Comparing the Notions of Optimality in Strategic Games and Soft Constraints

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Abstract

The notion of optimality naturally arises in many areas of applied mathematics and computer science concerned with decision making.

Here we consider this notion in the context of two formalisms used for different purposes in reasoning about multi-agent systems. One of them are strategic games that are used to capture the idea that agents interact with each other while pursuing their own interests. The other are soft constraints that are used to express preferences in presence of constraints and uncertainty.

To relate the notions of optimality in these formalisms we define two mappings. We show for a natural mapping from soft constraints to strategic games that in general no relation exists between the notions of an optimal solution and Nash equilibrium. However, for a class of soft constraints that includes weighted constraints every optimal solution is a Nash equilibrium. In turn, for a natural mapping from strategic games to soft constraints the notion that coincides with optimality for soft constraints is that of Pareto efficient joint strategy.

Introduction

The concept of optimality is prevalent in many areas of applied mathematics and computer science. It is of relevance whenever we need to choose among several alternatives that are not equally preferable. For example, in constraint optimization, each solution of a constraint problem has a quality level associated with it and the aim is to choose an optimal solution, that is, a solution with an optimal quality level.

The aim of this paper is to clarify the relation between the notions of optimality used in two areas within AI: game theory (commonly used to model multi-agent systems) and soft constraints. This allows us to gain new insights into these notions which hopefully will lead to further cross-fertilization between these areas.

Game theory, notably *strategic games*, captures the idea of an interaction between agents (players) by equipping each agent with a payoff function on the game outcomes and allowing the agents to take actions (in strategic games simultaneously) with the aim of maximizing their payoffs. The most commonly used concept of optimality is that of a Nash equilibrium. Intuitively, it is an outcome that is optimal for each player under the assumption that only he may reconsider his action. Strategic games form one of the main tools in the area of multi-agent systems since they formalize in a simple and powerful way the idea that the agents interact with each other while pursuing their own interests.

Soft constraints, see e.g. (Bistarelli, Montanari, & Rossi 1997), are used to express preferences in the presence of constraints and uncertainty. An example are fuzzy constraints, see (Dubois, Fargier & Prade 1993) and (Ruttkay 1994), for which the preference of a solution is the minimal preference computed over all the constraints, and an optimal solution is the one with the highest preference. The research in this area mainly focused on the algorithms for finding optimal solutions and on the relationship between modelling formalisms (see (Rossi, Meseguer, & Schiex 2006)).

Strategic games

Let us recall now the notion of a strategic game, see, e.g., (Myerson 1991). A strategic game for n players ($n > 1$) is a sequence $(S_1, \ldots, S_n, p_1, \ldots, p_n)$, where for each $i \in [1..n]$

- S_i is the non-empty set of ***strategies*** available to player i,
- p_i is the ***payoff function*** for the player i, so $p_i : S_1 \times \ldots \times S_n \rightarrow \mathcal{R}$, where $\mathcal{R}$ is the set of real numbers.

Given a sequence of non-empty sets $S_1, \ldots, S_n$ and $s \in S_1 \times \ldots \times S_n$ we denote the ith element of s by s_i, abbreviate $N \setminus \{i\}$ to $-i$, and use the following standard notation of game theory, where $i \in [1..n]$ and $I := i_1, \ldots, i_k$ is a subsequence of $1, \ldots, n$:

- $s_I := (s_{i_1}, \ldots, s_{i_k})$,
- $(s'_i, s_{-i}) := (s_1, \ldots, s_{i-1}, s'_i, s_{i+1}, \ldots, s_n)$, where we assume that $s'_i \in S_i$,
- $S_I := S_{i_1} \times \ldots \times S_{i_k}$.

A joint strategy s is called

- a (pure) ***Nash equilibrium*** if $p_i(s) \geq p_i(s'_i, s_{-i})$ for all $i \in [1..n]$ and all $s'_i \in S_i$,
- ***Pareto efficient*** if for no joint strategy s', $p_i(s') \geq p_i(s)$ for all $i \in [1..n]$ and $p_i(s') > p_i(s)$ for some $i \in [1..n]$.

