

# The Complexity of Planning Revisited — A Parameterized Analysis

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## Abstract

The early classifications of the computational complexity of planning under various restrictions in STRIPS (Bylander) and SAS<sup>+</sup> (Bäckström and Nebel) have influenced following research in planning in many ways. We go back and reanalyse their subclasses, but this time using the more modern tool of parameterized complexity analysis. This provides new results that together with the old results give a more detailed picture of the complexity landscape. We demonstrate separation results not possible with standard complexity theory, which contributes to explaining why certain cases of planning have seemed simpler in practice than theory has predicted. In particular, we show that certain restrictions of practical interest are tractable in the parameterized sense of the term, and that a simple heuristic is sufficient to make a well-known partial-order planner exploit this fact.

## 1 Introduction

Bylander (1994) made an extensive analysis of the computational complexity of propositional STRIPS under various restrictions, like limiting the number of preconditions or effects. Bäckström and Nebel (1995) made a similar analysis of planning with multi-valued state variables in the SAS<sup>+</sup> formalism, investigating the complexity of all combinations of the P, U, B and S restrictions introduced by Bäckström and Klein (1991). These were among the first attempts to understand why and when planning is hard or easy and have had heavy influence on recent research in planning, of which we list a few representative examples. Giménez and Jonsson (2008), Chen and Giménez (2010) as well as Katz and Domshlak (2008) have studied the complexity of planning for various restrictions on the causal graph, the latter also considering combinations with restrictions P and U. Katz and Domshlak further pointed out a particularly important usage of such results, saying:

Computational tractability can be an invaluable tool even for dealing with problems that fall outside all the known tractable fragments of planning. For instance, tractable fragments of planning provide the foundations for most (if not all) rigorous heuristic estimates employed in planning as heuristic search.

Two examples of slightly different ways to do this are the following. Helmert (2004) used a planning algorithm for a simpler restricted problem to compute heuristic values for subproblems and then combine these values. Similarly, the popular  $h^+$  heuristic (Hoffmann 2005) exploits Bylander's results that planning is simpler with only positive preconditions and uses this as a relaxation for computing a heuristic value. As a complement to such analyses of restricted planning languages, Helmert (2006) studied the complexity and inherent restrictions in a number of application problems.

We revisit these early classifications of STRIPS and of SAS<sup>+</sup>, but using parameterized complexity analysis rather than standard complexity analysis. Parameterized complexity analysis was invented to enable a more fine-grained analysis than standard complexity analysis allows, by treating a parameter as independent of the instance rather than being a part of it. Somewhat simplified, the idea is as follows. Consider some problem and let  $n$  denote the instance size. We usually consider a problem as tractable if it can be solved by some algorithm in  $O(n^c)$  time, that is, in polynomial time. For many problems, like the NP-hard problems, we do not know of any significantly faster way to solve them than doing brute-force search, which typically requires exponential, or at least super-polynomial, time in  $n$ . In practice the search is often not exponential in the size of the whole instance, but rather in some smaller hard part of it. In these cases the complexity may rather be something like  $O(2^k n^c)$  where  $k$  is a parameter that is typically independent of the instance size  $n$ . Thus, the combinatorial explosion is confined to the parameter  $k$ . We say that a problem is *fixed-parameter tractable (FPT)* if it can be solved in this way. This is the essence of parameterized complexity theory and provides a tractability concept which is more relaxed than the usual one, while correlating better with tractability in practice for real-world problems. The theory also offers various classes for problems that are not FPT, for example **W**[1] and **W**[2]. Parameterized complexity analysis has contributed fundamental new insights into complexity theory (Downey and Fellows 1999). It is nowadays a very common technique in many areas of computer science, including many subareas of AI, like non-monotonic reasoning (Gottlob, Pichler, and Wei 2006), constraints (Gaspers and Szeider 2011), social choice (Brandt, Brill, and Seedig 2011) and argumentation (Ordyniak and Szeider 2011). The examples in planning are rare,

however. Downey, Fellows and Stege (1999) proved that STRIPS planning is  $\mathbf{W}[1]$ -hard and conjectured that it is also complete for  $\mathbf{W}[1]$ . We disprove this conjecture and show that STRIPS planning is actually  $\mathbf{W}[2]$ -complete. There is also a result by Bäckström and Jonsson (2011) that STRIPS planning is  $\mathbf{FPT}$  under a certain restriction that deliberately lower-bounds the plan length, thus not contradicting our results. This restriction was motivated by a different agenda, studying the expressive power of planning languages in general rather than subclasses of a particular language.

The parameterized analyses of planning that we provide in this paper does not replace the earlier results or make them obsolete. Since the parameterized complexity classes and the standard ones are not comparable, our results must be viewed as supplementary, providing further information. If we consider the previous classifications together with our parameterized classification we get a more detailed and informative picture of planning complexity than by considering either of them alone. This sheds new light on the discrepancy between theoretical and practical results regarding the difficulty of planning. For instance, while Bäckström and Nebel proved that restriction  $\mathbf{U}$  (actions can change only one variable) does not make planning easier under standard analysis, we show that it is actually easier from a parameterized point of view. This is interesting since restriction  $\mathbf{U}$  has been considered acceptable in some practical applications of planning, for instance on-board planning in spacecrafts (Williams and Nayak 1997; Brafman and Domshlak 2003). Furthermore, Bäckström and Nebel showed that planning is  $\mathbf{NP}$ -hard under restriction  $\mathbf{P}$  (there are never two actions that set the same variable value) but did not provide any better upper bound than in the unrestricted case. We show that planning is actually  $\mathbf{FPT}$  under this restriction. We also show that a standard partial-order planning algorithm (McAllester and Rosenblitt 1991) can exploit this fact with a minor modification that could be implemented as a heuristic. This suggests that many successful applications of planning might be cases where the problem is “almost tractable” and the algorithm used happens to implicitly exploit this. This is in line with the claim by Downey et. al. (2008) that in many cases existing algorithms with heuristics turn out to already be  $\mathbf{FPT}$  algorithms.

