Implementing Euler/Venn Reasoning Systems

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Abstract

This paper proposes an implementation of a Euler/Venn reasoning system using directed acyclic graphs and shows that this implementation is correct with respect to a modified Shin/Hammer mathematical model of Euler/Venn Reasoning. In proving its correctness it will also be shown that the proposed implementation preserves or inherits the soundness and completeness properties of the mathematical model of the Euler/Venn system.

Introduction

In the following study, we will look at an implementation of a Euler/Venn mechanical reasoning system and show that this implementation captures the essential properties[1] of a system similar to the Shin/Hammer mathematical Euler/Venn system as given in (Shin 1996; Hammer & Danner 1996; Hammer 1995). To do this, we will first look at a modified Shin/Hammer formal mathematical system that is associated with Euler/Venn diagrams. Then a second diagrammatic system representing Euler/Venn reasoning, one lending itself naturally to implementation, will be proposed using DAG's[2], and the relations between this system and the formal mathematical system associated with Euler/Venn diagrams will be explored. It will be argued that this second representation is in fact true to the formal mathematical model of Euler/Venn reasoning and thereby preserves the properties of being sound and complete.

Formal Specification of Mathematical System

The mathematical formalization of the diagrammatic language of Euler/Venn, EV_F, is defined to be the three-tuple $\langle \Gamma, \Delta, \Sigma \rangle$, with Γ as the set of grammatical or well-formed formulae, Δ the deductive system, and Σ the semantics of the system. EV_F is defined to be a traditional Venn system with Euler like extensions (see below.) While this treatment was inspired by and is quite similar to that found in (Hammer 1995), there are a number of important differences that should be noted, the most important of which include that the grammar presented here adds more well-formed diagrams, and that the system's semantics have been changed to accommodate these new diagrams. By having a modified semantics and more well-formed diagrams, two new inference rules are introduced to maintain the completeness of the system.

The Vocabulary

1. Rectangles - Each rectangle denotes the domain of discourse to be represented by the diagram.
2. Closed Curves - A countably infinite set C_1, C_2, C_3 ... of closed curves. Each closed curve must not intersect itself. These curves denote sets.
3. Shading - The shading of any region denotes that the set represented by that region is empty.
4. $\otimes$ - A countably infinite set $\otimes_1, \otimes_2, \otimes_3, \ldots$ of individual constants.
5. Lines - Lines are used to connect individual constants $\otimes_n$ of the same n, in different regions to illustrate the uncertainty of which set contains that constant.

Γ - The Mathematical Grammar

Formation rules Formation Rules for well-formed diagrams V_{EV_F} of EV_F:

1. Any diagram containing only a Rectangle is a member of V_{EV_F}.
2. If $V \in V_{EV_F}$ then:
   (a) V with the addition of any closed curve C with unique label N completely within the rectangle of V so that the regions intersected by C are split into at most two new regions, is a member of V_{EV_F}.[3]

[1] One system *captures the essential properties* of another system if there is a translation or mapping between them that preserves deductive and semantic relations.

[2] A DAG is a Directed Acyclic Graph.

[3] This grammatical stipulation while more general than that used in (Hammer 1995) is still not as general as one

(b) V with the addition of a $\otimes_n$ of a new n within any region of a closed curve of V is a member of V_{EV_F}.

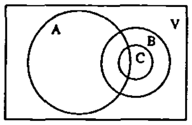
(c) V with the shading of any enclosed region is a member of V_{EV_F}.

(d) If V contains a certain $\otimes_n$ then the result of adding another $\otimes_n$ to any region not containing $\otimes_n$ and then connecting the two of them together with a line is a member of V_{EV_F}.

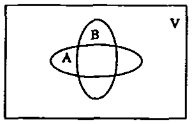
3. No other diagram is in V_{EV_F}.

Examples of diagrams of EV_F

Valid Diagram



Invalid Diagram



Notion of region A *region* is any area of a diagram completely enclosed by lines of that diagram. Any region of the diagram completely enclosed by one closed curve is referred to as a *basic region.* A *minimal region* is any region which is not the combination of other regions. The following set theoretic operations on regions will be allowed:

1. $\cup$ The union of two regions is the region containing both of those regions.
2. $\cap$ The intersection of two regions is the region that is common to both regions.
3. $\subset$ One region is the subset of another if that region is entirely contained within the other.
4. $-$ The difference of two regions is the regions of the first not contained by the second.
5. $\bar{r}$ The complement of a region is the region not contained in that region but still within the rectangle of the diagram.