Pareto efficiency can be alternatively defined by considering the following strict ***Pareto ordering*** $<_P$ on the n-tuples of reals: $(a_1, \ldots, a_n) <_P (b_1, \ldots, b_n)$ iff $\forall i \in [1..n]\ a_i \leq b_i$ and $\exists\ i \in [1..n]\ a_i < b_i$. Then a joint strategy s is Pareto efficient iff the n-tuple $(p_1(s), \ldots, p_n(s))$ is a maximal element in the $<_P$ ordering on such n-tuples of reals.

To clarify these notions consider the classical Prisoner's Dilemma game represented by the following bimatrix representing the payoffs to both players:

	C_2	N_2
C_1	$3,3$	$0,4$
N_1	$4,0$	$1,1$

So each player i has two strategies, C_i (cooperate) and N_i (not cooperate), the payoff to player 1 for the joint strategy (C_1, N_2) is 0, etc. Here the unique Nash equilibrium is (N_1, N_2), while the other three joint strategies (C_1, C_2), (C_1, N_2) and (N_1, C_2) are Pareto efficient.

Soft constraints

Soft constraints, see e.g. (Bistarelli, Montanari, & Rossi 1997), model problems with preferences using c-semirings. A ***c-semiring*** is a tuple $\langle A, +, \times, \mathbf{0}, \mathbf{1} \rangle$, where:

- A is a set, called the ***carrier*** of the semiring, and $\mathbf{0}, \mathbf{1} \in A$;
- $+$ is commutative, associative, idempotent, $\mathbf{0}$ is its unit element, and $\mathbf{1}$ is its absorbing element;
- $\times$ is associative, commutative, distributes over $+$, $\mathbf{1}$ is its unit element and $\mathbf{0}$ is its absorbing element.

Elements $\mathbf{0}$ and $\mathbf{1}$ represent, respectively, the highest and lowest preference. While the operator $\times$ is used to combine preferences, the operator + induces a partial ordering on the carrier A defined by $a \leq b$ iff $a + b = b$.

Given a c-semiring $S = \langle A, +, \times, \mathbf{0}, \mathbf{1} \rangle$, and a set of variables V, each variable x with a domain $D(x)$, a ***soft constraint*** is a pair $\langle def, con \rangle$, where $con \subseteq V$ and $def : \times_{y \in con} D(y) \rightarrow A$. So a constraint specifies a set of variables (the ones in con), and assigns to each tuple of values from $\times_{y \in con} D(y)$, the Cartesian product of the variable domains, an element of the semiring carrier A.

A ***soft constraint satisfaction problem (SCSP)*** (in short, a ***soft CSP***) is a tuple $\langle C, V, D, S \rangle$ where V is a set of variables, with the corresponding set of domains D, C is a set of soft constraints over V and S is a c-semiring. Given an SCSP a ***solution*** is an instantiation of all the variables. The ***preference*** of a solution s is the combination by means of the $\times$ operator of all the preference levels given by the constraints to the corresponding subtuples of the solution, or more formally, $\times_{c \in C} def_c(s \downarrow_{con_c})$, where $\times$ is the multiplicative operator of the semiring and $def_c(s \downarrow_{con_c})$ is the preference associated by the constraint c to the projection of the solution s on the variables in con_c.

A solution is called ***optimal*** if there is no other solution with a strictly higher preference.

Three widely used instances of SCSPs are:

- **Classical CSPs** (in short **CSPs**), based on the c-semiring $\langle \{0, 1\}, \vee, \wedge, 0, 1 \rangle$. They model the customary CSPs in which tuples are either allowed or not. So CSPs can be seen as a special case of SCSPs.
- **Fuzzy CSPs**, based on the ***fuzzy c-semiring*** $\langle [0, 1], max, min, 0, 1 \rangle$. In such problems, preferences are the values in $[0, 1]$, combined by taking the minimum and the goal is to maximize the minimum preference.
- **Weighted CSPs**, based on the ***weighted c-semiring*** $\langle \Re_+, min, +, \infty, 0 \rangle$. Preferences are costs ranging over non-negative reals, which are aggregated using the sum. The goal is to minimize the total cost.