The rest of the paper is laid out as follows. Section 2 defines some concepts of parameterized complexity theory and Section 3 defines the  $\mathbf{SAS}^+$  and STRIPS languages. The hardness results are collected in Section 4 and the membership results in Section 5, including the result on using an existing planning algorithm. Section 6 summarizes the results of the paper and discusses some observations and consequences. The paper ends with a discussion in Section 7.

## 2 Parameterized Complexity

We define the basic notions of Parameterized Complexity and refer to other sources (Downey and Fellows 1999; Flum and Grohe 2006) for an in-depth treatment. A *parameterized problem* is a set of pairs  $\langle \mathbb{I}, k \rangle$ , the *instances*, where  $\mathbb{I}$  is the main part and  $k$  the *parameter*. The parameter is usually a non-negative integer. A parameterized problem is *fixed-parameter tractable (FPT)* if there exists an algorithm that

solves any instance  $\langle \mathbb{I}, k \rangle$  of size  $n$  in time  $f(k)n^c$  where  $f$  is an arbitrary computable function and  $c$  is a constant independent of both  $n$  and  $k$ .  $\mathbf{FPT}$  is the class of all fixed-parameter tractable decision problems.

Parameterized complexity offers a completeness theory, similar to the theory of  $\mathbf{NP}$ -completeness, that allows the accumulation of strong theoretical evidence that some parameterized problems are not fixed-parameter tractable. This theory is based on a hierarchy of complexity classes

$$\mathbf{FPT} \subseteq \mathbf{W}[1] \subseteq \mathbf{W}[2] \subseteq \mathbf{W}[3] \subseteq \dots$$

where all inclusions are believed to be strict. Each class  $\mathbf{W}[i]$  contains all parameterized decision problems that can be reduced to a certain canonical parameterized problem (known as  $\mathbf{WEIGHTED } i\text{-NORMALIZED SATISFIABILITY}$ ) under *parameterized* reductions. A parameterized problem  $L$  reduces to a parameterized problem  $L'$  if there is a mapping  $R$  from instances of  $L$  to instances of  $L'$  such that

1.  $\langle \mathbb{I}, k \rangle$  is a  $\mathbf{YES}$ -instance of  $L$  if and only if  $\langle \mathbb{I}', k' \rangle = R(\mathbb{I}, k)$  is a  $\mathbf{YES}$ -instance of  $L'$ ,
2. there is a computable function  $g$  such that  $k' \leq g(k)$ , and
3. there is a computable function  $f$  and a constant  $c$  such that  $R$  can be computed in time  $O(f(k) \cdot n^c)$ , where  $n$  denotes the size of  $\langle \mathbb{I}, k \rangle$ .

Not much is known about the relationship between the parameterized complexity classes and the standard ones, except that  $\mathbf{P} \subseteq \mathbf{FPT}$ .

## 3 Planning Framework

Let  $V = \{v_1, \dots, v_n\}$  be a finite set of *variables* over a finite *domain*  $D$ . Implicitly define  $D^+ = D \cup \{\mathbf{u}\}$ , where  $\mathbf{u}$  is a special value not present in  $D$ . Then  $D^n$  is the set of *total states* and  $(D^+)^n$  is the set of *partial states* over  $V$  and  $D$ , where  $D^n \subseteq (D^+)^n$ . The value of a variable  $v$  in a state  $s \in (D^+)^n$  is denoted  $s[v]$ . A  $\mathbf{SAS}^+$  *instance* is a tuple  $\mathbb{P} = \langle V, D, A, I, G \rangle$  where  $V$  is a set of variables,  $D$  is a domain,  $A$  is a set of *actions*,  $I \in D^n$  is the *initial state* and  $G \in (D^+)^n$  is the *goal*. Each action  $a \in A$  has a *precondition*  $\text{pre}(a) \in (D^+)^n$  and an *effect*  $\text{eff}(a) \in (D^+)^n$ . We will frequently use the convention that a variable has value  $\mathbf{u}$  in a precondition/effect unless a value is explicitly specified. Let  $a \in A$  and let  $s \in D^n$ . Then  $a$  is *valid* in  $s$  if for all  $v \in V$ , either  $\text{pre}(a)[v] = s[v]$  or  $\text{pre}(a)[v] = \mathbf{u}$ . Furthermore, the *result of  $a$  in  $s$*  is a state  $t \in D^n$  defined such that for all  $v \in V$ ,  $t[v] = \text{eff}(a)[v]$  if  $\text{eff}(a)[v] \neq \mathbf{u}$  and  $t[v] = s[v]$  otherwise.

