

# Strategic Voting and Strategic Candidacy

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## Abstract

Models of strategic candidacy analyze the incentives of candidates to run in an election. Most work on this topic assumes that strategizing only takes place among candidates, whereas voters vote truthfully. In this paper, we extend the analysis to also include strategic behavior on the part of the voters. (We also study cases where only candidates or only voters are strategic.) We consider two settings in which strategic voting is well-defined and has a natural interpretation: majority-consistent voting with single-peaked preferences and voting by successive elimination. In the former setting, we analyze the type of strategic behavior required in order to guarantee desirable voting outcomes. In the latter setting, we determine the complexity of computing the set of potential outcomes if both candidates and voters act strategically.

## 1 Introduction

Voting has emerged as a topic of key interest among multi-agent systems researchers, as it provides a methodology for a group of agents with distinct preferences to reach a common decision. When analyzing voting rules, the set of candidates is usually assumed to be fixed. In a pathbreaking paper, Dutta, Jackson, and Le Breton (2001) have initiated the study of *strategic candidacy* by accounting for candidates' incentives to run in an election. They assumed that candidates have preferences over other candidates and defined a voting rule to be *candidate stable* if no candidate ever has an incentive not to run. In this model, it is assumed that every candidate prefers himself to all other candidates. Therefore, the winner of an election never has an incentive not to run. Non-winning candidates, on the other hand, might be able to alter the winner by leaving the election. Dutta, Jackson, and Le Breton (2001) showed that, under mild conditions, no non-dictatorial rule is candidate stable.<sup>1</sup>

This result naturally leads to the question of how voting outcomes are affected by candidates' incentives. It is

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<sup>1</sup>The notion of candidates dropping out naturally suggests that the candidates themselves are agents. But the model can also make sense in the context where candidates are (say) joint plans. For example, consider the setting where multiple groups bid to host a conference, and a board eventually votes over the submitted bids. In this context, one of the groups may well decide to drop its bid in order to increase the chances of another, perhaps close-by, group.

straightforward to model strategic candidacy as a two-stage game. At the first stage, each candidate decides whether to run in the election or not. At the second stage, each voter casts a ballot containing a ranking of the running candidates. When analyzing this game, an important ingredient is the assumed voter behavior. That is, what assumptions are made about the votes in the second stage, conditional on the set of running candidates?

Most papers on strategic candidacy assume that voters vote truthfully, i.e., their reported ranking for any given subset of candidates corresponds to their true preferences, restricted to that subset (Dutta, Jackson, and Le Breton 2001; Ehlers and Weymark 2003; Samejima 2005; Eraslan and McLennan 2004; Rodríguez-Álvarez 2006b; 2006a; Samejima 2007; Lang, Maudet, and Polukarov 2013). However, it is well known that this is an unrealistic assumption (Gibbard 1973; Satterthwaite 1975). It is therefore natural to account for strategic behavior on the part of the voters as well. Thus, in the models we consider, both candidates and voters act strategically.

The technical problem in accounting for strategic voting is that, generally speaking, too many voting equilibria exist (Myerson and Weber 1993; De Sinopoli 2000). If we only consider Nash equilibria, then any profile of votes for which no single voter can change the outcome is an equilibrium. In some cases, a straightforward refinement rules out many of the equilibria. For example, in a majority election between two candidates, it is natural to rule out the strange equilibria where some voters play the weakly dominated strategy of voting for their less-preferred candidate. But this reasoning does not generally extend to more than two candidates. In this paper, we focus on two settings that admit natural equilibrium refinements.

The first setting is that of single-peaked preferences (Black 1948). It is well known that, if the number of voters is odd, this domain restriction guarantees the existence of a Condorcet winner (namely, the median) and admits a strategyproof and Condorcet-consistent voting rule (namely, the median rule) (Moulin 1980). Dutta, Jackson, and Le Breton (2001) observed that any Condorcet-consistent rule is candidate stable in any domain that guarantees the existence of a Condorcet winner. Lang, Maudet, and Polukarov (2013) extended this result by showing that, in this setting, no *coalition* of candidates ever has an incentive to change

their strategies as long as the Condorcet winner is running. We study the effect of strategic candidacy with single-peaked preferences when the voting rule is *not* Condorcet-consistent. Our motivation is that the voting rules that are most widely used in practice, *plurality*, *plurality with runoff*, and *single transferable vote (STV)*, may fail to select the Condorcet winner, even for single-peaked preferences. We consider the class of *majority-consistent* voting rules, which are rules that, if there is a candidate that is ranked first by more than half the voters, will select that candidate. This class includes all Condorcet-consistent rules, but also other rules such as plurality, plurality with runoff, STV, and Bucklin. For this class, we show that under some assumptions on strategic behavior, the Condorcet winner does in fact end up being elected (though for other assumptions this does not hold).

The second setting is voting by successive elimination. This voting rule, which is often used in committees, proceeds by holding successive pairwise elections. In this setting, there is a particularly natural notion of strategic voting known as *sophisticated voting* (Farquharson 1969; Moulin 1983; Miller 1995). The outcomes of sophisticated voting (the so-called *sophisticated outcomes*) have been characterized by Banks (1985) for the case when all candidates run. Dutta, Jackson, and Le Breton (2002) extended the characterization result by Banks to the case of strategic candidacy. We study the computational complexity of the latter case and show that computing the set of sophisticated outcomes is NP-complete.

Due to space constraints, some proofs are omitted. They can be found in the full version of this paper.

