) = c$.

Notice that $\tau(a) \not\prec_\omega^\star \tau(b)$. As a simple exercise, we invite the reader to show by induction on the complexity of formulas ψ in $=$, $<_\epsilon$ and $<_1$ that for all assignments f on $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$, $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle \models_f \psi$ iff $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle \models_{\tau\circ f} \psi$. Hence, for all assignments f on $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$, $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle \models_f \phi(x,y)$ iff $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle \models_{\tau\circ f} \phi(x,y)$. Thus, for all assignments f on $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$, $f(x) \prec_\omega^\star f(y)$ iff $\tau(f(x)) \prec_\omega^\star \tau(f(y))$. If f is an assignment on $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$ such that $f(x) = a$ and $f(y) = b$ then $a \prec_\omega^\star b$ iff $\tau(a) \prec_\omega^\star \tau(b)$: a contradiction.
5. We demonstrate that $\prec_1^\star$ is not definable with $=$, $<_\epsilon$ and $<_\omega$ in $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$. Assume that there exists a formula $\phi(x,y)$ in $=$, $<_\epsilon$ and $<_\omega$ such that for all assignments f on $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$, $f(x) \prec_1^\star f(y)$ iff $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle \models_f \phi(x,y)$. Let $a, b \in {}^\star\mathbb{R}$ be such that $a \prec_1^\star b$. We need to consider a bijective mapping σ on $[a]_\epsilon$ onto $[b]_\epsilon$ preserving $\prec_\epsilon^\star$. Now, let τ be the bijective mappping on ${}^\star\mathbb{R}$ onto itself such that for all $c \in {}^\star\mathbb{R}$,

- if $c \in [a]_\epsilon$ then $\tau(c) = \sigma(c)$,
- if $c \in [b]_\epsilon$ then $\tau(c) = \sigma^{-1}(c)$,
- if $c \notin [a]_\epsilon$ and $c \notin [b]_\epsilon$ then $\tau(c) = c$.

Notice that $\tau(a) \not\prec_1^\star \tau(b)$. As a simple exercise, we invite the reader to show by induction on the complexity of formulas ψ in $=$, $<_\epsilon$ and $<_\omega$ that for all assignments f on $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$, $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle \models_f \psi$ iff $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle \models_{\tau\circ f} \psi$. Hence, for all assignments f on $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$, $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle \models_f \phi(x,y)$ iff $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle \models_{\tau\circ f} \phi(x,y)$. Thus, for all assignments f on $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$, $f(x) \prec_1^\star f(y)$ iff $\tau(f(x)) \prec_1^\star \tau(f(y))$. If f is an assignment on $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$ such that $f(x) = a$ and $f(y) = b$ then $a \prec_1^\star b$ iff $\tau(a) \prec_1^\star \tau(b)$: a contradiction. $\dashv$

Proof of proposition 9: By proposition 5, it suffices to consider definability in $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$.
1. It suffices to observe that for all assignments f on $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$, $f(x) = f(y)$ iff $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle \models_f \forall z\,(x <_{\epsilon\omega} z \leftrightarrow y <_{\epsilon\omega} z)$.
2. It suffices to observe that for all assignments f on $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$, $f(x) = f(y)$ iff $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle \models_f \forall z\,(x <_{\epsilon 1} z \leftrightarrow y <_{\epsilon 1} z)$.
3. By item 2 of proposition 8.
4. It suffices to observe that for all assignments f on $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$, $f(x) \prec_\epsilon^\star f(y)$ iff $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle \models_f x <_{\epsilon\omega} y \wedge \forall z\,(x <_{\epsilon\omega} z \rightarrow y = z \vee y <_{\epsilon\omega} z \vee z <_{\epsilon\omega} y$.
5. We demonstrate that $\prec_\epsilon^\star$ is not definable with $=$ and $<_{\epsilon 1}$ in $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$. Assume that there exists a formula $\phi(x,y)$ in $=$ and $<_{\epsilon 1}$ such that for all assignments f on $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$, $f(x) \prec_\epsilon^\star f(y)$ iff $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle \models_f \phi(x,y)$. Let $a \in {}^\star\mathbb{R}$ and $b', b'' \in [a]_1$ be such that $b' \prec_1^\star b''$. We need to consider a bijective mapping σ on $[a]_1$ onto itself preserving $\prec_{\epsilon 1}^\star$ and such that $\sigma(b') \prec_\epsilon^\star \sigma(b'')$. Now, let τ be the bijective mappping on ${}^\star\mathbb{R}$ onto itself such that for all $c \in {}^\star\mathbb{R}$,