Let  $s_0, s_\ell \in D^n$  and let  $\omega = \langle a_1, \dots, a_\ell \rangle$  be a sequence of actions. Then  $\omega$  is a *plan from  $s_0$  to  $s_\ell$*  if either

- 1)  $\omega = \langle \rangle$  and  $\ell = 0$  or
- 2) there are states  $s_1, \dots, s_{\ell-1} \in D^n$  such that for all  $i$ , where  $1 \leq i \leq \ell$ ,  $a_i$  is valid in  $s_{i-1}$  and  $s_i$  is the result of  $a_i$  in  $s_{i-1}$ . A state  $s \in D^n$  is a *goal state* if for all  $v \in V$ , either  $G[v] = s[v]$  or  $G[v] = \mathbf{u}$ . An action sequence  $\omega$  is a *plan for  $\mathbb{P}$*  if it is a plan from  $I$  to some goal state  $s \in D^n$ . We will study the following problem:

## BOUNDED SAS<sup>+</sup> PLANNING

*Instance:* A tuple  $\langle \mathbb{P}, k \rangle$  where  $\mathbb{P}$  is a SAS<sup>+</sup> instance and  $k$  is a positive integer.

*Parameter:* The integer  $k$ .

*Question:* Does  $\mathbb{P}$  have a plan of length at most  $k$ ?

We will consider the following four restrictions, originally defined by Bäckström and Klein (1991).

**P:** For each  $v \in V$  and each  $x \in D$  there is at most one  $a \in A$  such that  $\text{eff}(a)[v] = x$ .

**U:** For each  $a \in A$ ,  $\text{eff}(a)[v] \neq \mathbf{u}$  for exactly one  $v \in V$ .

**B:**  $|D| = 2$ .

**S:** For all  $a, b \in A$  and all  $v \in V$ , if  $\text{pre}(a)[v] \neq \mathbf{u}$ ,  $\text{pre}(b)[v] \neq \mathbf{u}$  and  $\text{eff}(a)[v] = \text{eff}(b)[v] = \mathbf{u}$ , then  $\text{pre}(a)[v] = \text{pre}(b)[v]$ .

For any set  $R$  of such restrictions we write  $R$ -BOUNDED SAS<sup>+</sup> PLANNING to denote the restriction of BOUNDED SAS<sup>+</sup> PLANNING to only instances satisfying the restrictions in  $R$ .

The propositional STRIPS language can be treated as the special case of SAS<sup>+</sup> satisfying restriction B. More precisely, this corresponds to the variant of STRIPS that allows negative preconditions.

## 4 Hardness Results

In this section we prove the two main hardness results of this paper. For the first proof we need the following **W[2]**-complete problem (Downey and Fellows 1999, p. 464).

### HITTING SET

*Instance:* A finite set  $S$ , a collection  $C$  of subsets of  $S$  and an integer  $k \leq |C|$ .

*Parameter:* The integer  $k$ .

*Question:* Is there a hitting set  $H \subseteq S$  such that  $|H| \leq k$  and  $H \cap c \neq \emptyset$  for every  $c \in C$ ?

**Theorem 1.**  $\{B, S\}$ -BOUNDED SAS<sup>+</sup> PLANNING is **W[2]**-hard, even when the actions have no preconditions.

*Proof.* By parameterized reduction from HITTING SET. Let  $\mathbb{I} = \langle S, C, k \rangle$  be an instance of this problem. We construct an instance  $\mathbb{I}' = \langle \mathbb{P}, k' \rangle$ , where  $\mathbb{P} = \langle V, D, A, I, G \rangle$ , of the  $\{B, S\}$ -BOUNDED SAS<sup>+</sup> PLANNING problem such that  $\mathbb{I}$  has a hitting set of size at most  $k$  if and only if there is a plan of length at most  $k' = k$  for  $\mathbb{I}'$  as follows. Let  $V = \{v_c \mid c \in C\}$  and let  $A = \{a_e \mid e \in S\}$  where  $\text{eff}(a_e)[v_c] = 1$  if  $e \in c$  and  $\text{eff}(a_e)[v_c] = \mathbf{u}$  otherwise. We set  $I = \langle 0, \dots, 0 \rangle$  and  $G = \langle 1, \dots, 1 \rangle$ . Clearly,  $\mathbb{P}$  satisfies restrictions B and S, and the actions have no preconditions. It is now routine to show that  $\mathbb{P}$  has a plan of length at most  $k'$  if and only if  $\mathbb{I}$  has a hitting set of size  $k$ .  $\square$

We continue with the second result. The following problem is **W[1]**-complete (Pietrzak 2003).

### PARTITIONED CLIQUE

*Instance:* A  $k$ -partite graph  $G = \langle V, E \rangle$  with partition  $V_1, \dots, V_k$  such that  $|V_i| = |V_j| = n$  for all  $i$ , where  $1 \leq i < j \leq k$ .

*Parameter:* The integer  $k$ .

*Question:* Are there nodes  $v_1, \dots, v_k$  such that  $v_i \in V_i$  for all  $i$ , where  $1 \leq i \leq k$  and,  $\{v_i, v_j\} \in E$  for all  $i, j$ , where  $1 \leq i < j \leq k$ ? (The graph  $\{\{v_1, \dots, v_k\}, \{\{v_i, v_j\} \mid 1 \leq i < j \leq k\}\}$  is a  $k$ -clique of  $G$ .)

**Theorem 2.**  $\{U, B, S\}$ -BOUNDED SAS<sup>+</sup> PLANNING is **W[1]**-hard, even for instances where every action has at most one precondition and one postcondition.

*Proof.* By parameterized reduction from PARTITIONED CLIQUE. Let  $G = \langle V, E \rangle$  be a  $k$ -partite graph where  $V$  is partitioned into  $V_1, \dots, V_k$ . Let  $k_2 = \binom{k}{2}$  and  $k' = 7k_2 + k$ . We define  $J_i = \{j \mid 1 \leq j \leq k \text{ and } j \neq i\}$  for every  $1 \leq i \leq k$ .