## 2 Preliminaries

This section introduces the concepts and notations that are used in the remainder of the paper. For a finite set  $X$ , let  $\mathcal{L}(X)$  denote the set of rankings of  $X$ , where a ranking is a binary relation on  $X$  that is complete, transitive, and antisymmetric. For a ranking  $R \in \mathcal{L}(X)$ ,  $\text{top}(R)$  denotes the top-ranked element according to  $R$ .

### 2.1 Players and Preferences

Let  $C$  be a finite set of candidates and  $V$  a finite set of voters. Throughout this paper, we assume that  $|V|$  is odd.<sup>2</sup> The set  $P$  of *players* is given by  $P = C \cup V$ . We assume that  $C \cap V = \emptyset$ .<sup>3</sup> Each player  $p \in P$  has preferences over the set of candidates, given by a ranking  $R_p \in \mathcal{L}(C)$ . For all candidates  $c \in C$ , we assume that the top-ranked candidate in  $R_c$  is  $c$  itself.<sup>4</sup> A preference profile  $R = (R_p)_{p \in P} \in \mathcal{L}(C)^P$

<sup>2</sup>When the number of voters is even, a Condorcet winner is not guaranteed to exist even if preferences are single-peaked. However, in this case there will always be at least one *weak* Condorcet winner. The results in Section 3 extend to the setting with an even number of voters, with the role of the Condorcet winner taken over by one of the weak Condorcet winners, namely the one in whose favor the tie is broken.

<sup>3</sup>For results without this assumption, see, e.g., (Dutta, Jackson, and Le Breton 2001; 2002).

<sup>4</sup>This assumption is known as *narcissism*. Without it, scenarios can arise where no candidate has an incentive to run.

contains preferences for all players. For a player  $p \in P$  and two candidates  $a, b \in C$ , we write  $a \succeq_p b$  if  $(a, b) \in R_p$  and  $a \succ_p b$  if  $a \succeq_p b$  and  $a \neq b$ .

For a preference profile  $R$  and a candidate  $c$ , let  $V_R(c)$  denote the set of voters that have  $c$  as their top-ranked candidate, i.e.,  $V_R(c) = \{v \in V : \text{top}(R_v) = c\}$ . Moreover, for a candidate  $d \neq c$ , let  $V_R(c, d)$  denote the set of voters that prefer  $c$  to  $d$ , i.e.,  $V_R(c, d) = \{v \in V : c \succ_v d\}$ . Candidate  $c$  is a *majority winner* in  $R$  if  $|V_R(c)| > |V|/2$ , and  $c$  is a *Condorcet winner* in  $R$  if  $|V_R(c, d)| > |V|/2$  for all  $d \in C \setminus \{c\}$ . Note that both concepts ignore the preferences of candidates. Every preference profile can have at most one majority winner and at most one Condorcet winner. If a candidate is a majority winner in  $R$ , then it is also a Condorcet winner in  $R$ .

### 2.2 Single-Peakedness

A well studied structural restriction on preferences is single-peakedness (Black 1958). Intuitively, preferences are single-peaked if the candidates can be ordered on a one-dimensional spectrum in such a way that every voter has an ideal (most preferred) point on this spectrum, and preference is declining when moving away from this ideal point. Settings in which the assumption of single-peakedness seems reasonable include elections in which candidates correspond to numerical values (e.g., voting over a tax rate) or elections in which the candidates can be assigned positions on a one-dimensional political spectrum (e.g., ranging from left-wing to right-wing political views). Our definition of a single-peaked preference profile requires not only the preferences of voters, but also the preferences of candidates to be single-peaked. The assumption that  $\text{top}(R_c) = c$  for every candidate  $c$  implies each candidate's ideal point coincides with their position on the spectrum.

Formally, let  $\triangleleft \in C \times C$  be a strict ordering of the candidates. A preference profile  $R = (R_p)_{p \in P}$  is *single-peaked with respect to*  $\triangleleft$  if the following condition holds for all  $a, b \in C$  and  $p \in P$ : if  $a \triangleleft b \triangleleft \text{top}(R_p)$  or  $\text{top}(R_p) \triangleleft b \triangleleft a$ , then  $b R_p a$ . For a preference profile  $R$  that is single-peaked with respect to  $\triangleleft$ , the *median* of  $R$  is defined as the unique candidate  $c$  for which both  $\sum_{a \in C: a \triangleleft c} |V_R(a)| < |V|/2$  and  $\sum_{a \in C: c \triangleleft a} |V_R(a)| < |V|/2$ . It is well known that the median is a Condorcet winner in  $R$ .

Let  $c_1 \triangleleft c_2 \triangleleft \dots \triangleleft c_m$  and let  $R$  be a preference profile that is single-peaked with respect to  $\triangleleft$ . The *peak distribution* of  $R$  with respect to  $\triangleleft$  is the vector of length  $m$  whose  $j$ -th entry is the number  $|V_R(c_j)|$  of voters that rank  $c_j$  highest.

### 2.3 Voting Rules

A *voting rule*  $f$  maps a non-empty subset  $B \subseteq C$  of candidates and a profile of votes  $r = (r_v)_{v \in V} \in \mathcal{L}(B)$  to a candidate  $f(B, r) \in B$ . A voting rule  $f$  is *majority-consistent* if  $f(B, (R_v)_{v \in V}) = c$  whenever  $c$  is a majority winner in  $R|_B$ , and  $f$  is *Condorcet-consistent* if  $f(B, (R_v)_{v \in V}) = c$  whenever  $c$  is a Condorcet winner in  $R|_B$ . Because majority winners are always Condorcet winners, (perhaps confusingly) Condorcet-consistency implies majority-consistency.