A simple example of a fuzzy CSP is the following one:

- three variables: x, y, and z, each with the domain $\{a, b\}$;
- two constraints: C_{xy} (over x and y) and C_{yz} (over y and z) defined by:
  $C_{xy} := \{(aa, 0.4), (ab, 0.1), (ba, 0.3), (bb, 0.5)\}$,
  $C_{yz} := \{(aa, 0.4), (ab, 0.3), (ba, 0.1), (bb, 0.5)\}$.

The unique optimal solution of this problem is *bbb* (an abbreviation for $x = y = z = b$). Its preference is 0.5.

From soft constraints to strategic games

In this and next section we relate the optimality notions in games and soft constraints. We shall see that the notion of optimality in soft constraints is not always related to the notion of Nash equilibria, but rather to the notion of Pareto efficient joint strategies.

Mapping soft constraints to graphical games

We define now a mapping from soft CSPs to a specific kind of games, and study the relation between the optimal outcomes in soft CSPs and Nash equilibria in the corresponding games.

Because soft constraints are defined by quantitative means it is natural to relate them to the original quantitative definition of a strategic game and not to the games with parametrized preferences. As in the case of CP-nets we shall identify the players with the variables. But the constraints link variables, so in the resulting game players are naturally connected. To capture this aspect we shall therefore use the graphical games.

A ***graphical game***, see (Kearns, Littman & Singh 2001), for n players with the corresponding strategy sets $S_1, \ldots, S_n$ with the payoffs being elements of a linearly ordered set A, is defined by assuming a neighbour relation *neigh* that given a player i yields its set of neighbours *neigh*(i). The payoff for player i is then a function from $\times_{j \in neigh(i) \cup \{i\}} S_j$ to A. We denote such a graphical game by $(S_1, \ldots, S_n, neigh, p_1, \ldots, p_n, A)$.

By using the canonic extensions of these payoff functions to the Cartesian product of all strategy sets one can then extend the previously introduced concepts to the graphical games. Further, when all pairs of players are neighbours a graphical game reduces to a strategic game.

Let us consider a first possible mapping from SCSPs to graphical games. In what follows we focus on SCSPs based on c-semirings with the carrier linearly ordered by $\leq$ (e.g. fuzzy or weighted) and on the concepts of optimal solutions in SCSPs and Nash equilibria in games.

Given a SCSP $P := \langle C, V, D, S \rangle$ we define the corresponding graphical game for $n = |V|$ players as follows:

- the players: one for each variable;

- the strategies of player i: all values in the domain of the corresponding variable x_i;
- the neighbourhood relation: $j \in neigh(i)$ iff the variables x_i and x_j appear together in some constraint from C;
- the payoff function of player i:
  Let $C_i \subseteq C$ be the set of constraints involving x_i and let X be the set of variables that appear together with x_i in some constraint in C_i (i.e., $X = \{x_j \mid j \in neigh(i)\}$.) Then given an assignment s to all variables in $X \cup \{x_i\}$ the payoff of player i w.r.t. s is defined by: $p_i(s) := \times_{c \in C_i} def_c(s \downarrow_{con_c})$.

We denote the resulting graphical game by $L(P)$ to emphasize the fact that the payoffs are obtained using a *local* information about each variable, by looking only at the constraints in which it is involved.

We now analyze the relation between the optimal solutions of a SCSP P and the Nash equilibria of the derived game $L(P)$.