Notion of counterpart Two regions of different diagrams are considered to be *counterparts* if both are directly enclosed by rectangles or if there is a subset of the labels of the diagrams such the regions are both the result of taking the intersection of the basic regions associated with this set of labels. Counterparts are preserved under union and complement. *Counterparts agree with respect to shading and $\otimes$ sequences* in two diagrams when for any two regions that are counterparts one is shaded iff the other is shaded, and one contains a link of a $\otimes_n$ sequence iff the other contains a $\otimes_n$ link of the same n.

Δ - The Mathematical Deductive System

Given diagrams V and V' of EV_F, V' can be inferred from V if V' is the result of applying any of the following rules [4] to V:

1. **Erasure of part of a $\otimes$ sequence** – V' is obtained by erasing a $\otimes_n$ of a $\otimes$ sequence of V where that $\otimes_n$ falls within a shaded region and provided that the possibly split $\otimes$ sequence is rejoined by a line if necessary.
2. **Extending a $\otimes$ sequence** – V' is the result of adding a new $\otimes_n$ link to a $\otimes$ sequence of V to a minimal region not already containing a link of that sequence.
3. **Erasure** – V' is obtained from V by erasing:
   (a) An entire $\otimes$ sequence
   (b) The shading of a region
   (c) A closed curve if the removal does not cause any counterpart regions to disagree with regard to shading or containment of links of a $\otimes$ sequence
4. **Introduction of a new curve** – V' is the result of adding a new curve to V, so that the other labels of V are left undisturbed and all counterparts agree with respect to shading and containment of links of a $\otimes$ sequence.
5. **Inconsistency** – V' of any form can obtained from V if V contains a region that is both shaded and has the one and only link of a $\otimes$ sequence.
6. **Adding shaded regions** – V' is the result of adding a new minimal region corresponding to the intersection of basic regions already existing in V provided that this new region is shaded and it drawn so that the region is contained within the basic regions to whose intersection it is intended to correspond.
7. **Removing shaded regions** – V' is the result of removing a shaded minimal but not basic region of V. To emphasize the fact that the region has been removed the lines enclosing the now non-existing region should be smoothed into curves, and the remaining curves should be spaced out to remove points of unintended intersection.

Unification – V' can be inferred from diagrams V_1 and V_2 if it is the case that:

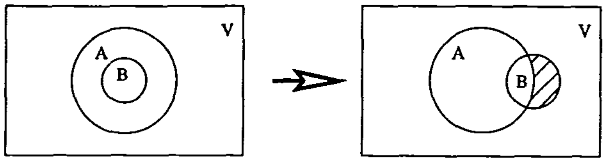
1. The set of labels of V' is the union of the labels of V_1 and V_2.
2. Counterparts in both V' and V_1 and V' and V_2 agree with respect to shading and containment of a link of a $\otimes$ sequence.

The following are figures to illustrate the use of the system's two new rules.

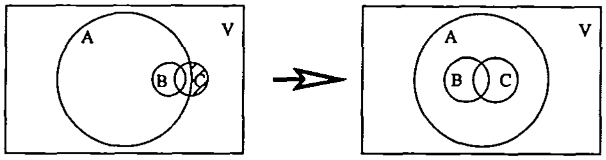
might like. But it can be argued that any valid Euler/Venn Diagram can be expressed under this restriction.

[4] Please note that the rules of **Adding shaded regions** and **Removing shaded regions** are the above mentioned new rules.

Adding Shaded Regions:



Removing Shaded Regions:



A diagram V is provable from the set of diagrams $\mathfrak{D}$ in EV_F, written as $\mathfrak{D} \vdash_{EF_V} V$, if there is a sequence of diagrams $V_1 \ldots V_n$ where V_n is equal to V and all $V_1 \ldots V_n$ are either members of $\mathfrak{D}$ or the result of applying one of the above rules of inference to a prior diagram in the sequence.

Σ - The Mathematical Semantics

The semantics of the system is given by the assignment of a domain to the diagram and subsets of this domain to each basic regions of the diagram. Formally this assignment is the pair (U, f) where U is the domain and f is a function associating a subset of U with each basic region. Basic regions of the same label are assigned the same subset of U by f.

Proposition 1 *(Hammer (Hammer 1995))*
If (U, f) is an assignment of U to basic regions then there is a unique set assignment (U, g) to minimal regions(where g is a function assigning subsets of U to the minimal regions of the diagram with minimal regions not existing in the diagram being assigned $\emptyset$). Given this (U, g) and (U, f), there is a unique model (U, I) s.t. it extends both of them. This model's interpretation function I preserves the counterpart relation.