A *scoring rule* is a voting rule that is defined by a sequence  $s = (s^n)_{n \geq 1}$ , where for each  $n \in \mathbb{N}$ ,  $s^n =$

$(s_1^n, \dots, s_n^n) \in \mathbb{R}^n$  is a *score vector* of length  $n$ . For a preference profile  $R$  on  $k$  candidates, the score vector  $s^k$  is used to allocate points to candidates: each candidate receives a score of  $s_j^k$  for each time it is ranked in position  $j$  by a voter. (Again, preferences of candidates are ignored.) The scoring rule then selects the candidate with maximal total score. In the case of a tie, a fixed tiebreaking ordering is used. Prominent examples of scoring rules are *plurality* ( $s^n = (1, 0, \dots, 0)$ ), *Borda's rule* ( $s^n = (n-1, n-2, \dots, 0)$ ), and *veto* ( $s^n = (0, \dots, 0, -1)$ ).

The plurality winner is a candidate maximizing  $|V_R(\cdot)|$ . Plurality is majority-consistent, but not Condorcet-consistent. Borda's rule and veto are not majority-consistent and (hence) not Condorcet-consistent.

## 2.4 Candidacy and Voting as a Two-Stage Game

We consider the following two-stage game. At the first stage, each candidate decides whether to run in the election or not. At the second stage, each voter casts a ballot containing a ranking of the running candidates. Throughout, we consider *complete-information* games: the preferences of the candidates and voters are common knowledge among the candidates and voters. Hence, we do not need to model games as (pre-)Bayesian and strategies do not have to condition on the player's type.

Let  $S_p$  be the set of strategies of player  $p$ . Then for each candidate  $c \in C$ , the set  $S_c$  is given by  $\{0, 1\}$ , with the convention that 1 corresponds to "running" and 0 corresponds to "not running." For each voter  $v \in V$ , the set  $S_v$  consists of all functions

$$s_v : 2^C \rightarrow \bigcup_{B \subseteq C} \mathcal{L}(B)$$

that map a subset  $B \subseteq C$  of candidates to a ranking  $s_v(B) \in \mathcal{L}(B)$ . The interpretation is that  $s_v(B)$  is the vote of voter  $v$  when the set of running candidates is  $B$ . In particular, each  $S_v$  contains a strategy that corresponds to *truthful voting* for voter  $v$ : this strategy maps every set  $B$  to the ranking  $R_v|_B$ . In general, however, a voter can rank two candidates differently depending on which other candidates run.

We are now ready to define the outcomes of the game. A *strategy profile*  $s = (s_p)_{p \in P}$  contains a strategy for every player. Given a strategy profile  $s$  and a voting rule  $f$ , define  $C(s) = \{c \in C : s_c = 1\}$  (the set of running candidates<sup>5</sup>) and  $r(s) = (s_v(C(s)))_{v \in V} \in \mathcal{L}(C(s))^V$  (the votes cast for this set of running candidates). The outcome  $o_f(s)$  of  $s$  under  $f$  is then given by  $o_f(s) = f(C(s), r(s))$ .

## 2.5 Equilibrium Concepts

Let  $s = (s_p)_{p \in P}$  be a strategy profile. For a subset  $\tilde{P} \subseteq P$  and a profile of strategies  $s'_{\tilde{P}} = (s'_p)_{p \in \tilde{P}}$  for players in  $\tilde{P}$ , let  $(s'_{\tilde{P}}, s_{-\tilde{P}})$  denote the strategy profile where each player  $p \in \tilde{P}$  plays strategy  $s'_p$  and all remaining players play the

<sup>5</sup>If  $C(s) = \emptyset$ , define  $o_f(s) = \top$ . We assume that each candidate prefers himself to the outcome  $\top$ . This assumption ensures that at least one candidate will run whenever candidates act strategically.

same strategy as in  $s$ . Fix a voting rule  $f$  and a preference profile  $R$ . For a strategy profile  $s$  and a subset  $\tilde{P} \subseteq P$  of players, say that  $s$  is  $(R, f)$ -*deviation-proof w.r.t.  $\tilde{P}$*  if for all  $s'_{\tilde{P}}$ , there exists  $p \in \tilde{P}$  such that

$$o_f(s) \succeq_p o_f(s'_{\tilde{P}}, s_{-\tilde{P}}).$$

For a strategy profile  $s = (s_p)_{p \in P}$ , we sometimes write  $s = (s_C, s_V)$ , where  $s_C = (s_c)_{c \in C}$  is the profile of candidate strategies and  $s_V = (s_v)_{v \in V}$  is the profile of voter strategies. We can now define equilibrium behavior for both candidates and voters.

**Definition 1.** Let  $R$  be a preference profile and let  $f$  be a voting rule. A strategy profile  $s = (s_C, s_V)$  is

- a  $C$ -equilibrium for  $R$  under  $f$  if  $s$  is  $(R, f)$ -deviation-proof w.r.t.  $\{c\}$  for all  $c \in C$ ;
- a strong  $C$ -equilibrium for  $R$  under  $f$  if  $s$  is  $(R, f)$ -deviation-proof w.r.t.  $C'$  for all  $C' \subseteq C$ ;
- a  $V$ -equilibrium for  $R$  under  $f$  if for every  $s'_C \in \{0, 1\}^C$ ,  $(s'_C, s_V)$  is  $(R, f)$ -deviation-proof w.r.t.  $\{v\}$  for all  $v \in V$ ;
- a strong  $V$ -equilibrium for  $R$  under  $f$  if for every  $s'_C \in \{0, 1\}^C$ ,  $(s'_C, s_V)$  is  $(R, f)$ -deviation-proof w.r.t.  $V'$  for all  $V' \subseteq V$ .