- if $c \in [a]_1$ then $\tau(c) = \sigma(c)$,
- if $b \notin [a]_1$ then $\tau(c) = c$.

Notice that $\tau(b') \prec_\epsilon^\star \tau(b'')$. As a simple exercise, we invite the reader to show by induction on the complexity of formulas ψ in $=$ and $<_{\epsilon 1}$ that for all assignments f on $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$, $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle \models_f \psi$ iff $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle \models_{\tau\circ f} \psi$. Hence, for all assignments f on $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$, $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle \models_f \phi(x,y)$ iff $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle \models_{\tau\circ f}$

$\phi(x,y)$. Thus, for all assignments f on $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$, $f(x) \prec_\epsilon^\star f(y)$ iff $\tau(f(x)) \prec_\epsilon^\star \tau(f(y))$. If f is an assignment on $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$ such that $f(x) = b'$ and $f(y) = b''$ then $b' \prec_\epsilon^\star b''$ iff $\tau(b') \prec_\epsilon^\star \tau(b'')$: a contradiction.
6. By item 3 of proposition 8.
7. By item 4.
8. By item 4 of proposition 8.
9. We demonstrate that $\prec_\omega^\star$ is not definable with $=$ and $<_{\omega 1}$ in $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$. Assume that there exists a formula $\phi(x,y)$ in $=$ and $<_{\omega 1}$ such that for all assignments f on $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$, $f(x) \prec_\omega^\star f(y)$ iff $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle \models_f \phi(x,y)$. Let $a \in {}^\star\mathbb{R}$ and $b', b'' \in {}^\star\mathbb{R} \setminus [a]_\epsilon$ be such that $b' \prec_1^\star b''$. We need to consider a bijective mapping σ on ${}^\star\mathbb{R} \setminus [a]_\epsilon$ onto itself preserving $\prec_{\omega 1}^\star$ and such that $\sigma(b') \prec_\omega^\star \sigma(b'')$. Now, let τ be the bijective mappping on ${}^\star\mathbb{R}$ onto itself such that for all $c \in {}^\star\mathbb{R}$,

- if $c \in {}^\star\mathbb{R} \setminus [a]_\epsilon$ then $\tau(c) = \sigma(c)$,
- if $b \notin {}^\star\mathbb{R} \setminus [a]_\epsilon$ then $\tau(c) = c$.