Diagram V is *true* in model $M = (U, I)$ of EV_F iff for every region r of V, if r is shaded then $I(r) = \emptyset$, and if r completely contains a $\otimes$ sequence then $I(r) \neq \emptyset$. When this is the case $M \models_{EV_F} V$ will be written. With $\mathfrak{D} \cup \{V\}$ a set of diagrams, V is a *logical consequence* of $\mathfrak{D}$ in EV_F iff every model which makes all of $\mathfrak{D}$ true in EV_F also makes V true. This is written as $\mathfrak{D} \models_{EV_F} V$.

Soundness and Completeness of EV_F

Theorem 1 *Soundness of EV_F (Extension of Hammer (Hammer 1995))*
For every set of diagrams $\mathfrak{D} \cup \{V\}$, if $\mathfrak{D} \vdash_{EV_F} V$ then $\mathfrak{D} \models_{EV_F} V$.

Proof Sketch:
It suffices to show that the two new rules of inference preserve soundness; this plus Hammer's Soundness proof will demonstrate the soundness of EV_F.

1. If V' is the result of applying the rule of **Adding a Shaded Region** to V, then $V \models_{EV_F} V'$. Suppose that $(U, I) \models_{EV_F} V$ then for all minimal regions r not existing in V $I(r) = \emptyset$. Thus since the newly added region is shaded then $I(r) = \emptyset$, and $(U, I) \models_{EV_F} V'$.
2. If V' is the result of applying the rule of **Removing a Shaded Region** to V, then $V \models_{EV_F} V'$. Suppose that $(U, I) \models_{EV_F} V$ then for all shaded regions r in V diagram $I(r) = \emptyset$. Thus since the removed minimal region does not exist in the diagram then $I(r) = \emptyset$, and $(U, I) \models_{EV_F} V'$.

Theorem 2 *Completeness of EV_F (Extension of Hammer (Hammer 1995))*
For every set of diagrams $\mathfrak{D} \cup \{V\}$, if $\mathfrak{D} \models_{EV_F} V$ then $\mathfrak{D} \vdash_{EV_F} V$.

Proof Sketch:
For this proof, Hammer's completeness proof found in (Hammer 1995) will again be greatly relied upon. First all diagrams in $\mathfrak{D}$ are extended to Venn diagrams and put into the set $\mathfrak{D}'$, through the repeated application of the **Adding shaded regions** inference rule. The same is done to V extending it to V'. From soundness and the transitivity of $\models_{EV_F}$ it is concluded that $\mathfrak{D}' \models_{EV_F} V'$. Now Hammer's completeness result will be used to show that $\mathfrak{D}' \vdash_{EV_F} V'$. We now only need to apply the rule of **Removing shaded regions** to show $V' \vdash_{EV_F} V$. Hence $\mathfrak{D} \vdash_{EV_F} V$.

To further clarify the above proof the following diagram has been provided.

$$\begin{array}{ccccccc}
\mathfrak{D} & \xrightarrow[\text{Soundness}]{\models_{EF_V}} & \mathfrak{D}' & \xrightarrow[\text{Transitivity}]{\models_{EF_V}} & V' & \xrightarrow[\text{Soundness}]{\models_{EF_V}} & V \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathfrak{D} & \xrightarrow[\text{Add Shade}]{\vdash_{EV_F}} & \mathfrak{D}' & \xrightarrow[\text{Completeness}]{\vdash_{EV_F}} & V' & \xrightarrow[\text{Remove Shade}]{\vdash_{EV_F}} & V
\end{array}$$

Formal Specifications of the DAG Implementation

The implementation of the diagrammatic language of Euler/Venn, EV_I, is defined to be the three-tuple $\langle \mathcal{G}, \mathcal{D}, \mathcal{S} \rangle$.

The Vocabulary of the Implementation

1. Nodes - Each node represents a set that can be expressed as the intersection of one or more of the sets represented by the diagram. Associated with each node are a number of attributes: a name, whether the set is empty (shading), and whether the set possibly contains any individual constants ($\otimes_n$). Regions are named so that the set that they represent is exactly the intersection of the sets associated with the letters of its name. Likewise a node can also be thought of as a region, not necessarily minimal, of the diagram.

2. Directed Edges - Directed edges connect nodes A and B, leading from A to B, expressing that their associated sets, $S(A)$ and $S(B)$, are such that $S(A)$ *covers*[5] $S(B)$. Likewise the edge relation can also be thought of in terms of region containment.

In the sections to follow the natural meaning of the predicates parent, child, ancestor, and descendent will be used.