We omit the reference to  $R$  and  $f$  if the preference profile or the voting rule is known from the context. In a  $C$ -equilibrium, no candidate can achieve a more preferred outcome by unilaterally changing their strategy. In a strong  $C$ -equilibrium, no coalition of candidates can change the outcome in such a way that every player in the coalition prefers the new outcome to the original one. Thus, (strong)  $C$ -equilibria correspond to (strong) Nash equilibria when strategies of voters are assumed to be fixed. For voters, the equilibrium notions are more demanding: In order to be considered a (strong)  $V$ -equilibrium, the strategies of voters are required to form a (strong) Nash equilibrium for every subset  $B \subseteq C$  of running candidates.

It is instructive to relate these definitions to established game-theoretic solution concepts for extensive-form games, such as subgame-perfect equilibrium and subgame-perfect strong equilibrium. A strategy profile  $s$  is a *subgame-perfect equilibrium* of a game  $G$  if for any subgame  $G' \subseteq G$ , the restriction of  $s$  to  $G'$  is a Nash equilibrium of  $G'$ , and it is a *subgame-perfect strong equilibrium* if for any subgame  $G' \subseteq G$ , the restriction of  $s$  to  $G'$  is a strong Nash equilibrium of  $G'$ . In the candidacy game, every subgame (other than the game itself) corresponds to a voting game that takes place after the candidates have decided whether or not to run. Thus, a proper subgame can be identified with the set of candidates that run in this subgame.

For candidates, playing a subgame-perfect equilibrium is not a stronger requirement than playing a Nash equilibrium, because the only subgame in which they play is the entire game itself. For voters, on the other hand, playing a subgame-perfect equilibrium entails playing a Nash equilibrium for every possible set of running candidates. Therefore, we have the following.

**Fact 1.** A strategy profile is a subgame-perfect equilibrium of the candidacy game if and only if it is both a  $C$ -equilibrium and a  $V$ -equilibrium.

For subgame-perfect strong equilibria, one implication is straightforward.

**Fact 2.** Every subgame-perfect strong equilibrium of the candidacy game is both a strong  $C$ -equilibrium and a strong  $V$ -equilibrium.

However, the other direction does not hold in general, because even if coalitions of either one type of players cannot successfully deviate, it is possible that a mixed coalition including players of both types can.

Splitting up the equilibrium definitions into separate requirements for  $C$  and  $V$  allows us to capture scenarios in which only players of one type (candidates or voters) act according to the corresponding equilibrium notion. In Section 3 we will analyze which combinations of equilibrium notions yield desirable outcomes. We will present both positive results, stating that a desirable outcome will be selected whenever a strategy profile meets a certain combination of equilibrium conditions, and negative results, stating that undesirable outcomes may be selected even if certain equilibrium conditions hold.

In sufficiently general settings, the existence of solutions is not guaranteed for any of the equilibrium concepts in Definition 1.<sup>6</sup> However, for all the positive results in Section 3, we also show that every preference profile admits a strategy profile that meets the corresponding equilibrium conditions.

### 3 Majority-Consistent Voting Rules and Single-Peaked Preferences

In this section, we assume that preference profiles are single-peaked with respect to a given order  $\triangleleft$ . (If the order is not part of the input, it can be computed in polynomial time (Bartholdi, III and Trick 1986; Escoffier, Lang, and Öztürk 2008).) Note that our definition of single-peakedness in Section 2.1 also requires the preferences of candidates to be single-peaked with respect to  $\triangleleft$ . Given that the preferences of voters are single-peaked with respect to  $\triangleleft$ , this does not appear to be an unreasonable assumption.

We are interested in the following question: which requirements on the strategies of players are sufficient for the Condorcet winner (which is guaranteed to exist) to be the outcome? For Condorcet-consistent rules, the answer to this question is relatively straightforward (Lang, Maudet, and Polukarov 2013). However, as we have argued in the introduction, most rules that are typically used in practice are majority-consistent, but not Condorcet-consistent. The simplest and most important such rule is plurality.

It is easy to construct a plurality election in which some candidates have an incentive not to run (assuming truthful voting).

<sup>6</sup>Subgame-perfect equilibria are guaranteed to exist if one allows for *mixed strategies* and extends the preferences of players to the set of all probability distributions over  $C \cup \{\top\}$  in an appropriate way.

**Example 1.** Consider a single-peaked preference profile with candidates  $a \triangleleft b \triangleleft c$  and peak distribution<sup>7</sup>  $(3, 2, 4)$ . Under truthful voting, the plurality winner is  $c$ . However, if candidate  $a$  does not run, the three voters in  $V_R(a)$  rank candidate  $b$  first, making  $b$  the plurality winner. By single-peakedness, candidate  $a$  prefers  $b$  to  $c$ .

This example also shows that plurality can fail to select the Condorcet winner when all candidates run and all voters vote truthfully. The next example shows that requiring both candidates and voters to play subgame-perfect equilibrium strategies is still not sufficient for the Condorcet winner to be chosen.

**Example 2.** Consider a single-peaked preference profile with candidates  $a \triangleleft b \triangleleft c \triangleleft d \triangleleft e$  and peak distribution  $(11, 3, 3, 3, 3)$ . The Condorcet winner is  $b$ . Let  $s$  be the strategy profile in which  $s_x = 1$  for all  $x \in \{a, b, c, d, e\}$  and  $s_v$  is “truthful voting” for all voters  $v$ . Then  $o_{\text{plurality}}(s) = a$  and no candidate other than  $a$  can change that outcome by unilaterally deviating. Therefore,  $s$  is a  $C$ -equilibrium. To see that  $s$  is also a  $V$ -equilibrium, we need to check that “truthful voting” is deviation-proof for every subset of running candidates. Deviation-proofness clearly holds whenever at most two candidates run. If at least three candidates run, single-peakedness implies that the leftmost running candidate has a plurality score of at least 11, whereas each other running candidate has a score of at most 9. Thus, no voter can change the outcome by unilaterally deviating.