General case In general, these two concepts are unrelated. Indeed, consider the fuzzy CSP defined at the end of Section . The corresponding game has:

- three players, x, y, and z;
- each player has two strategies, a and b;
- the neighbourhood relation is defined by: $neigh(x) := \{y\}$, $neigh(y) := \{x, z\}$, $neigh(z) := \{y\}$;
- the payoffs of the players are defined as follows:
  - for player x: $p_x(aa*) := 0.4$, $p_x(ab*) := 0.1$, $p_x(ba*) := 0.3$, $p_x(bb*) := 0.5$;
  - for player y: $p_y(aaa) := 0.4$, $p_y(aab) := 0.3$, $p_y(abb) := 0.1$, $p_y(bbb) := 0.5$, $p_y(bba) := 0.5$, $p_y(baa) := 0.3$, $p_y(bab) := 0.3$, $p_y(aba) := 0.1$;
  - for player z: $p_z(*aa) := 0.4$, $p_z(*ab) := 0.3$, $p_z(*ba) := 0.1$, $p_z(*bb) := 0.5$;

where $*$ stands for either a or b and where to facilitate the analysis we use the canonic extensions of the payoff functions p_x and p_z to the functions on $\{a, b\}^3$.

This game has two Nash equilibria: aaa and bbb. However, only bbb is an optimal solution of the fuzzy SCSP. One could thus think that in general the set of Nash equilibria is a superset of the set of optimal solutions of the corresponding SCSP. However, this is not the case. Indeed, consider a fuzzy CSP with as before three variables, x, y and z, each with the domain $\{a, b\}$, but now with the constraints:

$C_{xy} := \{(aa, 0.9), (ab, 0.6), (ba, 0.6), (bb, 0.9)\}$,
$C_{yz} := \{(aa, 0.1), (ab, 0.2), (ba, 0.1), (bb, 0.2)\}$.

Then aab, abb, bab and bbb are all optimal solutions but only aab and bbb are Nash equilibria of the corresponding graphical game.

CSPs with strictly monotonic $\times$ We now consider the case when the multiplicative operator $\times$ is strictly monotonic. Recall that given a c-semiring $\langle A, +, \times, \mathbf{0}, \mathbf{1}\rangle$, the operator $\times$ is ***strictly monotonic*** if for any $a, b, c \in A$ such that $a < b$ we have $c \times a < c \times b$. (The symmetric condition is taken care of by the commutativity of $\times$.)

Note for example that in the case of classical CSPs $\times$ is not strictly monotonic, as $a < b$ implies that $a = 0$ and $b = 1$ but $c \wedge a < c \wedge b$ does not hold then for $c = 0$. Also in fuzzy CSPs $\times$ is not strictly monotonic, as $a < b$ does not imply that $min(a, c) < min(b, c)$ for all c. In contrast, in weighted CSP $\times$ is strictly monotonic, as $a < b$ in the carrier means that $b < a$ as reals, so for any c we have $c+b < c+a$, i.e., $c \times a < c \times b$ in the carrier.

So consider now a c-semiring with a linearly ordered carrier and a strictly monotonic multiplicative operator. As in the previous case, given an SCSP P, it is possible that a Nash equilibrium of $L(P)$ is not an optimal solution of P. Consider for example a weighted SCSP P with

- two variables, x and y, each with the domain $D = \{a, b\}$;
- one constraint $C_{xy} := \{(aa, 3), (ab, 10), (ba, 10), (bb, 1)\}$.

The corresponding game $L(P)$ has:

- two players, x and y, who are neighbours of each other;
- each player has two strategies, a and b;
- the payoffs defined by: $p_x(aa) := p_y(aa) := 7$, $p_x(ab) := p_y(ab) := 0$, $p_x(ba) := p_y(ba) := 0$, $p_x(bb) := p_y(bb) := 9$.

Notice that, in a weighted CSP we have $a \le b$ in the carrier iff $b \le a$ as reals, so when passing from the SCSP to the corresponding game, we have complemented the costs w.r.t. 10, when making them payoffs. In general, given a weighted CSP, we can define the payoffs (which must be maximized) from the costs (which must be minimized) by complementing the costs w.r.t. the greatest cost used in any constraint of the problem.

