Mechanism Design for Multi-Type Housing Markets

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Abstract

We study multi-type housing markets, where there are \( p \geq 2 \) types of items, each agent is initially endowed one item of each type, and the goal is to design mechanisms without monetary transfer to (re)allocate items to the agents based on their preferences over bundles of items, such that each agent gets one item of each type. In sharp contrast to classical housing markets, previous studies in multi-type housing markets have been hindered by the lack of natural solution concepts, because the strict core might be empty.

We break the barrier in the literature by leveraging AI techniques and making natural assumptions on agents’ preferences. We show that when agents’ preferences are lexicographic, even with different importance orders, the classical top-trading-cycles mechanism can be extended while preserving most of its nice properties. We also investigate computational complexity of checking whether an allocation is in the strict core and checking whether the strict core is empty. Our results convey an encouragingly positive message: it is possible to design good mechanisms for multi-type housing markets under natural assumptions on preferences.

Introduction

In this paper, we ask the following question: is it possible at all to design good mechanisms for multi-type housing markets? In multi-type housing markets (Moulin 1995; Konishi, Quint, and Wako 2001; Wako 2005; Klaus 2008), there are multiple types of items, for example houses and cars, each agent is initially endowed with one item of each type. The goal is to design mechanisms without monetary transfer to (re)allocate items to the agents based on their preferences over bundles of items, such that each agent gets one item of each type.

Multi-type housing markets are often described using examples of houses and cars as metaphors for indivisible items. However, the allocation problem is applicable to many other types of items and scarce resources. For example, students may want to exchange papers and dates for presentations (Mackin and Xia 2016); in cloud computing, agents may want to allocate multiple types of resources, including CPU, memory, and storage (Ghodsi et al. 2011; Abdulkadiroğlu and Sönmez 1999). Therefore, as Sönmez and Ünver (2011) noted: “Positive results of this section on housing markets no longer hold in an economy in which one agent can consume multiple houses or multiple types of houses”. This is the problem we address in this paper and provide a number of positive results, a first in this field.

Our contributions

In this paper, we present novel algorithms building on AI techniques in preference representation and reasoning for allocation in multi-type housing markets. We assume that agents’ preferences are represented by arbitrary acyclic CP-nets (Boutilier et al. 2004). Different agents may have arbitrarily different CP-net structure. We also assume that agents’ preferences are lexicographic, meaning that agents have arbitrary importance orders over item types.

We propose the following natural extension of TTC, which we call Multi-type TTC (MTTC) for multi-type hous-
ing markets. MTTC builds a directed bipartite graph in each round, where agents and items are two separated groups of vertices. There is an edge \((a, b_i)\), where \(a\) is an agent and \(b_i\) is agent \(b\)'s initial type-\(i\) endowment, if (1) the most important type for agent \(a\), from which she has not obtained an item, is type \(i\), and (2) \(b_i\) is agent \(a\)'s top-ranked type-\(i\) item among remaining type-\(i\) items. For any type \(i\) and any agent \(b\), there is an edge \((b, b_i)\). Then, MTTC finds and implements all cycles as TTC does, but only removes the items in the cycle. The agents always remain in the graph.

We note that in MTTC, a cycle may involve items of multiple types. Our main theorem is the following:

**Theorem 1.** For lexicographic preferences, MTTC runs in polynomial-time and satisfies strict-core-selection (which implies Pareto optimality and individual rationality), non-bossiness, and strong group strategy-proofness when agents cannot lie about importance orders over types.

We note that Theorem 1’s assumption on lexicographic preferences are not only very mild given agents can have arbitrarily different CP-net structure and importance orders, but also an extension of naturalistic decision making structures studied in cognitive science literature for ordering items based on multiple criteria (Gigerenzer and Goldstein 1996). This positive result is especially surprising when it is put in comparison with similar research in combinatorial voting, where negative results emerge as soon as agents do not share similar CP-net structures (Lang and Xia 2016).