We go on to show that the Condorcet winner will be chosen if we require stronger equilibrium notions. We first analyze strong  $V$ -equilibria. Note that this result does not require single-peaked preferences.<sup>8</sup>

**Theorem 1.** Let  $R$  be a preference profile with Condorcet winner  $c^*$  and let  $f$  be a majority-consistent voting rule.

- (i) If  $R|_B$  has a Condorcet winner for every nonempty subset  $B \subseteq C$ , then there exists a subgame-perfect strong equilibrium (and hence a strategy profile that is both a strong  $C$ -equilibrium and a strong  $V$ -equilibrium) for  $R$  under  $f$  in which all candidates run.
- (ii) If  $s$  is a strong  $V$ -equilibrium for  $R$  under  $f$  with  $s_{c^*} = 1$ , then  $o_f(s) = c^*$ .

The proof of part (i) consists in showing that the following strategy profile is a subgame-perfect strong equilibrium: all candidates run and all voters, when faced with a set  $B \subseteq C$  of running candidates, rank the Condorcet winner in  $R|_B$  first. The main idea of the proof of (ii) is that a strategy profile  $s$  with  $s_{c^*} = 1$  and  $o_f(s) = x \neq c^*$  cannot be a strong  $V$ -equilibrium, as it is not deviation-proof w.r.t.  $V_R(c^*, x)$ . The following example illustrates this.

**Example 3.** Let  $R$  be a single-peaked preference profile with candidates  $a \triangleleft b \triangleleft c \triangleleft d$  and peak distribution

<sup>7</sup>We often simplify examples with single-peaked preference profiles by specifying the peak distribution only. This piece of information is clearly sufficient to identify both the Condorcet winner and, in the absence of ties, the plurality winner.

<sup>8</sup>In particular, note that Theorem 1 does not make any assumptions on the preferences of candidates (other than narcissism).

(2, 1, 2, 4). The Condorcet winner is  $c$ . Consider the strategy profile  $s$  in which all candidates run and all voters vote truthfully. Then  $o_{\text{plurality}}(s) = d$ . If all voters in  $V_R(c, d) = V_R(a) \cup V_R(b) \cup V_R(c)$  deviate and rank  $c$  first, the outcome changes to  $c$ .

We remark that part (ii) of Theorem 1 can be generalized<sup>9</sup> by observing that it is sufficient for  $f$  to satisfy the following condition, which is considerably weaker than majority-consistency:

Whenever a set  $V' \subseteq V$  of voters forms a majority (i.e.,  $|V'| > |V|/2$ ), then for every candidate  $a \in C$  that is running and every profile of votes for voters in  $V \setminus V'$ , the voters in  $V'$  can vote in such a way that candidate  $a$  is chosen.

It can be shown that all unanimous C2 functions (Fishburn 1977) satisfy this property.

The following corollary summarizes the consequences of Theorem 1 for single-peaked preference profiles.

**Corollary 1.** *Let  $R$  be a single-peaked preference profile with Condorcet winner  $c^*$  and let  $f$  be a majority-consistent voting rule.*

- (i) *There exists a subgame-perfect strong equilibrium (and hence a strategy profile that is both a strong  $V$ -equilibrium and a strong  $C$ -equilibrium) for  $R$  under  $f$ .*
- (ii) *If  $s$  is a strong  $V$ -equilibrium and a  $C$ -equilibrium (strong or not) for  $R$  under  $f$ , then  $o_f(s) = c^*$ .*

We provide two examples to show that the statements of Corollary 1 do not hold for rules that are not majority-consistent.

**Example 4.** *Let  $R$  be a single-peaked preference profile with candidates  $a \triangleleft b \triangleleft c$  and peak distribution  $(5, 0, 4)$ . If  $f$  is Borda's rule, there does not exist a strong  $V$ -equilibrium (and hence no subgame-perfect strong equilibrium). To see this, consider the case where all candidates run. Observe that in any strong  $V$ -equilibrium, the outcome would have to be  $a$ . (Suppose the outcome is not  $a$ . Then, the five voters in  $V_R(a)$  can jointly deviate and change the outcome to  $a$ . They can do this by having one voter voting  $a \succ b \succ c$ , and the remaining four voters voting exactly the opposite rankings of the voters in  $V_R(c)$ .) However, there is no strong  $V$ -equilibrium that yields outcome  $a$ . This is because the voters in  $V_R(c)$  prefer both other alternatives to  $a$ , and—no matter how the voters in  $V_R(a)$  vote—the voters in  $V_R(c)$  can jointly deviate and achieve an outcome other than  $a$ . (One of  $b$  and  $c$  will obtain a score of at least 3 from the voters in  $V_R(a)$ ). Without loss of generality, suppose it is  $b$ . Then the voters in  $V_R(c)$  can all vote  $b \succ c \succ a$ , making  $b$  win.)*

<sup>9</sup>Sertel and Sanver (2004) prove a similar result in the (standard) setting where all candidates are assumed to run. A further strengthening of part (ii) of Theorem 1 was pointed out to us by François Durand: Instead of requiring that voters play a strong  $V$ -equilibrium for every subset of running candidates, it is sufficient to require voters to play a strong  $V$ -equilibrium *only in those subgames that actually allow strong  $V$ -equilibria* (and to not make any assumptions on voter behavior otherwise).