Notice that $\tau(b') \prec_\omega^\star \tau(b'')$. As a simple exercise, we invite the reader to show by induction on the complexity of formulas ψ in $=$ and $<_{\omega 1}$ that for all assignments f on $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$, $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle \models_f \psi$ iff $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle \models_{\tau\circ f} \psi$. Hence, for all assignments f on $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$, $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle \models_f \phi(x,y)$ iff $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle \models_{\tau\circ f} \phi(x,y)$. Thus, for all assignments f on $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$, $f(x) \prec_\omega^\star f(y)$ iff $\tau(f(x)) \prec_\omega^\star \tau(f(y))$. If f is an assignment on $\langle {}^\star\mathbb{R}, \prec_\epsilon^\star, \prec_\omega^\star, \prec_1^\star\rangle$ such that $f(x) = b'$ and $f(y) = b''$ then $b' \prec_\omega^\star b''$ iff $\tau(b') \prec_\omega^\star \tau(b'')$: a contradiction.
10. By item 5 of proposition 8.
11. By item 5.
12. By item 9.
Proof of proposition 10: Simple exercise. $\dashv$
Proof of proposition 11: Simple exercise. $\dashv$
Proof of proposition 12: Simple exercise. $\dashv$
Proof of proposition 13: Simple exercise. $\dashv$
Proof of proposition 14: Simple application of the fact that for all $a, b \in {}^\star\mathbb{R}$, if there exists $c \in {}^\star\mathbb{R}$ such that the pair $\langle a, c\rangle$ satisfies the relation $R \in \mathcal{P}(\mathcal{B}_{HY})$ and the pair $\langle c, b\rangle$ satisfies the relation $S \in \mathcal{P}(\mathcal{B}_{HY})$ then $\langle a, b\rangle$ satisfies $R \circ S$ (respectively: $\langle a, b\rangle$ satisfies $\mathbf{I}(R \circ S)$). $\dashv$
Proof of proposition 15: Let $\mathcal{N} = \langle V, C\rangle$ be a hyperpoint network. Suppose that $\mathcal{N}$ is weakly path-consistent, we demonstrate that $\mathbf{I}(\mathcal{N})$ is path-consistent. Assume that there exists $x, y, z \in V$ such that $\mathbf{I}(C(x,z)) \circ \mathbf{I}(C(z,y)) \not\supseteq \mathbf{I}(C(x,y))$. By item 3 of proposition 10, $\mathbf{I}(C(x,z) \circ C(z,y)) \not\supseteq \mathbf{I}(C(x,y))$. By item 1 of proposition 10, $\mathbf{I}(C(x,z) \circ C(z,y)) \not\supseteq C(x,y)$. Hence, $\mathcal{N}$ is not weakly path-consistent: a contradiction. $\dashv$
Proof of proposition 16: Let $\mathcal{N} = \langle V, C\rangle$ be a hyperpoint network. Suppose that $\mathcal{N}$ is convex and path-consistent, we demonstrate that either it is empty or it admits a consistent instantiation of maximal dimension. Assume that for all $x, y \in V$, $C(x,y) \neq \emptyset$. Let $(x_1, \ldots, x_k)$ be a list of all variables in V, i.e. $k = Card(V)$. In order to demonstrate that there exists a list $(\mid \mathbf{a}_1 \mid, \ldots, \mid \mathbf{a}_k \mid)$ of hyperreal numbers such that for all positive integers i_1, i_2, if $i_1 \leq k$ and $i_2 \leq k$ then the primitive relation in hyperpoint algebra linking $\mid \mathbf{a}_{i_1} \mid$ and $\mid \mathbf{a}_{i_2} \mid$ is of maximal dimension in $C(x_{i_1}, x_{i_2})$, we demonstrate that there exists a list $((s_1, t_1, u_1), \ldots, (s_k, t_k, u_k))$ of triples of real numbers such that for all positive integers i_1, i_2, if $i_1 \leq k$ and $i_2 \leq k$ then the primitive relation in hyperpoint algebra linking $(s_{i_1}, t_{i_1}, u_{i_1})$ and $(s_{i_2}, t_{i_2}, u_{i_2})$ is of maximal dimension in $C(x_{i_1}, x_{i_2})$. In this respect, we use the following step-by-step construction. Let k' be a positive integer such that $k' \leq k$ and there exists a list $((s_1, t_1, u_1), \ldots, (s_{k'-1}, t_{k'-1}, u_{k'-1}))$ of triples of real numbers such that for all positive integers i_1, i_2, if $i_1 \leq k'-1$ and $i_2 \leq k'-1$ then the primitive relation in hyperpoint algebra linking $(s_{i_1}, t_{i_1}, u_{i_1})$ and $(s_{i_2}, t_{i_2}, u_{i_2})$ is of maximal dimension in $C(x_{i_1}, x_{i_2})$. For all positive integers i, if $i \leq k'-1$ then let $reg(i)$ be the set of all $(s, t, u) \in \mathbb{R}^3$ such that the primitive relation in hyperpoint algebra linking (s_i, t_i, u_i) and (s, t, u) is in $C(x_i, x_{k'})$. Since $\mathcal{N}$ is convex, then for all positive integers i, if $i \leq k'-1$ then

- $reg(i)$ is the Cartesian product of 3 intervals in $\mathbb{R}$.