We also prove an impossibility theorem to show that Theorem 1 cannot be improved by allowing agents to lie about the importance order. As for computing other strict cores, we prove that, it is coNP-complete to check whether a given allocation is in the strict core, even for two types of items. When agents have separable and lexicographic preferences, MTTC implements all cycles as TTC does, but only removes the items in the cycle. The agents always remain in the graph.

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We also prove an impossibility theorem to show that Theorem 1 cannot be improved by allowing agents to lie about the importance order. As for computing other strict cores, we prove that, it is coNP-complete to check whether a given allocation is in the strict core, even for two types of items and agents with separable preferences w.r.t. the same importance order. As for computing other strict cores, we prove that, it is coNP-complete to check whether a given allocation is in the strict core, even for two types of items and agents with separable preferences w.r.t. the same importance order. This hardness result is unexpected due to related agents with separable preferences w.r.t. the same importance order. As for computing other strict cores, we prove that, it is coNP-complete to check whether a given allocation is in the strict core, even for two types of items and agents with separable and additive preferences. This assumption is characterized strategy-proof mechanisms for multi-type house allocation, when agents’ preferences include lexicographic preferences. To the best of our knowledge, our paper is the first time that CP-nets and lexicographic preferences are investigated in multi-type housing markets.

Konishi, Quint, and Wako (2001) assumed that agents have separable and additive preferences. This assumption is in general incomparable to our assumptions on agents’ preferences, because preferences represented by CP-nets are generally not separable, except when the CP-net has no cycles. Therefore, our positive results are not obtained by putting further restrictions on agents’ preferences.

There has been very little work on multi-type housing markets after Konishi, Quint, and Wako’s negative results (Wako 2005; Klaus 2008). These papers focus on a different solution concept called coordinate-wise core rule, which are composed of type-wise strict-core allocations. We show that this naturally corresponds to the output of MTTC when agents have separable and lexicographic preferences with a common importance order.

**Preliminaries**

We consider a market consisting of a set \(N = \{1, \ldots, n\}\) of agents with \(p \geq 2\) types of indivisible items. For any \(i \leq p\), there are \(n_i\) items of type \(i\), denoted by \(T_i = \{i_1, \ldots, i_{n_i}\}\). For each item \(o\), Type\((o)\) is the type of \(o\), that is, \(o \in T_{\text{type}(o)}\). Each agent \(j \in N\) initially owns exactly one item of each type, and her endowment is denoted by a \(p\)-vector \(O(j)\). W.l.o.g. in this paper we let \(O(j) = (j_1, \ldots, j_p)\). Let \(T = T_1 \times \cdots \times T_p\) be the set of all bundles, each of which is represented by a \(p\)-vector. We will often use vectors such as \(\vec{d}\) and \(\vec{c}\) to represent bundles, and for any \(i \leq p\), let \([\vec{d}]_i\) denote the type-\(i\) item in \(\vec{d}\). A multi-type housing market \(M\) is given by the tuple \((N, \{T_1, \ldots, T_p\}, O)\).

Each agent desires to consume exactly one item of each type, and her preferences are represented by a linear order over \(T\). A preference profile \(P = (R_1, \ldots, R_n)\) is a collection of agents’ preferences. In any multi-type housing mar-
ket $M$, an allocation $A$ is a mapping from $N$ to $\mathcal{T}$ such that for any $j \leq n$, $A(j)$ is the bundle allocated to $j$. Since no item is allocated twice, we have that for any $j \neq j'$ and any $i \leq p$, $[A(j)]_i \neq [A(j')]_i$. Given a market $M$, a mechanism $f$ is a function that maps agents' profile $P$ to an allocation in $M$.