For the  $\{U, B, S\}$ -BOUNDED SAS<sup>+</sup> PLANNING instance  $\mathbb{P}$  we introduce four kinds of variables:

1) For every  $e \in E$  we introduce an *edge variable*  $x(e)$ .

2) For every  $1 \leq i \leq k$  and  $v \in V_i$  we introduce  $k - 1$  *vertex variables*  $x(v, j)$  where  $j \in J_i$ .

3) For every  $1 \leq i \leq k$  and every  $j \in J_i$  we introduce a *checking variable*  $x(i, j)$ .

4) For every  $v \in V$ , we introduce a *clean-up variable*  $x(v)$ .

We also introduce five kinds of actions:

1) For every  $e \in E$  we introduce an action  $a^e$  such that  $\text{eff}(a^e)[x(e)] = 1$ .

2) For every  $e = \{v_i, v_j\} \in E$  where  $v_i \in V_i$  and  $v_j \in V_j$ , we introduce two actions  $a_i^e$  and  $a_j^e$  such that  $\text{pre}(a_i^e)[x(e)] = 1$ ,  $\text{eff}(a_i^e)[x(v_i, j)] = 1$ ,  $\text{pre}(a_j^e)[x(e)] = 1$  and  $\text{eff}(a_j^e)[x(v_j, i)] = 1$ .

3) For every  $v \in V_i$  and  $j \in J_i$ , we introduce an action  $a_j^v$  such that  $\text{pre}(a_j^v)[x(v, j)] = 1$  and  $\text{eff}(a_j^v)[x(i, j)] = 1$ .

4) For every  $v \in V$ , we introduce an action  $a_v$  such that  $\text{eff}(a_v)[x(v)] = 1$ .

5) For every  $v \in V_i$ , for some  $1 \leq i \leq k$ , and  $j \in J_i$ , we introduce an action  $a_v^j$  such that  $\text{pre}(a_v^j)[x(v)] = 1$  and  $\text{eff}(a_v^j)[x(v, j)] = 0$ .

Let  $A_1, \dots, A_5$  be sets of actions corresponding to these five groups, and let  $A = A_1 \cup \dots \cup A_5$  be the set of all actions. Let  $I = \langle 0, \dots, 0 \rangle$  and define  $G$  such that all checking variables  $x(i, j)$  are 1, all vertex variables  $x(v, j)$  are 0 and the rest are  $\mathbf{u}$ .

We now need to prove that  $G$  has a  $k$ -clique if and only if there is a plan for  $\mathbb{P}$  of length at most  $k'$ . We sketch the leftward direction; the opposite is similar. Assume  $G$  has a  $k$ -clique  $K = \langle V_K, E_K \rangle$  where  $V_K = \{v_1, \dots, v_k\}$  with  $v_i \in V_i$  for every  $1 \leq i \leq k$ . For all  $1 \leq i < j \leq k$ , we apply the actions  $a_{\{v_i, v_j\}} \in A_1$  and  $a_i^{\{v_i, v_j\}}, a_j^{\{v_i, v_j\}} \in A_2$ . This gives  $3k_2$  actions. Then for each checking variable  $x(i, j)$ , for every  $1 \leq i \leq k$  and  $j \in J_i$ , we apply  $a_j^{v_i} \in A_3$ . This gives  $2k_2$  actions. Now we have all checking variables set to the required value 1, but the vertex variables  $x(v_i, j)$ , for  $1 \leq i \leq k$  and  $j \in J_i$ , still bear the value 1 which will have to be set back to 0 in the goal state. So we need some actions to “clean up” the values of these vertex variables. First we set up a cleaner for each vertex  $v_i$  by applying  $a_{v_i} \in$



$A_4$ . This gives  $k$  actions. Then we use  $a_{v_i}^j \in A_5$  for all  $j \in J_i$  to set the vertex variables  $x(v_i, j)$  to 0. This requires  $2k_2$  actions. We observe that all the checking variables are now set to 1, and all vertex variables are set to 0. The goal state is therefore reached from the initial state by the execution of exactly  $k' = k + 7k_2$  actions, as required.  $\square$

## 5 Membership Results

Our membership results are based on *first-order (FO) model checking* (Sec. 5.1) and *partial-order planning* (Sec. 5.2).

### 5.1 Model Checking

For a class of FO formulas  $\Phi$  we define the following parameterized decision problem.

$\Phi$ -FO MODEL CHECKING

*Instance:* A finite structure  $\mathcal{A}$ , an FO formula  $\varphi \in \Phi$ .

*Parameter:* The length of  $\varphi$ .

*Question:* Does  $\varphi$  have a model?

Let  $\Sigma_1$  be the class of all FO formulas of the form  $\exists x_1 \dots \exists x_t. \varphi$  where  $t$  is arbitrary and  $\varphi$  is a quantifier-free FO formula. For arbitrary positive integer  $u$ , let  $\Sigma_{2,u}$  denote the class of all FO formulas of the form  $\exists x_1 \dots \exists x_t \forall y_1 \dots \forall y_u. \varphi$  where  $t$  is arbitrary and  $\varphi$  is a quantifier-free FO formula. Flum and Grohe (2006, Theorem 7.22) have shown the following result.