**Example 5.** *Let  $R$  be a single-peaked preference profile with candidates  $a \triangleleft b \triangleleft c$  and five voters: three voters have preferences  $a \succ b \succ c$  and two voters have preferences  $b \succ c \succ a$ . The Condorcet winner is  $a$ . Let  $f$  be the voting rule veto<sup>10</sup> and let  $s$  be the strategy profile where all candidates run and all voters vote truthfully. Then,  $o_f(s) = b$ . Moreover,  $s$  is a strong  $C$ -equilibrium and a strong  $V$ -equilibrium. The former holds because any deviation involving  $a$  does not change the outcome (provided  $b$  still runs), and  $c$  can only change the outcome to the less preferred alternative  $a$ . For the latter, the only interesting case is when all three candidates run. In this case, the two voters in  $V_R(b)$  have no incentive to deviate from truthful voting (their favorite candidate is winning) and there is no way for the three voters in  $V_R(a)$  to jointly deviate and achieve outcome  $a$ . (They can change the outcome to  $c$  by voting  $a \succ c \succ b$ , but they prefer  $b$  to  $c$ .) It can furthermore be shown that, when all candidates run, every strong  $V$ -equilibrium yields outcome  $b$ .*

We now move to the case where candidates play a strong equilibrium. If voters vote truthfully, the outcome will be the Condorcet winner.

**Theorem 2.** *Let  $R$  be a single-peaked preference profile with Condorcet winner  $c^*$  and let  $f$  be a majority-consistent voting rule.*

- (i) *There exists a strong  $C$ -equilibrium for  $R$  under  $f$  where all voters vote truthfully.*
- (ii) *If  $s$  is a strong  $C$ -equilibrium for  $R$  under  $f$  where all voters vote truthfully, then  $o_f(s) = c^*$ .*

*Proof.* For (i), let  $s$  be the strategy profile in which only  $c^*$  runs and all voters vote truthfully. We show that this is a strong  $C$ -equilibrium for  $R$  under  $f$ . Suppose, for the sake of contradiction, that  $\tilde{C} \subseteq C$  is a coalition of candidates that can, by changing its strategies, make alternative  $a \neq c^*$  win, and moreover that all candidates in  $\tilde{C}$  prefer  $a$  to  $c^*$ . Define  $C^- = \{c \in C : c \triangleleft c^*\}$  and  $C^+ = \{c \in C : c^* \triangleleft c\}$ , and without loss of generality suppose that  $a \in C^-$ . Because candidates' preferences are single-peaked and they rank themselves first, it follows that  $\tilde{C} \subseteq C^-$ . But this implies that still, no candidate in  $C^+$  runs. Hence, all voters with  $\text{top}(R_v) \in C^+ \cup \{c^*\}$  still rank  $c^*$  first (since they vote truthfully), and because  $f$  is majority-consistent, it follows that  $c^*$  wins. This gives us the desired contradiction.

For (ii), let  $s$  be a strong  $C$ -equilibrium for  $R$  under  $f$  where all voters vote truthfully. Consider the set  $C(s)$  of candidates that are running under  $s$ . Define  $C_s^- = \{c \in C(s) : c \triangleleft c^*\}$  and  $C_s^+ = \{c \in C(s) : c^* \triangleleft c\}$ . Assume for the sake of contradiction that  $o_f(s) = a \neq c^*$ . Without loss of generality, suppose that  $a \in C_s^-$ . Consider the set  $\tilde{C} = C_s^+ \cup \{c^*\}$ . Define  $s'_{\tilde{C}} = (s'_c)_{c \in \tilde{C}}$  by

$$s'_c = \begin{cases} 1 & \text{if } c = c^* \\ 0 & \text{if } c \in C_s^+ \end{cases}$$

<sup>10</sup>Veto does not only violate majority-consistency, but also the weaker property defined after Theorem 1.

and observe that  $o_f(s'_{\tilde{C}}, s_{-\tilde{C}}) = c^*$ . The reason for the latter is that (1) the set of voters  $v$  with  $top(R_v) = c^*$  or  $c^* \triangleleft top(R_v)$  forms a majority, (2) all of these voters satisfy  $top(R_v|_{C(s'_{\tilde{C}}, s_{-\tilde{C}})}) = c^*$ , and (3) all voters vote truthfully by assumption. Moreover, single-peakedness implies that all candidates in  $\tilde{C}$  prefer  $c^*$  to  $a$ . Therefore,  $s$  is not  $(R, f)$ -deviation-proof w.r.t.  $\tilde{C}$ , contradicting the assumption that  $s$  is a strong  $C$ -equilibrium.  $\square$

**Example 6.** Consider again the preference profile  $R$  and the strategy profile  $s$  from Example 3. If both  $a$  and  $b$  deviate to “not running,” the outcome (under plurality) changes from  $d$  to  $c$ . Therefore,  $s$  is not a strong  $C$ -equilibrium.

Similar to the case of Theorem 1, we now provide examples that show that Theorem 2 cannot be generalized in certain ways. Example 7 shows that Theorem 2 does not hold for Borda’s rule (which is not majority-consistent), and Example 8 shows that Theorem 2 does not hold if the preferences of candidates are not single-peaked.

**Example 7.** Consider a single-peaked preference profile with candidates  $a \triangleleft b \triangleleft c$  and five voters: three voters have preferences  $a \succ b \succ c$  and two voters have preferences  $b \succ c \succ a$ . The Condorcet winner is  $a$ . Let  $s$  be the strategy profile where  $s_a = s_b = s_c = 1$  and  $s_v$  is “truthful voting” for all voters  $v$ . It is easily verified that  $s$  is a strong  $C$ -equilibrium and  $o_{Borda}(s) = b$ . In fact, it can be checked that the Condorcet winner is not chosen in any strong  $C$ -equilibrium with truthful voting. (The only other strong  $C$ -equilibrium under truthful voting has candidates  $b$  and  $c$  running and also yields outcome  $b$ .)