**Axiomatic Properties**

A mechanism $f$ satisfies **individually rationality** if for any profile $P$, no agent prefers her initial endowment to her allocation by $f$. $f$ satisfies **Pareto optimality** if for any profile $P$, there does not exist an allocation $A$ such that (1) every agent weakly prefers her allocation in $A$ to her allocation in $f(P)$, and (2) some agent strictly prefers her allocation in $A$ to that in $f(P)$. $f$ is **non-bossy** if for any profile $P$, no agent can change any other agent's allocation without changing her own by reporting differently. $f$ is **strategy-proof** if for each agent, falsely reporting her preferences is not beneficial. A mechanism satisfies **strong group strategy-proofness** if there is no group of agents $S$ who can falsely report their preferences so that (1) every agent in $S$ gets a weakly preferred bundle, and (2) at least one agent in $S$ gets a strictly preferred bundle.

An allocation $A$ is said to be **weakly blocked** by a coalition $S \subseteq N$, if the agents in $S$ can find an allocation $B$ of their initial endowments so that each agent weakly prefers allocation in $B$ to that in $A$, and some agent is strictly better off in $B$ than in $A$. The **strict core** of a market is the set of all allocations that are not weakly blocked by any coalition. A mechanism $f$ is **strict-core-selecting**, if for any profile $P$, $f(P)$ is always in the strict core.

**CP-nets and Lexicographic Preferences**

A (directed) CP-net $\mathcal{N}$ over $\mathcal{T}$ is defined by (i) a directed graph $G = (\{T_1, \ldots, T_p\}, E)$, called the **dependency graph**, and (ii) for each $T_i$, there is a conditional preference table $\text{CP}^i_T$, which contains a linear order $\succ^i_T$ over $T_i$ for each valuation $\bar{u}$ of the parents of $T_i$ in $G$, denoted $\text{Pa}(T_i)$. Each CP-entry $\succ^i_T$ carries the following meaning: my preferences over type $i$ is $\succ^i_T$ given that I get items $\bar{u}$, and these preferences are independent of other items I get. An agents' preferences are **lexicographic**, if for any profile $P$, $f(P)$ is always in the strict core.

Each CP-net $\mathcal{N}$ represents a partial order $\succ^{}_{\mathcal{N}}$, which is the transitive closure of preference relations represented by all CP entries, which are $\{(a_i, \bar{u}, \bar{z}) \succ^{}_{\mathcal{N}} (b_i, \bar{u}, \bar{z}) : i \leq p; a_i, b_i \in T_i; \bar{u} \in \text{Pa}(T_i); \bar{z} \in T_-(\text{Pa}(T_i) \cup \{1, \ldots, i\})\}$.

For example, Figure 1 illustrates a separable CP-net. There are two types: houses (H) and cars (C), with two items per type. Since the preferences are separable, there is no edge in the dependency graph (in the left of the figure). The CPTs are shown in the middle of the Figure 1, and the partial order represented by the CP-net is shown in the right.

Let $\mathcal{O} = \{T_1 \triangleright \cdots \triangleright T_p\}$ be a linear order over the types. A CP-net is $\mathcal{O}$-legal, if there is no edge $(T_k, T_l)$ with $k > l$ in its dependency graph. A **lexicographic extension** of an $\mathcal{O}$-legal CP-net $\mathcal{N}$ is a linear order $V$ over $\mathcal{T}$, where for any $i \leq p$, any $\bar{x} \in T_1 \times \cdots \times T_{i-1}$, any $a_i, b_i \in T_i$, and any $\bar{y}, \bar{z} \in T_{i+1} \times \cdots \times T_p$, if $a_i \succ^i_V b_i$ in $\mathcal{N}$, then $(\bar{x}, a_i, \bar{y}) \succ_V^{} (\bar{x}, b_i, \bar{z})$. In other words, the agent believes that type $T_1$ is most important type to her, $T_2$ is the second most important type, etc. In a lexicographic extension, $\mathcal{O}$ is called the **importance order**.