**Proposition 1.** *The problem  $\Sigma_1$ -FO MODEL CHECKING is  $\mathbf{W}[1]$ -complete. For every positive integer  $u$  the problem  $\Sigma_{2,u}$ -FO MODEL CHECKING is  $\mathbf{W}[2]$ -complete.*

We will reduce planning to model checking, so for an arbitrary planning instance  $\mathbb{I} = \langle \mathbb{P}, k \rangle$  (where  $\mathbb{P} = \langle V, D, A, I, G \rangle$ ) of the BOUNDED SAS<sup>+</sup> PLANNING problem we need a relational structure  $\mathcal{A}(\mathbb{P})$  defined as:

- The universe of  $\mathcal{A}(\mathbb{P})$  is  $V \cup A \cup D^+$ .
- $\mathcal{A}(\mathbb{P})$  contains the unary relations  $\text{Var} = V$ ,  $\text{Act} = A$ , and  $\text{Dom} = D^+$  together with the following relations of higher arity:
  - $\text{Init} = \{ \langle v, x \rangle \in V \times D \mid I[v] = x \}$ ,
  - $\text{Goalv} = \{ \langle v, x \rangle \in V \times D \mid G[v] = x \neq \mathbf{u} \}$ ,
  - $\text{Pre} = \{ \langle a, v \rangle \in A \times V \mid \text{pre}(a)[v] \neq \mathbf{u} \}$ ,
  - $\text{Eff} = \{ \langle a, v \rangle \in A \times V \mid \text{eff}(a)[v] \neq \mathbf{u} \}$ ,
  - $\text{Prev} = \{ \langle a, v, x \rangle \in A \times V \times D \mid \text{pre}(a)[v] = x \neq \mathbf{u} \}$
  - $\text{Effv} = \{ \langle a, v, x \rangle \in A \times V \times D \mid \text{eff}(a)[v] = x \neq \mathbf{u} \}$ .

**Theorem 3.** BOUNDED SAS<sup>+</sup> PLANNING is in  $\mathbf{W}[2]$ .

*Proof.* By parameterized reduction to the  $\mathbf{W}[2]$ -complete problem  $\Sigma_{2,2}$ -FO MODEL CHECKING. Let  $\mathbb{I} = \langle \mathbb{P}, k \rangle$  (where  $\mathbb{P} = \langle V, D, A, I, G \rangle$ ) be an instance of BOUNDED SAS<sup>+</sup> PLANNING. We construct an instance  $\mathbb{I}' = \langle \mathcal{A}(\mathbb{P}), \varphi \rangle$  of  $\Sigma_{2,2}$ -FO MODEL CHECKING such that  $\mathbb{I}$  has a solution if and only if  $\mathbb{I}'$  has a solution and the size of the formula  $\varphi$  is bounded by some function that only depends on  $k$ . Assume without loss of generality that  $A$  contains a dummy action  $\hat{a}$  with no preconditions and no effects. To define  $\varphi$  we first need the following definitions.

We define a formula  $\text{value}(\langle a_1, \dots, a_i \rangle, v, x)$  such that  $\text{value}(\langle \rangle, v, x) = \text{Init}(v, x)$  and  $\text{value}(\langle a_1, \dots, a_i \rangle, v, x) = (\text{value}(\langle a_1, \dots, a_{i-1} \rangle, v, x) \wedge \neg \text{Eff}(a_i, v)) \vee \text{Effv}(a_i, v, x)$  for every  $0 \leq i \leq k$ , which holds if applying  $a_1, \dots, a_i$  in state  $I$  results in a state  $s$  such that  $s[v] = x$ .

We also define a formula  $\text{check-pre}(\langle a_1, \dots, a_i \rangle, v, x) = \text{Prev}(a_i, v, x) \rightarrow \text{value}(\langle a_1, \dots, a_{i-1} \rangle, v, x)$  for all  $1 \leq i \leq k$ , that is,  $\forall v \forall x. \text{Var}(v) \wedge \text{Dom}(x) \wedge \text{check-pre}(\langle a_1, \dots, a_i \rangle, v, x)$  holds if all preconditions of action  $a_i$  are satisfied after actions  $a_1, \dots, a_{i-1}$  have been executed in state  $I$ . We similarly define a formula  $\text{check-pre-all}(\langle a_1, \dots, a_k \rangle, v, x) = \bigwedge_{i=1}^k \text{check-pre}(\langle a_1, \dots, a_i \rangle, v, x)$  that “checks” the preconditions of all actions in a sequence.

Finally, define  $\text{check-goal}(\langle a_1, \dots, a_k \rangle, v, x) = \text{Goalv}(v, x) \rightarrow \text{value}(\langle a_1, \dots, a_k \rangle, v, x)$ . The formula  $\forall v \forall x. \text{Var}(v) \wedge \text{Dom}(x) \wedge \text{check-goal}(\langle a_1, \dots, a_k \rangle, v, x)$  holds if the goal state is reached after the execution of the sequence  $a_1, \dots, a_k$  in the state  $I$ .

We can now define the formula  $\varphi$  itself as:

$$\begin{aligned} \varphi = & \exists a_1 \dots \exists a_k \forall v \forall x. \\ & (\bigwedge_{i=1}^k \text{Act}(a_i)) \wedge \\ & (\text{Var}(v) \wedge \text{Dom}(x) \rightarrow \\ & \quad \text{check-pre-all}(\langle a_1, \dots, a_k \rangle, v, x) \wedge \\ & \quad \text{check-goal}(\langle a_1, \dots, a_k \rangle, v, x)). \end{aligned}$$

Evidently  $\varphi \in \Sigma_{2,2}$ , the length of  $\varphi$  is bounded by some function that only depends on  $k$  and  $\mathcal{A}(\mathbb{P}) \models \varphi$  if and only if  $\mathbb{P}$  has a plan of length at most  $k$ . The dummy action guarantees that there is a plan exactly of length  $k$  if there is a shorter plan.  $\square$

The proof of the next theorem resembles the previous proof but the details are a bit involved. Thus, we only provide a high-level description of it.