**Example 8.** Consider the following preference profile with candidates  $a, b, c$  and 14 voters: four voters have preferences  $a \succ b \succ c$ , four voters have preferences  $b \succ a \succ c$ , and six voters have preferences  $c \succ b \succ a$ . The preferences of the candidates are such that  $a$  prefers  $c$  over  $b$  and  $b$  prefers  $c$  over  $a$ . Whereas the preferences of the voters are single-peaked with respect to the ordering  $a \triangleleft b \triangleleft c$ , this is not true for the preferences of the candidates. (Therefore, this profile is not single-peaked according to the definition in Section 2.1.) The Condorcet winner is  $b$  and the Condorcet loser is  $c$ . Let  $s$  be the strategy profile where all candidates run and all voters vote truthfully. It is easily verified that  $s$  is a strong  $C$ -equilibrium and  $o_{plurality}(s) = c$ . In fact, “everybody running” is the only strong  $C$ -equilibrium under truthful voting.

Since Theorem 1 already covers the case where both voters and candidates play a strong (subgame-perfect) equilibrium, only one case is left to consider: candidates playing a strong  $C$ -equilibrium, and voters merely playing a  $V$ -equilibrium. The following example shows that these requirements are *not* sufficient for the Condorcet winner to be chosen.

**Example 9.** Consider a single-peaked preference profile with candidates  $a \triangleleft b \triangleleft c$  and peak distribution  $(1, 1, 1)$ . The Condorcet winner is  $b$ . Let  $s$  be a strategy profile with

	strong $V$ -eq.	$V$ -eq.	truthful voting
strong $C$ -equilibrium	yes (Cor. 1)	no (Ex. 9)	yes (Thm. 2)
$C$ -equilibrium	yes (Cor. 1)	no (Ex. 2)	no (Ex. 2)
naive candidacy ( $s_c = 1 \ \forall c$ )	yes (Thm. 1)	no (Ex. 2)	no (Ex. 1 & 2)

Table 1: Overview of results. A table entry is “yes” if every strategy profile that satisfies the corresponding row and column conditions yields the Condorcet winner under every majority-consistent voting rule. Moreover, for every “yes” entry, a strategy profile satisfying the conditions is guaranteed to exist.

$s_c = 1$  and voter strategies  $s_v$  that satisfy

$$top(s_v(B)) = \begin{cases} c & \text{if } c \in B \\ top(R_v|_B) & \text{otherwise} \end{cases}$$

for each  $B \subseteq C$ . That is, all three voters rank  $c$  first whenever  $c$  runs, and vote truthfully otherwise.<sup>11</sup> Obviously,  $o_{plurality}(s) = c$ . We claim that  $s$  is both a  $V$ -equilibrium and a strong  $C$ -equilibrium. For the former, we distinguish two cases: If  $c$  runs, then all voters rank  $c$  first and no voter can change the outcome by unilaterally deviating. If  $c$  does not run, then at most two candidates run and no voter can benefit by voting for their less preferred candidate. For the latter, no coalition of candidates can change the outcome in such a way that all members of the coalition prefer the new outcome to  $c$ . (Such a coalition would need to include candidate  $c$ , who has no incentive to deviate.)

The phenomenon illustrated in this example is perhaps somewhat surprising: Assuming that candidates play a strong  $C$ -equilibrium, both truthful voting and strong  $V$ -equilibrium voting yields the desirable outcome; however,  $V$ -equilibrium voting—a notion of sophistication that might appear to be “in between” the other two notions—does not. Table 1 summarizes the results of this section.

## 4 Computing the Candidate Stable Set

In this section, we study a voting rule known as *voting by successive elimination (VSE)*. In particular, we will be interested in the computational complexity of computing outcomes under VSE if both candidates and voters act strategically. We do not require single-peaked preferences, but in order to avoid majority ties, we still assume that the number

<sup>11</sup>Note that the voter  $v$  with  $top(R_v) = a$  plays a weakly dominated strategy, because  $c$  is her least preferred alternative. This can be avoided by introducing a fourth candidate  $d$  with  $c \triangleleft d$  and  $V_R(d) = \emptyset$ .

of voters is odd. VSE takes as input an ordering  $\sigma \in \mathcal{L}(C)$  of the candidates. The rule proceeds by holding successive pairwise elections. In a pairwise election, there are two candidates  $a$  and  $b$  and every voter  $v \in V$  votes for exactly one of the two candidates. The winner of a pairwise election is the candidate that gets the majority of votes.

For a given subset  $B \subseteq C$  of candidates with  $|B| \geq 2$ , VSE works as follows. Label the candidates such that  $\sigma|_B = (c_1, c_2, \dots, c_{|B|})$ . In the first round, there is a pairwise election between  $c_1$  and  $c_2$ . The winner of this election proceeds to the second round, where he faces  $c_3$ . The winner of this election then faces  $c_4$ , and so on. VSE selects the winner of round  $|B| - 1$ .