In this paper an agent’s preferences are lexicographic, which means that each agent’s ranking is a lexicographic extension of a CP-net. We note that the CP-net does not need to be separable and the importance order can be different.

**Example 1.** Suppose the agent’s preferences is lexicographic w.r.t. the separable CP-net in Figure 1 and the importance order $H \triangleright C$, then her preferences are $(1_H, 1_C) \succ (1_H, 2_C) \succ (2_H, 1_C) \succ (2_H, 2_C)$. If her importance order is $C \triangleright H$, then her preferences are $(1_H, 1_C) \succ (2_H, 1_C) \succ (1_H, 2_C) \succ (2_H, 2_C)$.

**The Multi-Type TTC Mechanism**

We propose the **multi-type TTC (MTTC)** mechanism as Algorithm 1. MTTC assumes that agents’ preferences are lexicographic (w.r.t. possibly different importance orders and possibly different CP-net structures).

**Example 2.** Consider the market with 3 agents and 2 types in Figure 2. The two types are Houses (H) and Cars (C) with items $\{1_H, 2_H, 3_H\}$ and $\{1_C, 2_C, 3_C\}$, respectively. The initial endowments of each agent $j$ is $(j_H, j_C)$. Figure 2 shows agents’ lexicographic preferences and the execution of MTTC. In particular, Figure 2 shows the graphs $G_1, G_2, G_3$ constructed in each round, and all the four cycles implemented in MTTC. The output is the allocation $A$ where $A(1) = (2_H, 1_C), A(2) = (3_H, 2_C), A(3) = (1_H, 3_C)$.

It is not hard to see that when all agents have the same importance order, the output of MTTC is the same as the output of the following $p$-step process. W.l.o.g. let the importance order be $T_1 \triangleright T_2 \cdots \triangleright T_p$. For each step $i$, we ask the agents to report their preferences over $T_i$, conditioned on items they got in previous steps; then we use TTC to allocate the items in $T_i$, and move on to the next step. In particular, when all agents’ preference are separable, MTTC coincides with the coordinate-wise core rule (Konishi, Quint, and Wako 2001).

**Theorem 1.** For lexicographic preferences, MTTC runs in polynomial-time and satisfies strict-core-selection (which implies Pareto optimality and individual rationality), non-bossiness, and strong group strategy-proofness when agents cannot lie about importance orders over types.
Algorithm 1 MTTC
1: **Input:** A multi-type housing market \( M \) and a profile \( P \) of lexicographic preferences.
2: \( t \leftarrow 1 \). Let \( L \leftarrow \cup_{i \leq t} T_i \) be the set of unallocated items.
   Let \( A \) be the empty assignment. For each \( j \leq n \), let \( i_j^* \) be agent \( j \)'s most desirable type.
3: **while** \( L \neq \emptyset \) **do**
4: \( G \leftarrow (N \cup L, E) \). For every \( j \in N, j_i \in L \), add edge \((j, j_i)\) to \( E \). For every \( j \in N \), add edge \((j, j_i^*)\) to her most preferred item in \( L \) of type \( i_j^* \) to \( E \).
5: **Implement cycles in** \( G \). For each cycle \( C \), for every \((j, j_i^*) \in C \), assign \([A(j)]_{i_j^*} = j_i^*\).
6: Remove assigned items from \( L \).
7: For any agent \( j \) who is assigned an item, set \( i_j^* \) to be the next type according to \( j \)'s importance order.
8: \( t \leftarrow t + 1 \).
9: **end while**
10: **Output:** The allocation \( A \).

**Proof sketch:** Our proof follows the following steps. All omitted proofs can be found in the ArXiv version.

**Step 1:** MTTC outputs an allocation in polynomial time. This follows after the observation that the number of rounds is no more than \( np \).