$\mathcal{G}$ - The Grammar of the Implementation

Formation Rules for proper DAG's D_{EV_I} of EV_I:

1. Any DAG containing only one node named V and no edges is a member of D_{EV_I}.
2. If $D \in D_{EV_I}$, then D with the addition of one new node N such that:
   (a) N is connected to at least one other node N', and does not cause a cycle in the DAG.
   (b) N's name contains all of the letters of the names of its parents with at most one additional letter.
   (c) N's name does not contain, for any letter L, L and $\overline{L}$.

   is member of D_{EV_I}.
3. If $D \in D_{EV_I}$, then the modification of D such that either:
   (a) Some node along with all of its descendents are shaded.
   (b) Some terminal node and all of its ancestors contain a $\otimes_n$.

   is member of D_{EV_I}.
4. No other DAG is a member of D_{EV_I}.

Notion of region A *region* is represented by a node of the DAG. A region is referred to as a *basic region* if its label contains a letter with no bar, not contained in the labels of any of its parents. This letter is referred to as the basic region's *identifying letter*. A *minimal region* is any terminal node of the DAG. As before the set theoretic operations $\cup$, $\cap$, $\subset$, $-$, $\overline{r}$ will be allowed on regions.

Notion of counterpart Two regions are considered to be *counterparts* if both of their names contain the same set of letters. Counterparts are once again preserved under union and complement.

$\mathcal{D}$ - The Deductive System of the Implementation

Given DAG's D and D' of EV_I, D' can be inferred from D if it is the case that D' is the result of applying any of the following rules to D.

1. **Erasure of part of a $\otimes$ sequence** – D' is obtained by removing a $\otimes_n$ of a $\otimes$ sequence from a minimal region provided that this minimal region is shaded. The $\otimes_n$ is also removed from its ancestors not having a different descendent also containing a $\otimes_n$ of the same n.
2. **Extending a $\otimes$ sequence** – D' is the result of adding a new $\otimes_n$ link to a $\otimes$ sequence of D in a minimal region not already containing a link of that sequence. The same link is added to all of that node's ancestors.
3. **Erasure** – D' is obtained from D by erasing:
   (a) An entire $\otimes$ sequence, removing all $\otimes_n$'s of a certain n occuring in any node of the DAG.
   (b) The shading of a minimal region, and the shading of any of its parents not having all shaded descendants.
   (c) A basic region and all regions containing that region's identifying letter or its complement, provided that the removal does not cause any counterpart regions to disagree with regard to shading or containment of links of a $\otimes$ sequence.
4. **Introduction of a new curve** – D' is the result of adding a new basic region to D as specified by the Inductive Construction Technique (defined below) and the other labels of D are left undisturbed and all counterparts agree with respect to shading and containment of links of a $\otimes$ sequence.
5. **Inconsistency** – Any D' can be obtained from D if D contains a minimal region that is both shaded and has the one and only link of a $\otimes$ sequence.
6. **Adding shaded regions** – D' is the result of adding a new minimal region not existing in D as specified by the Direct Construction Technique (defined below) and provided that this minimal region is shaded and is not a basic region.
7. **Removing shaded regions** – D' is the result of removing a minimal but not basic region that is shaded from D and re-arranging the DAG as specified by step 4 of the Direct Construction Technique.

Unification – D' can be inferred from DAG's D_1 and D_2 if it is the case that:

1. The set of basic regions of D' is the union of the basic regions of D_1 and D_2.
2. Counterparts in both D' and D_1 and D' and D_2 agree with respect to shading and containment of a link of a $\otimes$ sequence.

A DAG D is provable from the set of DAG's $\mathfrak{D}$, written as $\mathfrak{D} \vdash_{EV_I} D$, if there is a sequence of DAG's $D_1 \ldots D_n$ where D_n is equal to D and all $D_1 \ldots D_n$ are either members of $\mathfrak{D}$ or the result of applying one of the above rules of inference to a prior DAG in that sequence.

[5] "*A covers B*" iff $B \subsetneq A$ and there is no C such that $B \subsetneq C$ and $C \subsetneq A$.

$\mathcal{S}$ - The Semantics of the Implementation

The semantics of the system is given by the assignment of a domain to the root of the DAG, and subsets of this domain to each basic region of the DAG. Formally this assignment is the pair (U, f), U being the domain and f being a function associating with each basic node a subset of U. Nodes of the same identifying letter are assigned the same subset of U. Non-basic regions not existing in the DAG are assigned $\emptyset$ by f. Once again Proposition 1 is used to establish that given (U, f) there is a unique assignment to minimal regions (U, g) and a unique model (U, I) extending them both.

Diagram D is *true* in model $M = (U, I)$ of EV_I iff for every region r of D, if r is shaded then $I(r) = \emptyset$, and if r or its descendants contain an entire $\otimes$ sequence then $I(r) \neq \emptyset$. When this is the case $M \models_{EV_I} D$ will be written. With $\mathfrak{D} \cup \{D\}$ a set of diagrams, D is a *logical consequence* of $\mathfrak{D}$ in EV_I iff it is true in every model which makes all of $\mathfrak{D}$ true in EV_I. This is written as $\mathfrak{D} \models_{EV_I} D$.