Here $L(P)$ has two Nash equilibria, aa and bb, but only bb is an optimal solution. Thus, as in the fuzzy case, we have that there can be a Nash equilibrium of $L(P)$ that is not an optimal solution of P. However, in contrast to the fuzzy case, when the multiplicative operator of the SCSP is strictly monotonic, the set of Nash equilibria of $L(P)$ is a superset of the set of optimal solutions of P.

Theorem 1 *Consider a SCSP P defined on a c-semiring $\langle A, +, \times, \mathbf{0}, \mathbf{1}\rangle$, where A is linearly ordered and $\times$ is strictly monotonic, and the corresponding game $L(P)$. Then every optimal solution of P is a Nash equilibrium of $L(P)$.*

Classical CSPs The above result does not hold for classical CSPs. Indeed, consider a CSP with:

- three variables: x, y, and z, each with the domain $\{a, b\}$;
- two constraints: C_{xy} (over x and y) and C_{yz} (over y and z) defined by:
  $C_{xy} := \{(aa, 1), (ab, 0), (ba, 0), (bb, 0)\}$,
  $C_{yz} := \{(aa, 0), (ab, 0), (ba, 1), (bb, 0)\}$.

This CSP has no solutions, i.e., each optimal solution, in particular baa, has preference 0. However baa is not a Nash equilibrium of the resulting graphical game, since the payoff of player x increases when he switches to the strategy a.

However, if we restrict the domain of L to consistent CSPs, that is, CSPs with at least one solution with value 1, then the discussed inclusion does hold.

Theorem 2 *Consider a consistent CSP P and the corresponding game $L(P)$. Then every solution of P is a Nash equilibrium of $L(P)$.*

The reverse inclusion does not need to hold. Indeed, consider the following CSP:

- three variables: x, y, and z, each with the domain $\{a, b\}$;
- two constraints: C_{xy} and C_{yz} defined by:
  $C_{xy} := \{(aa, 1), (ab, 0), (ba, 0), (bb, 0)\}$,
  $C_{yz} := \{(aa, 1), (ab, 0), (ba, 0), (bb, 0)\}$.

Then aaa is a solution, so the CSP is consistent. But bbb is not an optimal solution, while it is a Nash equilibrium of the resulting game.

So for consistent CSPs our mapping L yields games in which the set of Nash equilibria is a, possibly strict, superset of the set of solutions of the CSP.

However, there are ways to relate CSPs and games so that the solutions and the Nash equilibria coincide. This is what is done in (Gottlob, Greco & Scarcello 2005), where the mapping is from the strategic games to CSPs. Notice that our mapping goes in the opposite direction and it is not the reverse of the one in (Gottlob, Greco & Scarcello 2005). In fact, the mapping in (Gottlob, Greco & Scarcello 2005) is not reversible.

Another mapping Other mappings from SCSPs to games can be defined. While our mapping L is in some sense 'local', since it considers the neighbourhood of each variable, we can also define an alternative 'global' mapping that considers all constraints. More precisely, given a SCSP $P = \langle C, V, D, S \rangle$, with a linearly ordered carrier A of S, we define the corresponding game on $n = |V|$ players, $GL(P) = (S_1, \ldots, S_n, p_1, \ldots, p_n, A)$ by using the following payoff function p_i for player i:

- given an assignment s to *all* variables in V, $p_i(s) := \times_{c \in C} def_c(s \downarrow_{con_c})$.

Notice that in the resulting game the payoff functions of all players are the same.

Theorem 3 *Consider an SCSP P over a linearly ordered carrier, and the corresponding game $GL(P)$. Then every optimal solution of P is a Nash equilibrium of $GL(P)$.*

The opposite inclusion does not need to hold. Indeed, consider again the weighted SCSP with

- two variables, x and y, each with the domain $D = \{a, b\}$;
- one constraint, $C_{xy} := \{(aa, 3), (ab, 10), (ba, 10), (bb, 1)\}$.