**Step 2:** Strict-core-selecting. Suppose for the sake of contradiction, there exists a coalition \( S \) and an allocation \( B \) that blocks \( A = \text{MTTC}(P) \). Let \( t^* \) be the first round in which MTTC’s assignments restricted to \( S \) at the end of the round differs from the assignments in \( B \). We then prove that there is some \( j \in S \) and a type \( i \) such that \( j \) is assigned a different item of type \( i \) in \( B \) than \( A \). Let \( j_i^* = \lfloor B(j) \rfloor_i \neq [A(j)]_i = j_i. Because j \)'s preferences are lexicographic and \( j \) get the same items in more important types in \( B \) as in \( A \),

we have \( j_i^* \succ_j j_i \). This means that \( j_i^* \) is unavailable at round \( t^* \). Therefore, \( j_i^* \) must already have been assigned to another agent in a strictly earlier round \( t' < t^* \), which is a contradiction to the choice of \( t^* \).

**Step 3:** MTTC*, the single-cycle-elimination variant of MTTC. MTTC* is similar to MTTC except that in each round a single trading cycle is implemented.

**Definition 1.** Given a housing market \( M \) and any profile \( P \), let \( \text{MTTC}^*(P) \) denote the set of algorithms, each of which is a modification of MTTC (Algorithm 1), where instead of implementing all cycles in each round, the algorithm implements exactly one available cycle in each round.

We note that MTTC* depends on \( P \) because for different profiles the cycles in each round might be different. For each \( A \in \text{MTTC}^*(P) \), we let Order(\( A \)) denote the linear order over the cycles that \( A \) implements. That is, if \( \text{Order}(A) = C_1 \succ \cdots \succ C_k \), then it means that for any \( t \leq k \), \( C_t \) is the cycle implemented by \( A \) in round \( t \).

**Example 3.** Let \( M, P \) be defined as in Figure 2. Let \( C_1, C_2, C_3, C_4 \) be the same cycles as in Figure 2. An \( A \) in \( \text{MTTC}^* \) can be defined by following \( C_1 \succ C_2 \succ C_3 \succ C_4 \) to implement the cycles.

**Definition 2.** For any multi-type housing market \( M \), let \( \text{Cycles}(P) \) denote the set of cycles implemented in the execution of MTTC on \( P \). We define a partial order \( \text{PO}(P) \) over \( \text{Cycles}(P) \) as follows. For every pair of cycles \( C_k, C_i \), \( C_k \succ C_i \) in \( \text{PO}(P) \) if (1) there is an agent who gets an item of a more important type in \( C_k \) than in \( C_i \), or (2) there is an agent in \( C_i \) who prefers an item in \( C_k \) over the item she is pointing to in \( C_i \) of the same type, conditioned on the items she got in previous rounds.

**Example 4.** Continuing Example 3, \( \text{PO}(P) = \{ C_1 \succ C_2, C_1 \succ C_3, C_1 \succ C_4, C_3 \succ C_4 \} \). For all \( 2 \leq i \leq 4 \), \( C_1 \succ C_i \) because houses are more important than cars to all agents. \( C_3 \succ C_4 \) because agent 1 has a more preferred car in \( C_3 \) than in \( C_4 \).
We now present two key lemmas.

**Lemma 1.** For any multi-type housing market $M$ and any profile $P$, we have $\text{Order}(\text{MTTC}\star(P)) = \text{Ext}(P)$.

**Lemma 2.** For any multi-type housing market $M$, any profile $P$, any agent $j$, and any $P'$ obtained from $P$ by changing agent $j$’s top-ranked bundle to be $\text{MTTC}(P)(j)$, then $\text{MTTC}(P') = \text{MTTC}(P)$.