Relationships Between $\langle \Gamma, \Delta, \Sigma \rangle$ and $\langle \mathcal{G}, \mathcal{D}, \mathcal{S} \rangle$

Relationship Between Γ and $\mathcal{G}$

This section explains the grammatical relation between the formal mathematical representation of a Euler/Venn diagram and its corresponding DAG. It will be shown that there is a translation process that results in a bijection between classes of isomorphic Euler/Venn Diagrams and DAG's. This translation process will be given in two forms one inductive and the other direct. Each method is needed to explain algorithms used in the deductive system of the implementation(namely the rules of **Introduction of a new curve, Adding shaded regions,** and **Removing shaded regions.**)

Translating Euler/Venn diagrams into DAG's, inductive construction Knowing that an Euler/Venn Diagram V can be constructed by a sequence of adding circles in a certain way to an empty diagram, it suffices to define the translation technique inductively on this sequence.[6]

1. Base - The empty diagram is the DAG with one node V and no edges.
2. Induction - When adding circle A to an existing Euler/Venn diagram V' and its corresponding DAG D' proceed as follows:

   (a) Identify the region Y that covers A and the region X that covers $\overline{A}$. Add nodes YA and $X\overline{A}$ to D' directly below Y and X with an edge from Y to YA and from X to $X\overline{A}$.[7]

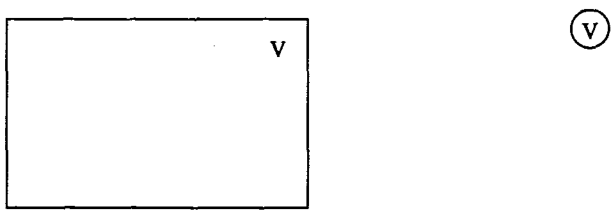
   (b) Identify all regions represented by nodes in the DAG crossed by YA and $X\overline{A}$ (previously known as A and $\overline{A}$ respectively.)[8] For each of these crossed regions W, add to D' WYA and $WX\overline{A}$ with edges from W to each of them and edges from YA to WYA and from $X\overline{A}$ to $WX\overline{A}$. Then any duplicate letters that might occur in an individual name are removed.

   (c) Starting with the top of the DAG, determine if any region is now covered by or covers one of the newly created regions Z, all regions whose name contains A or $\overline{A}$. Assume that it is W that now covers Z. Remove the edge leading to W from its previously covering parent connect this old parent to Z and draw a new edge from Z to W.[9] Appropriately rename each of these nodes W to include the letter(s) of their new parent Z not previously in W. After doing this the new letters are added to each of W's descendents, also adding new edges to their possible new parents. This is done to retain only edges between covering regions.

3. Final Step - To finish the construction shading and $\otimes$ sequence information needs to be included. Shade all minimal regions of the DAG which correspond to shaded minimal regions of the diagram. Likewise add a $\otimes_n$ to a minimal region of the DAG if its corresponding region of the diagram has a link of a $\otimes$ sequence of a certain n. Then starting at the bottom of the DAG and working up shade any node having all shaded children and put an $\otimes_n$ into any node having any children with a $\otimes_n$ of a certain n.

Example of inductive construction

1. The empty diagram.



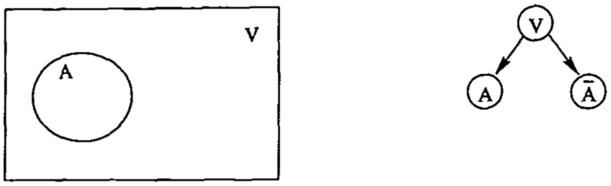
2. After adding the set A.

[6] If given an already constructed Euler/Venn Diagram this sequence can be arbitrarily chosen. The order of the placement of the circles on the page makes no difference as long as the two resulting diagrams are equivalent.

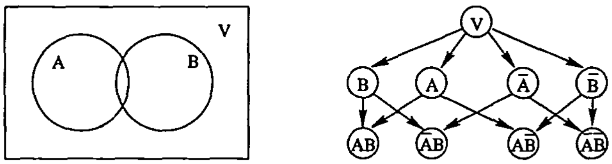
[7] In the case that Y is V leave out the V from the names of the nodes to make our DAG's easier to read.

[8] From the grammar Γ it is known that each of these regions is crossed once creating two new minimal regions.

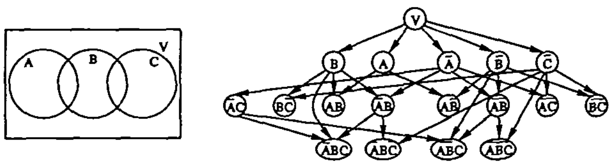
[9] Here since it is known that there is a unique covering parent so at most one edge needs to be changed for each node.