Since there is one constraint, the mappings L and GL coincide. Thus we have that aa is a Nash equilibrium of $GL(P)$ but is not an optimal solution of P.

From strategic games to soft constraints

Let us now consider the question of reversibility of our mappings. Both mappings defined in the previous section do not yield all graphical games, since the generated games are of a special kind. In particular, if two players have the same neighbourhood, then, for any given joint strategy, they have the same payoff. Thus, we cannot hope to reverse our mapping if we start from the set of all games.

Therefore we shall rather define a natural mapping from the graphical games to SCSPs for which we relate Nash equilibria and Pareto efficient joint strategies in games to optimal solutions in SCSPs.

In order to define a mapping from the graphical games to SCSPs, we consider SCSPs defined on c-semirings which are the Cartesian product of linearly ordered c-semirings. For example, the c-semiring $\langle [0, 1] \times [0, 1], (max, max), (min, min), (\mathbf{0}, \mathbf{0}), (\mathbf{1}, \mathbf{1}) \rangle$ is the Cartesian product of two fuzzy c-semirings. In a SCSP based on such a c-semiring, preferences are pairs, e.g. (0.1,0.9), combined using the min operator on each component, e.g. $(0.1, 0.8) \times (0.3, 0.6)$=$(0.1, 0.6)$. The ordering induced by using the max operator on each component is a partial ordering, e.g. $(0.1, 0.6) < (0.2, 0.8)$, while $(0.1, 0.9)$ is incomparable to $(0.9, 0.1)$.

Given a graphical game $G = (S_1, \ldots, S_n, neigh, p_1, \ldots, p_n, A)$ we define the corresponding SCSP $L'(G) = \langle C, V, D, S \rangle$, as follows:

- each variable x_i corresponds to a player i;
- the domain $D(x_i)$ of the variable x_i consists of the set of strategies of player i, i.e., $D(x_i) := S_i$;
- the c-semiring is $\langle A_1 \times \cdots \times A_n, (+_1, \ldots, +_n), (\times_1, \ldots, \times_n), (\mathbf{0}_1, \ldots, \mathbf{0}_n), (\mathbf{1}_1, \ldots, \mathbf{1}_n) \rangle$, the Cartesian product of n *arbitrary* linearly ordered semirings;
- soft constraints: for each variable x_i, one constraint $\langle def, con \rangle$ such that:
  - $con = neigh(x_i) \cup \{x_i\}$;
  - $def : \times_{y \in con} D(y) \to A_1 \times \cdots \times A_n$ such that for any $s \in \times_{y \in con} D(y)$, $def(s) := (d_1, \ldots, d_n)$ with $d_j = \mathbf{1}_j$ for every $j \neq i$ and $d_i = f(p_i(s))$, where $f : A \to A_i$ is an order preserving mapping from payoffs to preferences (i.e., if $r > r'$ then $f(r) > f(r')$ in the c-semiring's ordering).

To illustrate it consider again the example of the Prisoner's Dilemma game described in Section . Recall that in this game the only Nash equilibrium is (N_1, N_2), while the other three joint strategies are Pareto efficient.

We shall now construct a corresponding SCSP based on the Cartesian product of two weighted semirings. This SCSP according to the mapping L' has:[1]

- two variables: x_1 and x_2, each with the domain $\{c, n\}$;
- two constraints, both on x_1 and x_2:
  - constraint c_1 with $def(cc) := \langle 7, 0 \rangle$, $def(cn) := \langle 10, 0 \rangle$, $def(nc) := \langle 6, 0 \rangle$, $def(nn) := \langle 9, 0 \rangle$;

[1]Recall that in the weighted semiring **1** equals 0.