**Step 4: Non-bossiness.** Suppose for the sake of contradiction $\text{MTTC}$ does not satisfy non-bossiness. W.l.o.g. let $P$ and $P'$ denote profiles where only agent 1’s preferences are different, $\text{MTTC}(P)(1) = \text{MTTC}(P')(1)$, yet $\text{MTTC}(P) \neq \text{MTTC}(P')$. Let $\hat{P}$ denote the market obtained from $P$ by letting agent 1’s top-ranked bundle to be $\text{MTTC}(P)(1)$. By Lemma 2, $\text{MTTC}(\hat{P}) = \text{MTTC}(P)$ and $\text{MTTC}(P') = \text{MTTC}(P)$, which is a contradiction.

**Step 5: Strong group strategy-proofness when the agents cannot lie about the importance orders.** Suppose for the sake of contradiction, there exist truthful profile $P$, $S \subseteq N$, and a beneficial false profile $P'$. Let $\hat{P}$ denote the profile obtained from $P'$ by letting the top-ranked bundle of all agents $j \in S$ be $\text{MTTC}(P)(j)$. By sequentially applying Lemma 2 to agents in $S$, we have that $\text{MTTC}(\hat{P}) = \text{MTTC}(P')$.

We then compare side by side two parallel runs of two $\text{MTTC}\star$ algorithms: $A \in \text{MTTC}\star(P)$ and $\tilde{A} \in \text{MTTC}\star(\hat{P})$. We will define $A$ and $\tilde{A}$ dynamically. Starting with $t = 0$, let $G_t$ and $\tilde{G}_t$ denote the graphs of $\text{MTTC}\star$ at the beginning of round $t$ for input $P$ and input $\hat{P}$, respectively. If there is a common cycle $C$ in $G_t$ and $\tilde{G}_t$, then we let both $A$ and $\tilde{A}$ implement $C$, and move on to the next round. Let $t^*$ be the earliest round where $G_{t^*}$ and $\tilde{G}_{t^*}$ do not have a common cycle. We can prove that there exists $j \in S$ such that $(j, o_i) \in C$ and $(j, s_i) \in G_{t^*}$, where $o_i \neq s_i$. Agent j must strictly prefers $o_i$ to $s_i$, given her allocations in previous rounds. Also agent j must get $s_i$ in $\text{MTTC}(\hat{P})$ because this is how we define $\hat{P}$. This contradicts the assumption that none of agents in $S$ is strictly worse off in $\text{MTTC}(\hat{P})$.

**Proposition 1.** $\text{MTTC}$ is not strategy-proof w.r.t. only misreporting the importance order (i.e. without misreporting local preferences over types).

**Proof.** Consider the preferences in Figure 2. We recall that when agents are truthful, the output of $\text{MTTC}$ is $((2_H, 1_C), (3_H, 2_C), (1_H, 3_C))$ to agents 1, 2, 3, respectively (Example 2). If agent 1 misreports the importance order as $C \succ H$ without misreporting any preferences over types, then the output of $\text{MTTC}$ is $((2_H, 3_C), (3_H, 2_C), (1_H, 1_C))$. We note that agent 1 prefers $(2_H, 3_C)$ to $(2_H, 1_C)$.

**Proposition 2.** The strict core of a multi-type housing market can be multi-valued, even when agents’ preferences are separable and lexicographic w.r.t. the same order.

**Proof.** Consider the preferences in Figure 2. Let $B$ denote the allocation such that $B(1) = (2_H, 3_C), B(2) = (3_H, 2_C), B(3) = (1_H, 1_C)$. For the sake of contradiction, suppose $S$ be a blocking coalition to $B$. We can observe that agents 1, 2 each receive their top bundles in $B$ and have no incentive to participate in a coalition. Therefore $S = N$. However, it can be verified that $B$ is Pareto optimal. Further, agent 3 cannot benefit by not participating since $(1_H, 1_C) \succ (3_H, 3_C)$. This means that there is no coalition that blocks $B$, which is a contradiction.

**Theorem 2.** For any multi-type housing market $M$ with $n \geq 3$ and $p \geq 2$, no mechanism satisfies individual rationality, Pareto optimality, and strategy-proofness, even when agents’ preferences are lexicographic and separable.