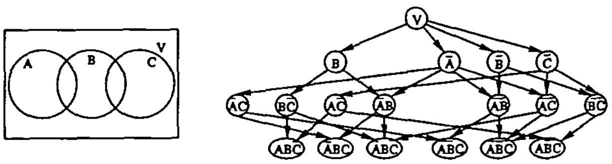
3. After adding the set B. B and $\overline{B}$ are added below V and both A and $\overline{A}$ are split.



4. After adding the set C. C is added below $\overline{A}$ and renamed $\overline{A}C$, $\overline{C}$ is added below V. B, $\overline{A}$, $\overline{B}$, $\overline{A}B$, and $\overline{A}\,\overline{B}$ are split. Duplicates $\overline{A}C$, $\overline{A}BC$, and $\overline{A}\,\overline{B}C$ are removed.



5. Re-order, A becomes $A\overline{C}$, AB becomes $AB\overline{C}$, $A\overline{B}$ becomes $A\overline{B}\,\overline{C}$, BC becomes $\overline{A}BC$, and $\overline{B}C$ becomes $\overline{A}\,\overline{B}C$. Edges from $\overline{C}$ to $\overline{A}\,\overline{B}\,\overline{C}$ and from B to $\overline{A}BC$ are removed and appropriately replaced. Finally remove duplicates $\overline{A}BC$ and $\overline{A}\,\overline{B}C$.



Translating Euler/Venn diagrams into DAG's, direct construction Given any Euler/Venn diagram V do the following to translate it into a corresponding DAG D:[10]

1. First identify all minimal regions of the Euler/Venn Diagram V. To each of these regions associate a name which contains the letters of all of the sets of the diagram or their complements. Thus for any minimal region R start by naming the region Λ (the empty string) and iteratively look at each of the sets represented in the diagram asking whether the minimal region is a subset of that set or its complement and concatenating the appropriate letter to its current name. Finally to make the names easier to read alphabetize the letters of the name.

2. Start by adding to an empty DAG one node appropriately named for each minimal region of the Euler/Venn diagram V.

3. Let N be the number of sets in the Euler/Venn diagram, thus the name of each of the minimal nodes consists of exactly N letters.[11] For each of these minimal nodes add to the diagram as their parents nodes with names consisting of N choose $N-1$, sometimes written as $\binom{N}{N-1}$, letters from each of their names, while being careful not to duplicate any node. Thus for each minimal region with a name of N letters construct parents for that node all of which have names of length $N-1$ and are subsets of the name of the minimal region. (See Figure 1 for further clarity.) If a duplicate occurs connect that minimal region to the already existing node. Continue this process for each of the nodes of $N-1$ letters and so on, until only nodes consisting of one letter are added. Finally add the node V as the parent of each of these nodes consisting on only one letter.

4. Starting with the minimal nodes and working up the DAG, eliminate all nodes who only have one child, and following this delete any nodes with no descendent minimal regions. If a node only has one child it is the union of one region thus equal to its child, and if it has no descendent minimal regions it is null.

5. Lastly shade and add $\otimes_n$'s to the minimal nodes of the DAG and then the entire DAG as done in the last step of the Inductive Construction.

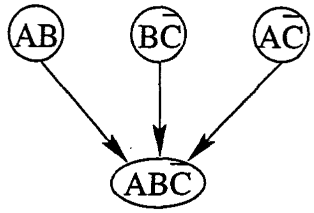
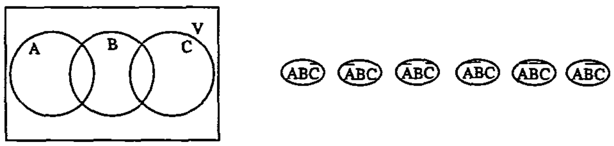


Figure 1: Example of rule 3. for one minimal node

Example of direct construction

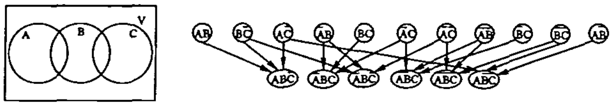
1. After first two steps, a partial DAG with only minimal regions.



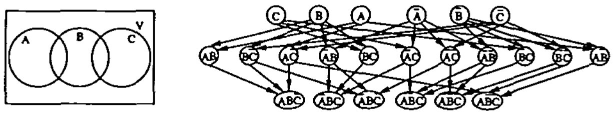
[10]Note, the intermediate DAG's used in this construction may not be well-formed in terms of the above grammar G.