**Theorem 4.**  $\{U\}$ -BOUNDED SAS<sup>+</sup> PLANNING is in  $\mathbf{W}[1]$ .

*Proof sketch:* In order to show  $\mathbf{W}[1]$ -membership of  $\{U\}$ -BOUNDED SAS<sup>+</sup> PLANNING we will reduce this problem to  $\Sigma_1$ -FO MODEL CHECKING and the basic idea is fairly close to the proof of Theorem 3. However, we cannot directly express within  $\Sigma_1$  that all the preconditions of an action are satisfied, as this would require a further universal quantification and thus move the formula to  $\Sigma_{2,u}$ . Hence, we avoid the universal quantification with a trick: we observe that the preconditions only need to be checked with respect to at most  $k$  “important” variables, that is, the variables in which the preconditions of an action differ from the initial state. If the precondition differs in more than  $k$  variables from the initial state, then it cannot be used in any plan of length  $k$ . It is now possible to guess the important variables with existential quantifiers.

It remains to check that all the significant variables are among these guessed variables. We do this without universal quantification by adding dummy elements  $d_1, \dots, d_k$  and a relation  $\text{Diff-act}$  to the relational structure  $\mathcal{A}(\mathbb{P})$ . The relation associates with each action exactly  $k$  different elements. These elements consist of all the important variables of the action, say the number of these variables is  $k'$ , plus  $k - k'$

dummy elements. Hence, by guessing these  $k$  elements and eliminating the dummy elements, the formula knows all the significant variables of the action and can check the preconditions without a universal quantification.  $\square$

## 5.2 Partial-order Planning

To prove that  $\{P\}$ -BOUNDED SAS<sup>+</sup> PLANNING is in FPT we use a slight modification of the well-known planning algorithm by McAllester and Rosenblitt (1991), which we refer to as MAR. It appears in Figure 1, combining the original and the modified versions into one. The only modification is the value of  $L'$ , which could easily be implemented as a heuristic for the original algorithm. The algorithm is generalized to SAS<sup>+</sup> rather than propositional STRIPS, which is straightforward and appears in the literature (Bäckström 1994). We only explain the algorithm and our notation, referring the reader to the original paper for details.

The algorithm works on a partially ordered set of *action occurrences*, each occurrence being a unique copy of an action. For each precondition  $\text{pre}(o_c)[v] \neq \mathbf{u}$  of an occurrence  $o_c$ , the algorithm uses a *causal link*  $o_p \xrightarrow{v=x} o_c$  to explicitly keep track of which other occurrence  $o_p$  with  $\text{eff}(o_p)[v] = x$  guarantees this precondition. An occurrence  $o_t$  is a *threat* to  $o_p \xrightarrow{v=x} o_c$  if  $\text{eff}(o_t)[v] \neq \mathbf{u}$  and  $o_p \neq o_t \neq o_c$ . A *plan structure* for a planning instance  $\mathbb{P} = \langle V, D, A, I, G \rangle$  is a tuple  $\Theta = \langle \mathbf{O}, \mathbf{P}, \mathbf{L} \rangle$  where  $\mathbf{O}$  is a finite set of action occurrences over  $A$ ,  $\mathbf{P}$  is a binary relation over  $\mathbf{O}$  and  $\mathbf{L}$  is a set of causal links. We write  $o \prec o'$  for  $\langle o, o' \rangle \in \mathbf{P}$ . Furthermore,  $\mathbf{O}$  always contains the two special elements  $o_I, o_G$ , where  $\text{eff}(o_I) = I$ ,  $\text{pre}(o_G) = G$  and  $o_I \prec o_G \in \mathbf{P}$ . An *open goal* in  $\Theta$  is a tuple  $\langle o, v, x \rangle$  such that  $o \in \mathbf{O}$ ,  $\text{pre}(o)[v] = x \neq \mathbf{u}$  and there is no  $o' \in \mathbf{O}$  such that  $o' \xrightarrow{v=x} o \in \mathbf{L}$ .

We say  $\Theta$  is *complete* if both the following conditions hold: 1) For all  $o_c \in \mathbf{O}$  and all  $v \in V$  such that  $\text{pre}(o_c)[v] = x \neq \mathbf{u}$ , there is a causal link  $o_p \xrightarrow{v=x} o_c \in \mathbf{L}$ . 2) For every  $o_p \xrightarrow{v=x} o_c \in \mathbf{L}$  and every threat  $o_t \in \mathbf{O}$  to  $o_p \xrightarrow{v=x} o_c$ , either  $o_t \prec o_p \in \mathbf{P}$  or  $o_c \prec o_t \in \mathbf{P}$ . McAllester and Rosenblitt proved that if starting with  $\Theta = \langle \{o_I, o_G\}, \{o_I \prec o_G\}, \emptyset \rangle$  then the algorithm fails if there is no plan and otherwise returns a plan structure  $\langle \mathbf{O}, \mathbf{P}, \mathbf{L} \rangle$  such that any topological sorting of  $\mathbf{O} - \{o_I, o_G\}$  consistent with  $\mathbf{P}$  is a plan.