**Proof.** For the sake of contradiction, let $f$ be such a mechanism. Consider the agents’ lexicographic and separable preferences $P$ as in Figure 3. Explicitly, $P$ is the following:

1: $(2_H, 1_C) \succeq (2_H, 3_C) \succeq (2_H, 2_C) \succeq (1_H, 1_C) \succeq$ others,
2: $(3_H, 2_C) \succeq (2_H, 2_C) \succeq$ others,
3: $(1_H, 1_C) \succeq (1_H, 3_C) \succeq (3_H, 1_C) \succeq (3_H, 3_C) \succeq$ others.

We can verify that the only allocations that satisfy individual rationality and Pareto optimality are $((2_H, 3_C), (3_H, 2_C), (1_H, 1_C))$ and $((2_H, 1_C), (3_H, 2_C), (1_H, 3_C))$. If $f(P) = ((2_H, 3_C), (3_H, 2_C), (1_H, 1_C))$, then we show that agent 1 has incentive misreports her lexicographic order as $2 \succ 1$, leading to the only individually rational and Pareto optimal allocation $((2_H, 1_C), (3_H, 2_C), (1_H, 3_C))$. If $f(P) = ((2_H, 1_C), (3_H, 2_C), (1_H, 3_C))$, then we show that agent 3 has incentive misreports her lexicographic order as $2 \succ 1$ and her local preferences over type 1 as $3_H \succ 1_H \succ 2_H$, leading to the only individually rational and Pareto optimal allocation $((2_H, 3_C), (3_H, 2_C), (1_H, 1_C))$. Either case leads to a contradiction with strategy-proofness.

**Computing the Membership of the Strict Core**

**Definition 3** (InStrictCore). Given a multi-type housing market $M$, agents’ preferences $P$, and an allocation $A$, we are asked whether $A$ is a strict core allocation w.r.t. $M$.

**Theorem 3.** InStrictCore is co-NPC even when agents have separable lexicographic preferences over $p = 2$ types.

**Proof sketch:** Membership in co-NPC is easy to verify. For co-NP hardness, we sketch a reduction from 3-SAT. An instance of 3-SAT is given by a formula $F$ in 3-CNF,
consisting of clauses $c_1, \ldots, c_n$ involving Boolean variables $x_1, \ldots, x_m$ that can take on values in $\{0, 1\}$, and we are asked whether there is a valuation of the variables that satisfies $F$. We will use $F = (x_1 \lor x_2 \lor x_3) \land (\neg x_1 \lor \neg x_2 \lor \neg x_3)$ to illustrate the proof.

Given an arbitrary instance $I$ of 3-SAT, we construct an instance $J$ of InStrictCore for $p = 2$ as follows. There are four classes of agents (1) $\{c_1, \ldots, c_n\}$; (2) for every $i \leq m$ and $j \leq n$, we have agents $1_i^j, 0_i^j$, corresponding to the values that might be used to satisfy the clauses; (3) for every $i \leq m$, we have three agents $x_i, \bar{x}_i, d_i$; (4) two extra agents $e_1, e_2$. For each agent $a$, her initial endowments are $([a]_1, [a]_2)$. Agents’ preferences are separable and lexicographic w.r.t. $1 \succ 2$. Preferences and the allocation $A$ are illustrated in Figure 4 (graph $G_1$) and Figure 5 (graph $G_2$), which correspond to type 1 and type 2, respectively.

For each $k \in \{1, 2\}$, a dashed edge $(a, b)$ in $G_k$ represents (i) the preference $[b]_k \succ [a]_k$, and (ii) the allocation $[A(a)]_k = [a]_k$. A solid edge $(a, b)$ indicates a strict preference $[b]_k \succ [A(a)]_k$. The absence of an edge $(a, b)$ indicates $[b]_k \prec [a]_k$.