– constraint c_2 with $def(cc) := \langle 0, 7\rangle$, $def(cn) := \langle 0, 6\rangle$, $def(nc) := \langle 0, 10\rangle$, $def(nn) := \langle 0, 9\rangle$;

The optimal solutions of this SCSPs are: cc, with preference $\langle 7, 7\rangle$, nc, with preference $\langle 10, 6\rangle$, cn, with preference $\langle 6, 10\rangle$. The remaining solution, nn, has a lower preference in the Pareto ordering. Indeed, its preference $\langle 9, 9\rangle$ is dominated by $\langle 7, 7\rangle$, the preference of cc (since preferences are here costs and have to be minimized). Thus the optimal solutions coincide here with the Pareto efficient joint strategies of the given game. This is true in general.

Theorem 4 *Consider a game G and a corresponding SCSP $L'(G)$. Then the optimal solutions of $L'(G)$ coincide with the Pareto efficient joint strategies of G.*

As mentioned above, in (Gottlob, Greco & Scarcello 2005) a mapping is defined from the graphical games to CSPs such that Nash equilibria coincide with the solutions of CSP. Instead, our mapping is from the graphical games to SCSPs, and is such that Pareto efficient joint strategies and the optimal solutions coincide.

Since CSPs can be seen as a special instance of SCSPs, where only **1**, **0**, the top and bottom elements of the semiring, are used, it is possible to add to any SCSP a set of hard constraints. Therefore we can merge the results of the two mappings into a single SCSP, which contains the soft constraints generated by L' and also the hard constraints generated by the mapping in (Gottlob, Greco & Scarcello 2005), Below we denote these hard constraints by $H(G)$. If we do this, then the optimal solutions of the new SCSP with the preference higher than **0** are the Pareto efficient Nash equilibria of the given game.

Theorem 5 *Consider a game G and the corresponding SCSP $L'(G)$. If the optimal solutions of $L'(G)$ have global preference greater than **0**, they correspond to the Pareto efficient Nash equilibria of G.*

For example, in the Prisoner's Dilemma game, the mapping in (Gottlob, Greco & Scarcello 2005) would generate just one constraint on x_1 and x_2 with nn as the only allowed tuple. In our setting, when using as the linearly ordered c-semirings the weighted semirings, this would become a soft constraint with $def(cc) := def(cn) := def(nc) = \langle\infty, \infty\rangle$, $def(nn) := \langle 0, 0\rangle$. With this new constraint, all solutions have the preference $\langle\infty, \infty\rangle$, except for nn which has the preference $\langle 9, 9\rangle$ and thus is optimal. This solution corresponds to the joint strategy (N_1, N_2) with the payoff $(1, 1)$ (and thus preference $(9, 9)$) that is the only Pareto efficient Nash equilibrium.

This method allows us to identify among Nash equilibria the 'best' ones. One may also be interested in knowing whether there exist Nash equilibria which are also Pareto efficient joint strategies. For example, in the Prisoners' Dilemma example, there are no such Nash equilibria. To find any such joint strategies we can use the two mappings separately, to obtain, given a game G, both an SCSP $L'(G)$ and a CSP $H(G)$ (using the mapping in (Gottlob, Greco & Scarcello 2005)). Then we should take the intersection of the set of optimal solutions of $L'(G)$ and the set of solutions of $H(G)$.

Conclusions

In this paper we related two formalisms that are commonly used to reason about optimal outcomes: strategic games and soft constraints. In partcular we considered the relation between strategic games and various classes soft constraints. We showed that for a natural mapping from soft CSPs to strategic games in general no relation exists between the notions of optimal solutions of soft CSPs and Nash equilibria.

For the reverse direction we showed that for a natural mapping from strategic games to soft CSPs optimal solutions coincide not with Nash equilibria but with Pareto efficient joint strategies. Moreover, if we add suitable hard constraints to the soft constraints, optimal solutions coincide with Pareto efficient Nash equilibria.