[11]Each minimal region has a name of length N from part 1. of the current construction.

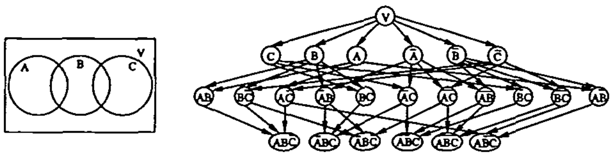
2. After first iteration of step 3, adding nodes $\{AB, B\overline{C}, A\overline{C}, \overline{A}B, BC, \overline{A}C, \overline{A}\,\overline{C}, \overline{A}\,\overline{B}, \overline{B}C, \overline{B}\,\overline{C}, A\overline{B}\}$



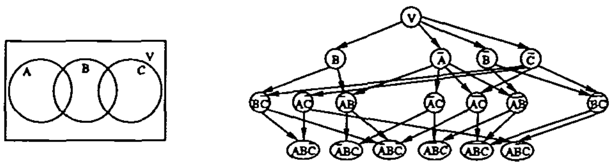
3. After second iteration of step 3, adding nodes $\{C, B, A, \overline{A}, \overline{B}, \overline{C}\}$.



4. After third iteration of step 3, adding node $\{V\}$.

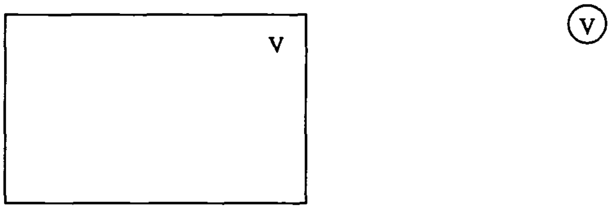


5. Final diagram after step 4, removing nodes $\{AB, BC, \overline{B}C, C, A\}$.

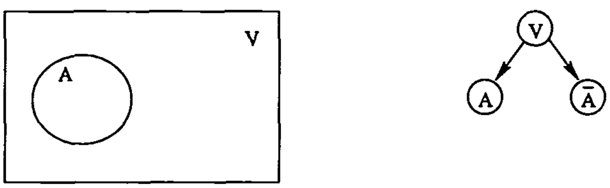


Examples of well-formed diagrams and their translations

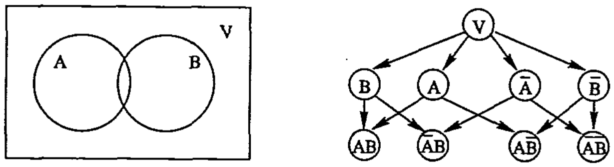
1. The empty diagram:



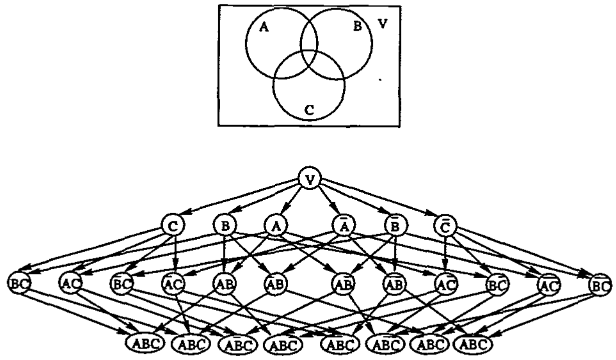
2. The diagram with one set:



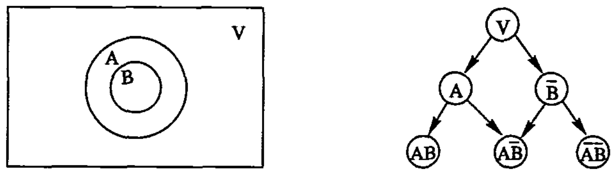
3. Two intersecting sets:



4. Three intersecting sets:



5. A set completely contained in another:



The Capturing of Essential Properties

Lemma 1 *Any Euler/Venn Diagram V can be translated into at least one DAG D.*

This is direct from either of the above construction techniques.