That the modified variant of MAR is correct for SAS<sup>+</sup>-P instances is based on the following observation about the original variant applied to such instances. Consider three occurrences  $o_1, o_2, o_3$  such that  $o_1$  has preconditions  $v = x$  and  $w = y$  which are both effects of  $o_2$ . If  $v = x$  is also an effect of  $o_3$ , then also  $w = y$  must be an effect of  $o_3$  due to restriction P. However, the algorithm must link both conditions from the same occurrence, either  $o_2$  or  $o_3$ , since it would otherwise add both  $o_2 \prec o_3$  and  $o_3 \prec o_2$ , causing it to fail. The set of possible outcomes for the two variants are thus identical, but the modified variant is an FPT algorithm.

**Theorem 5.**  $\{P\}$ -BOUNDED SAS<sup>+</sup> PLANNING is in FPT.

*Proof.* Consider the modified version of MAR. All nodes in the search tree run in polynomial time in the instance size. The search tree contains two types of nodes: leafs that terminate in either line 2 or 3 and nodes that make a nondeter-

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1 function Plan( $\Theta = \langle \mathbf{O}, \mathbf{P}, \mathbf{L} \rangle, k$ )
2 if  $\langle \mathbf{O}, \mathbf{P} \rangle$  is not acyclic or  $|\mathbf{O}| > k + 2$  then fail
3 elseif  $\Theta$  is complete then return  $\Theta$ 
4 elseif there is an  $o_p \xrightarrow{v=x} o_c \in \mathbf{L}$  with a threat  $o_t \in \mathbf{O}$ 
   and neither  $o_t \prec o_p \in \mathbf{P}$  nor  $o_c \prec o_t \in \mathbf{P}$  then
5a choose either of
5b return Plan( $\langle \mathbf{O}, \mathbf{P} \cup \{o_t \prec o_p\}, \mathbf{L} \rangle, k$ )
5c return Plan( $\langle \mathbf{O}, \mathbf{P} \cup \{o_c \prec o_t\}, \mathbf{L} \rangle, k$ )
6 else arbitrarily choose an open goal  $g = \langle o_c, v, x \rangle$ 
7a nondeterministically do either
7b 1) nondeterministically choose an  $o_p \in \mathbf{O}$ 
   such that  $\text{eff}(o_p)[v] = x$ 
7c 2) if there is an  $a \in A$  such that  $\text{eff}(a)[v] = x$ 
   then let  $o_p$  be a new occurrence of  $a$ 
8 if original algorithm then
    $\mathbf{L}' := \{o_p \xrightarrow{v=x} o_c\}$ 
9 if modified algorithm then
    $\mathbf{L}' := \{o_p \xrightarrow{w=y} o_c \mid \text{eff}(o_p)[w] = \text{pre}(o_c)[w] = y,$ 
    $y \neq \mathbf{u} \text{ and } \langle o_c, w, y \rangle \text{ is an open goal} \}$ 
10 return Plan( $\langle \mathbf{O} \cup \{o_p\}, \mathbf{P} \cup \{o_p \prec o_c\}, \mathbf{L} \cup \mathbf{L}' \rangle, k$ )

```

Figure 1: The MAR algorithm.

ministic choice either in line 5 or in line 7 and then make a recursive call. The latter nodes correspond to branching points in the search tree, and we analyse their contribution to the search-tree size separately.

Each time line 5 is visited, it adds a new element to  $\mathbf{P}$ , which thus grows monotonically along every branch in the search tree. We can thus visit line 5 at most  $(k + 2)^2$  times along any branch since  $|\mathbf{P}| \leq (k + 2)^2$ . There are two choices in line 5 so it contributes at most a factor  $2^{(k+2)^2}$  to the size of the search tree. Also  $\mathbf{O}$  grows monotonically along every branch and  $|\mathbf{O}| \leq k + 2$ . At any visit to lines 6–10 there are thus at most  $k + 1$  occurrences with open goals and at most  $k + 1$  different occurrences to link these goals to. That is, the preconditions of each occurrence are partitioned into at most  $k + 1$  parts, each part having all its elements linked at once in line 9. Lines 6–10 can thus be visited at most  $(k + 1)^2$  times along any branch in the search tree. Since there are at most  $k + 1$  existing occurrences to link to and at most one action to instantiate as a new occurrence, the branching factor is  $k + 2$ . The contribution of this to the size of the search tree is thus at most a factor  $(k + 2)^{(k+1)^2}$ . Hence, the total search-tree size is at most  $2 \cdot 2^{(k+2)^2} (k + 2)^{(k+1)^2}$  where the factor 2 accounts for the leaves. This does not depend on the instance size and each node is polynomial-time in the instance size so the modified MAR algorithm is an FPT algorithm.  $\square$

## 6 Summary of Results

The complexity results for the various combinations of restrictions P, U, B and S are displayed in Figure 2. Solid lines denote separation results by Bäckström and Nebel (1995), using standard complexity analysis, while dashed lines denote separation results from our parameterized analysis. The

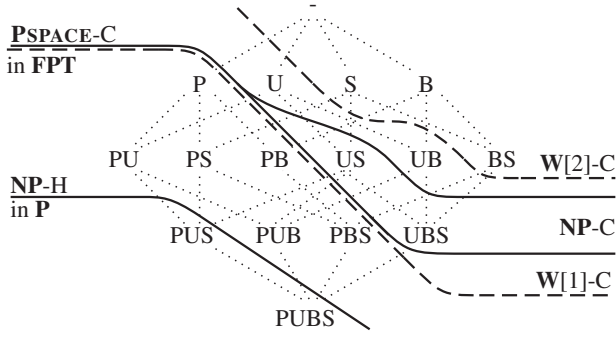


Figure 2: Complexity of BOUNDED SAS<sup>+</sup> PLANNING for all combinations of restrictions P, U, B and S.

W[2]-completeness results follow from Theorems 1 and 3, the W[1]-completeness results follow from Theorems 2 and 4, and the FPT results follow from Theorem 5.

Bylander (1994) studied the complexity of STRIPS under varying numbers of preconditions and effects, which is natural to view as a relaxation of restriction U in SAS<sup>+</sup>. Table 1 shows such results (for arbitrary domain sizes  $\geq 2$ ) under both parameterized and standard analysis. The parameterized results are derived as follows. For actions with an arbitrary number of effects, the results follow from Theorems 1 and 3. For actions with at most one effect, we have two cases: With no preconditions the problem is trivially in P. Otherwise, the results follow from Theorems 2 and 4.