Truthful voting for a voter  $v$  with preferences  $R_v$  corresponds to the strategy that, in every pairwise election between two candidates  $a$  and  $b$ , the voter votes for  $\text{top}(R_v|_{\{a,b\}})$ . It is well known that, under VSE, voters can benefit from voting strategically. Moreover, there is a particularly natural notion of strategic voting called *sophisticated voting* (Farquharson 1969; Moulin 1983; Miller 1995). Sophisticated voting assumes that voters' preferences are common knowledge and applies a backward induction argument: In the last round of VSE, there is no incentive to vote strategically and thus the majority winner of the remaining two candidates will be chosen. Anticipating that, in the second-to-last round, voters are able to compare which outcome would eventually result from either one of the current candidates winning this round, and vote accordingly; etc. In the absence of majority ties, sophisticated voting yields a unique winning candidate, the *sophisticated outcome*. The sophisticated outcome corresponds to the outcome that results when voters iteratively eliminate weakly dominated strategies.

In order to determine both the truthful outcome and the sophisticated outcome, it is sufficient to know the truthful outcome of pairwise elections between all pairs of the candidates. This information is captured by the majority relation. For a preference profile  $R$ , the *majority relation*  $R_M \subset C \times C$  is defined by

$$a R_M b \quad \text{if and only if} \quad V_R(a, b) > \frac{|V|}{2}.$$

Shepsle and Weingast (1984) defined an algorithm that, given a majority relation  $R_M$ , an ordering  $\sigma$ , and a subset  $B \subseteq C$  of the candidates, computes the sophisticated outcome when the set of running candidates is given by  $B$ . Moreover, Banks (1985) characterized the set of candidates that, for given  $R_M$  and  $B \subseteq C$ , are the sophisticated outcome for *some* ordering  $\sigma$ . This set is known as the *Banks set*  $BA(B, R_M)$ . In the notation<sup>12</sup> developed in this paper,  $BA(B, R_M)$  corresponds to  $\bigcup_{\sigma} \sigma_{VSE(\sigma)}(s)$ , where  $s_c = 1$  if  $c \in B$  and  $s_v$  is “sophisticated voting” for all voters.

Dutta, Jackson, and Le Breton (2002) analyzed how the set of sophisticated outcomes changes when strategic candidacy is accounted for. Consider a strategy profile  $s =$

$(s_C, s_V)$ , where  $s_C = (s_c)_{c \in C}$  and  $s_V = (s_v)_{v \in V}$  and say that  $s$  is an *entry equilibrium* if it is a  $C$ -equilibrium and  $s_v$  is “sophisticated voting” for all voters  $v \in V$ . The *candidate stable set*  $CS(R)$  of a preference profile  $R$  is defined as the set of all candidates that are the sophisticated outcome for some collection of candidate preferences and for some ordering  $\sigma$ , when the set of running candidates is given by  $C(s)$  for some entry equilibrium  $s$ . Thus, the candidate stable set is the analog of the Banks set when strategic candidacy is taken into account. Since  $CS(R)$  only depends on the majority relation  $R_M$  of  $R$ , we usually write  $CS(R_M)$ .

Dutta, Jackson, and Le Breton (2002) have provided an elegant characterization of  $CS(R_M)$ . We use this characterization to show that computing the candidate stable set is intractable. More precisely, we show that the following decision problem is NP-complete: Given a preference profile  $R$  and a candidate  $c \in C$ , is it the case that  $c \in CS(R_M)$ ?

**Theorem 3.** *Computing the candidate stable set is NP-complete.*

The proof adapts a construction that was used by Brandt et al. (2010) to show that computing the Banks set is NP-hard.

## 5 Conclusion

We have analyzed the combination of strategic candidacy and strategic voting in two settings that allow meaningful voting equilibria. In both settings, the set of equilibrium outcomes under strategic candidacy (given that voters are sufficiently sophisticated) has an elegant characterization: the Condorcet winner (in the single-peaked, majority-consistent rule setting with strong  $V$ -equilibria or with truthful voting and strong  $C$ -equilibria) and the candidate stable set (in the VSE setting with sophisticated voting). Whereas Condorcet winners are easy to compute, we have shown that the candidate stable set is computationally intractable.

It seems likely that the positive results in Section 3 extend to settings where preferences are single-peaked on a tree. It would also be interesting to check whether similar results can be obtained for related domain restrictions such as single-crossing or value-restricted preferences.

The positive results in Section 3 rely on finding the right level of equilibrium refinement (strong  $V$ -equilibrium, or strong  $C$ -equilibrium with truthful voting). If we move away from restricted domains, is there another type of equilibrium refinement (Dutta and Laslier 2010; Thomson et al. 2013; Obraztsova, Markakis, and Thomson 2014) that allows us to arrive at meaningful equilibria by ruling out “unnatural” ones?

Equilibrium *dynamics* (Meir et al. 2010) is another topic for future research. For example, in the setting with single-peaked preferences and a majority-consistent rule, are there natural dynamics that are guaranteed to lead us to an equilibrium choosing the Condorcet winner?

On a higher level, one might wonder to what extent the phenomena exhibited in candidacy games can be related to other problems that involve altering the set of candidates, such as control problems (see (Lang, Maudet, and Polukarov 2013), Section 5) cloning (Tideman 1987), and nomination of alternatives (Dutta and Pattanaik 1978; Dutta 1981).

<sup>12</sup>Strategies, outcomes, and equilibrium notions for VSE can be defined similarly to the definitions in Section 2. We omit the details since they are not important for our result. For formal definitions of the concepts considered in this section, we refer to (Dutta, Jackson, and Le Breton 2002).

## Acknowledgments

We would like to thank John Duggan, François Durand, and the anonymous reviewers at COMSOC and AAI for helpful comments. This work was supported by NSF and ARO under grants CCF-1101659, IIS-0953756, CCF-1337215, W911NF-12-1-0550, and W911NF-11-1-0332, and by a Feodor Lynen research fellowship of the Alexander von Humboldt Foundation.

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