Lemma 2 *Any Euler/Venn Diagram V can be translated into a unique DAG D.*

Proof Sketch:
First we notice that the regions of a Euler/Venn diagram can be ordered using the subset relation into a partial order. Also any partial order or poset can be described uniquely up to isomorphism by its comprising covers relations. We then shade the nodes of the poset if that region is empty and put a $\otimes_n$ in the node if it contains a link of the $\otimes$ sequence of the same n. We now have that for each Euler/Venn diagram there is a unique characterizing poset. By above Lemma 1 and looking closely at the above construction technique it is seen that the DAG being constructing is, with the directed edges interpreted as spatial relations, just this poset. This can be shown inductively, focusing on the inductive construction technique. For the base case we look at the empty diagram, this has by definition

a unique DAG. Assume Euler/Venn Diagram V has a unique DAG and show that V with the addition of one set has exactly one new node for each new region and that the edge relation preserves the covers ordering. Here by rule 2(a) two nodes are added, one for the new circle and another for its complement, so that there are at least two new regions one corresponding to the new set and the other to its negation. Due to the grammar, each region crossed by the new set is divided into two new regions, and by rule 2(b) exactly those nodes are added to the DAG. Hence exactly the right number of nodes are being added. Finally by rule 2(c) the new DAG is re-ordered to preserve the covering ordering. Trivially it is noted that the DAG and the diagram both have the same nodes shaded and containing a links of a $\otimes$ sequences, since the same rules are used to shade the poset and the DAG. Thus for all Euler/Venn diagrams our translation results in a unique DAG.

Lemma 3 *Each DAG D is the translation of a unique class of isomorphic Euler/Venn diagrams.*

Proof Sketch:
First we observe that each DAG has a unique set of terminal nodes with shading and $\otimes$ information corresponding to the minimal regions of the diagram. We next realize that each class of isomorphic Euler/Venn diagrams is characterized by a unique set of minimal regions with shading and $\otimes$ information.[12] Hence any DAG is the translation of a unique class of isomorphic Euler/Venn diagrams.

Lemma 4 *The translation of a single Euler/Venn diagram into its corresponding DAG by the inductive and non-inductive techniques stated above results in two equivalent DAG's.*

Theorem 3 *For any set of Euler/Venn diagrams* $\{\mathfrak{V} \cup V\} \subset D_{EV_F}$*, there exists a unique corresponding set of DAG's* $T(\{\mathfrak{V} \cup V\}) \subset D_{EV_I}$ *such that:* $\mathfrak{V} \vdash_{EV_F} V$ *iff* $T(\mathfrak{V}) \vdash_{EV_I} T(V)$ *and* $\mathfrak{V} \models_{EV_F} V$ *iff* $T(\mathfrak{V}) \models_{EV_I} T(V)$.

The proof of this theorem uses the above Lemma 2 and Lemma 3 and then demonstrates the close relationship between the deductive and semantic systems of EV_F and EV_I.

Soundness and Completeness of EV_I

Theorem 4 *Soundness*
For every set of diagrams $\mathfrak{D} \cup D$*, if* $\mathfrak{D} \vdash_{EV_I} D$ *then* $\mathfrak{D} \models_{EV_I} D$.

[12]This can be seen from the fact that all of the diagrams in one isomorphism class can be shown to be equivalent to a Venn diagram with the shading of certain minimal regions and with $\otimes_n$'s in certain minimal regions. Thus it is by either the shaded or unshaded regions, since one is the complement of the other, and which minimal regions contain $\otimes_n$'s that the class is characterized.

Proof:
Given $\mathfrak{D} \vdash_{EV_I} D$ we know that $T^{-1}(\mathfrak{D}) \vdash_{EV_F} T^{-1}(D)$ from Theorem 3. From this it is concluded that $T^{-1}(\mathfrak{D}) \models_{EV_F} T^{-1}(D)$ from the soundness of EV_F. Lastly, again using Theorem 3, $\mathfrak{D} \models_{EV_I} D$ is concluded. ■

Theorem 5 *Completeness*
For every set of diagrams $\mathfrak{D} \cup D$*, if* $\mathfrak{D} \models_{EV_I} D$ *then* $\mathfrak{D} \vdash_{EV_I} D$.

Proof:
Given $\mathfrak{D} \models_{EV_I} D$ we know that $T^{-1}(\mathfrak{D}) \models_{EV_F} T^{-1}(D)$ from Theorem 3. From this it is concluded that $T^{-1}(\mathfrak{D}) \vdash_{EV_F} T^{-1}(D)$ from the completeness of EV_F. Lastly, again using Theorem 3, $\mathfrak{D} \vdash_{EV_I} D$ is concluded. ■

Future Directions

In future work, it will be shown that thinking of Euler/Venn diagrams as DAG's allows one to more easily define classes of Euler/Venn diagrams and algorithms for determining membership in these classes. A few such classifications are the class of Euler/Venn diagrams that do not have any intersecting regions, only containment sometimes called transition diagrams, and the class of Euler/Venn diagrams that can be drawn as a valid Euler/Venn diagrams without any shading.

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I would like to thank Jon Barwise, Gerald Allwein, Eric Hammer, Kathi Fisler, and the other members of the Indiana University Visual Inference Lab for their patience and help with this work. Special thanks also goes to the US Dept. of Education whose Grant # P200A502367 provided support for this research.

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