We are left with the case when the number of effects is bounded by some constant  $m_e > 1$ . Bäckström (1992, proof of Theorem 6.7) presented a polynomial time reduction of this class of SAS<sup>+</sup> instances to the class of instances with one effect. It is easy to verify that his reduction is a parameterized reduction so we have membership in W[1] by Theorem 4. When  $m_p \geq 1$ , then we also have W[1]-hardness by Theorem 2. For the final case ( $m_p = 0$ ), we have no corresponding parameterized hardness result.

All non-parameterized hardness results in Table 1 follow directly from Bylander’s (1994, Fig. 1 and 2) complexity results for STRIPS. Note that we use results both for bounded and unbounded plan existence, which is justified since the unbounded case is (trivially) polynomial-time reducible to the bounded case. The membership results for PSPACE are immediate since BOUNDED SAS<sup>+</sup> PLANNING is in PSPACE. The membership results for NP (when  $m_p = 0$ ) follow from Bylander’s (1994) Theorem 3.9, which says that every solvable STRIPS instance with  $m_p = 0$  has a plan of length  $\leq m$  where  $m$  is the number of actions. It is easy to verify that the same bound holds for SAS<sup>+</sup> instances.

Since W[1] and W[2] are not directly comparable to the standard complexity classes we get interesting separations from combining the two methods. For instance, we can single out restriction U as making planning easier than in the general case, which is not possible with standard analysis. Since restrictions B and S remain as hard as the general case even under parameterized analysis, this shows that U is a more interesting and important restriction than the other two. Even more interesting is that planning is in FPT under

|               | $m_e = 1$          | fix $m_e > 1$      | arb. $m_e$         |
|---------------|--------------------|--------------------|--------------------|
| $m_p = 0$     | in P<br>in P       | in W[1]<br>NP-C    | W[2]-C<br>NP-C     |
| $m_p = 1$     | W[1]-C<br>NP-H     | W[1]-C<br>NP-H     | W[2]-C<br>PSPACE-C |
| fix $m_p > 1$ | W[1]-C<br>NP-H     | W[1]-C<br>PSPACE-C | W[2]-C<br>PSPACE-C |
| arb. $m_p$    | W[1]-C<br>PSPACE-C | W[1]-C<br>PSPACE-C | W[2]-C<br>PSPACE-C |

Table 1: Complexity of BOUNDED SAS<sup>+</sup> PLANNING, restricting the number of preconditions ( $m_p$ ) and effects ( $m_e$ )

restriction P, making it easier than the combination restriction US, while it seems to be rather the other way around for standard analysis where restriction P is only known to be hard for NP. In general, we see that there are still a number of open problems of this type in both Figure 2 and Table 1 for the standard analysis, while there is only one single open problem for the parameterized analysis: hardness for the case where  $m_p = 0$  and  $m_e$  is fixed.

## 7 Discussion

This work opens up several new research directions. We briefly discuss some of them below.

Although a modification was needed to make MAR an FPT algorithm for restriction P, no modification is necessary if also the number of preconditions of each action is bounded by a constant  $c$ . Then we can even relax P, such that for some constant  $d$  there can be at most  $d$  actions with the same effect. The proof is similar to the one for Theorem 5, using that the total number of causal links is bounded by  $c(k + 1)$  and the branching factor in line 7 is  $k + 1 + d$ . This is an important observation since many application and example problems in planning satisfy these constraints, for instance, many variants of the LOGISTICS domain used in the international planning competitions. Since planners like NONLIN and SNLP are practical variants of MAR, this may help to explain the gap between empirical and theoretical results for many applications.

The use of parameterized analysis in planning is by no means restricted to using plan length as parameter. We did so only to get results that are as comparable as possible with the previous results. For instance, Downey et. al. (1999) show that STRIPS planning can be recast as the SIGNED DIGRAPH PEBBLING problem which is modelled as a special type of graph. They analyse the parameterized complexity of this problem considering also the treewidth of the graph as a parameter. As another example, Chen and Giménez (2010) show that planning is in P if the size of the connected components in the causal graph is bounded by a constant, and otherwise unlikely to be in P. It seems natural to study this also from a parameterized point of view, using the component size as the parameter. It should also be noted that the parameter need not be a single value; it can itself be a combination of two or more other parameters.



There are close ties between model checking and planning and this connection deserves further study. For instance, model-checking traces can be viewed as plans and vice versa (Edelkamp, Leue, and Visser 2007), and methods and results have been transferred between the two areas in both directions (Edelkamp 2003; Wehrle and Helmert 2009; Edelkamp, Kellershoff, and Sulewski 2010). Our reductions from planning to model-checking suggest that the problems are related also on a more fundamental level than just straightforward syntactical translations.

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