Since $\mathcal{N}$ is path-consistent, then for all positive integers i_1, i_2, if $i_1 \leq k'-1$ and $i_2 \leq k'-1$ then

- $reg(i_1) \cap reg(i_2) \neq \emptyset$.

It follows immediately from Helly's theorem [Chvatal (1983)] that $reg(1) \cap \ldots \cap reg(k'-1)$ is the Cartesian product of 3 nonempty intervals in $\mathbb{R}$. Let $[s^-, s^+]$, $[t^-, t^+]$ and $[u^-, u^+]$ be these 3 nonempty intervals in $\mathbb{R}$. If $s^- \neq s^+$ then $reg(1) \cap \ldots \cap reg(k'-1)$ is a subset of $\mathbb{R}^3$ of dimension 3 and we can choose a triple $(s_{k'}, t_{k'}, u_{k'})$ of real numbers avoiding any finite set of triples of real numbers. If $s^- = s^+$ and $t^- \neq t^+$ then $reg(1) \cap \ldots \cap reg(k'-1)$ is a subset of $\mathbb{R}^3$ of dimension 2 and we can choose a triple $(s_{k'}, t_{k'}, u_{k'})$ of real numbers such that $s_{k'} = s^- = s^+$ and avoiding any finite set of triples of real numbers. If $s^- = s^+$, $t^- = t^+$ and $u^- \neq u^+$ then $reg(1) \cap \ldots \cap reg(k'-1)$ is a subset of $\mathbb{R}^3$ of dimension 1 and we can choose a triple $(s_{k'}, t_{k'}, u_{k'})$ of real numbers such that $s_{k'} = s^- = s^+$, $t_{k'} = t^- = t^+$ and avoiding any finite set of triples of real numbers. If $s^- = s^+$, $t^- = t^+$ and $u^- = u^+$ then $reg(1) \cap \ldots \cap reg(k'-1)$ is a subset of $\mathbb{R}^3$ of dimension 0 and we have no choice other than taking the triple $(s_{k'}, t_{k'}, u_{k'})$ of real numbers such that $s_{k'} = s^- = s^+$, $t_{k'} = t^- = t^+$ and $u_{k'} = u^- = u^+$. As a simple exercise, we invite the reader to show that for all positive integers i, if $i \leq k'-1$ then the primitive relation in hyperpoint algebra linking (s_i, t_i, u_i) and $(s_{k'}, t_{k'}, u_{k'})$ is of maximal dimension in $C(x_i, x_{k'})$. $\dashv$
Proof of proposition 17: A deterministic algorithm using a polynomial amount of time to solve the consistency problem consists, given a hyperpoint network $\mathcal{N} = \langle V, C\rangle$, in applying the weak path-consistency method to $\mathcal{N}$. By proposition 13, this method provides an equivalent weakly path-consistent hyperpoint subnetwork, say $\mathcal{N}'$, to the initial hyperpoint network. By proposition 14, $\mathbf{I}(\mathcal{N}')$ is path-consistent. By proposition 15, either it is empty or it admits a consistent instantiation of maximal dimension. Hence, either $\mathcal{N}'$ is empty or $\mathcal{N}'$ admits a consistent instantiation of maximal dimension. It is immediate from this construction that $\mathcal{N}$ is consistent iff $\mathcal{N}'$ admits a consistent instantiation of maximal dimension. $\dashv$