The results of this paper clarify the relationship between various notions of optimality used in strategic games and soft constraints. These results can be used in many ways. One obvious way is to try to exploit computational results existing for one of these areas in another. This has been pursued already in (Gottlob, Greco & Scarcello 2005) for games versus hard constraints. Using our results this can also be done for strategic games versus soft constraints. For example, finding a Pareto efficient joint strategy involves mapping a game into a soft CSP and then solving it. Similar approach can also be applied to Pareto efficient Nash equilibria, which can be found by solving a suitable soft CSP.

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Appendix: Proofs

Theorem 1 *Consider a SCSP P defined on a c-semiring $\langle A, +, \times, \mathbf{0}, \mathbf{1}\rangle$, where A is linearly ordered and $\times$ is strictly monotonic, and the corresponding game $L(P)$. Then every optimal solution of P is a Nash equilibrium of $L(P)$.*

Proof. We prove that if a joint strategy s is not a Nash equilibrium of game $L(P)$, then it is not an optimal solution of SCSP P.

Let a be the strategy of player x in s, and let $s_{neigh(x)}$ and s_Y be, respectively, the joint strategy of the neighbours of x, and of all other players, in s. That is, $V = \{x\} \cup neigh(x) \cup Y$ and $s = ((x, a)s_{neigh(x)}s_Y)$.

By assumption there is a strategy b for x such that the payoff $p_x(s')$ for the joint strategy $s' := ((x, b)s_{neigh(x)}s_Y)$ is higher than $p_x(s)$. (We use here the canonic extension of p_x.)

So by the definition of the mapping L

$$\times_{c \in C_x} def_c(s \downarrow_{con_c}) < \times_{c \in C_x} def_c(s' \downarrow_{con_c}),$$

where C_x is the set of all the constraints involving x in SCSP P. But the preference of s and s' is the same on all the constraints not involving x and $\times$ is strictly monotonic, so we conclude that

$$\times_{c \in C} def_c(s \downarrow_{con_c}) < \times_{c \in C} def_c(s' \downarrow_{con_c}).$$

This means that s is not an optimal solution of P. □

Theorem 2 *Consider a consistent CSP P and the corresponding game $L(P)$. Then every solution of P is a Nash equilibrium of $L(P)$.*

Proof. Consider a solution s of P. In the resulting game $L(P)$ the payoff to each player is maximal, namely 1. So the joint strategy s is a Nash equilibrium in game $L(P)$. □

Theorem 3 *Consider an SCSP P over a linearly ordered carrier, and the corresponding game $GL(P)$. Then every optimal solution of P is a Nash equilibrium of $GL(P)$.*

Proof. An optimal solution of P, say s, is a joint strategy for which all players have the same, highest, payoff. So no other joint strategy exists for which some player is better off and consequently s is a Nash equilibrium. □

Theorem 4 *Consider a game G and a corresponding SCSP $L'(G)$. Then the optimal solutions of $L'(G)$ coincide with the Pareto efficient joint strategies of G.*

Proof. In the definition of the mapping L' we stipulated that the mapping f maintains the ordering from the payoffs to preferences. As a result each joint strategy s corresponds to the n-tuple of preferences $(f(p_1(s)), \ldots, f(p_n(s)))$ and the Pareto orderings on the n-tuples $(p_1(s), \ldots, p_n(s))$ and $(f(p_1(s)), \ldots, f(p_n(s)))$ coincide. Consequently a sequence s is an optimal solution of the SCSP $L'(G)$ iff $(f(p_1(s)), \ldots, f(p_n(s)))$ is a maximal element of the corresponding Pareto ordering. □

Theorem 5 *Consider a game G and the corresponding SCSP $L'(G)$. If the optimal solutions of $L'(G)$ have global preference greater than **0**, they correspond to the Pareto efficient Nash equilibria of G.*

Proof. Given any solution s, let p be its preference in $L'(G)$ and p' in $L'(G) \cup H(G)$. By the construction of the constraints $H(G)$ we have that p' equals p if s is a Nash equilibrium and p' equals **0** otherwise. The remainder of the argument is as in the proof of Theorem 4. □