![Figure 4: $G_1$: preferences and allocations for type 1.](image)

![Figure 5: $G_2$: preferences and allocations for type 2.](image)

$G_1$ is used to ensure that the valuation of variables is consistent, and $G_2$ is used to ensure that all clauses are satisfied. If the 3-SAT instance is satisfiable, the blocking coalition $S$ includes all $c_i$’s, $e_1, e_2$, and the agents correspond to the valuation. The allocation $B$ corresponds to cycles in $G_1$ and $G_2$, respectively. On the other hand, if $A$ is not a strict core, we can prove that any blocking coalition corresponds to a satisfying assignment to the 3-SAT instance.

**Definition 4 (StrictCoreNonEmpty).** Given a multi-type housing market $M$, agents’ preferences $P$, and an allocation $A$, we are asked whether the strict core of $M$ is non-empty.

**Theorem 4.** StrictCoreNonEmpty is NP-hard in general for even $p = 2$ types.

**Proof sketch:** The hardness is proved by a reduction from 3-SAT. Given a 3-SAT instance $F$, we construct a StrictCoreNonEmpty instance with the following agents: for every clause $c_j$, there are three agents $c_j^1, c_j^2, c_j^3$; for every variable $x_i$ there is an agent $x_i$ and two agents $1_i^j, 0_i^j$ for every clause $c_j$. Agents’ preferences will be defined by leverage the delicate example with 3 agents and 2 types, where the strict core is empty (Konishi, Quint, and Wako 2001).

**Example 5.** [Example 2.2 in (Konishi, Quint, and Wako 2001)] The strict core is empty for the following multi-type housing market with 3 agents and 2 types.

<table>
<thead>
<tr>
<th>Agent</th>
<th>Preferences</th>
<th>Allocation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Agent 1: 13</td>
<td>$\succ$ 33 $\succ$ 12 $\succ$ others.</td>
<td>$d_1$ $\rightarrow$ $d_2$ $\rightarrow$ $d_3$</td>
</tr>
<tr>
<td>Agent 2: 23</td>
<td>$\succ$ 21 $\succ$ 33 $\succ$ 31 $\succ$ 22 $\succ$ others.</td>
<td>$c_1$ $\rightarrow$ $c_2$ $\rightarrow$ $c_3$</td>
</tr>
<tr>
<td>Agent 3: 21</td>
<td>$\succ$ 22 $\succ$ 31 $\succ$ 11 $\succ$ 32 $\succ$ 12 $\succ$ 23 $\succ$ 33 $\succ$ 13 $\succ$ others.</td>
<td>$e_1$ $\rightarrow$ $e_2$ $\rightarrow$ $e_3$</td>
</tr>
</tbody>
</table>

The preferences of each triple of $c_j^1, c_j^2, c_j^3$ is similar to Example 5, except that $c_j^1$’s preferences are: $(1_j^1, c_j^1+1_j^1) \succ (1_j^2, c_j^2+1_j^2) \succ (1_j^3, c_j^3+1_j^3)$. Similarly, $c_j^2$’s preferences are: $(c_j^1, c_j^2) \succ (c_j^2, c_j^3) \succ (c_j^3, c_j^1)$. Others. For $k \in \{1, 2, 3\}$, if $x_{j_k} = x_{j_k}$, then we replace $1_j^k$, with $0_j^k$. This modification is to ensure that if the strict core is non-empty, agent $c_j^3$ must get the type-2 endowment of agent $c_j^1$ and the type-1 endowment of a satisfying valuation; otherwise there is no strict core even restrict to agents $c_j^1, c_j^2, c_j^3$. Based on this we can prove the NP-hardness of StrictCoreNonEmpty.

**Summary and Future Work**

We propose MTTC for multi-type housing markets with lexicographic preferences, and prove that it satisfies many desirable axiomatic properties. There are many future directions in mechanism design for multi-type housing markets. Are there good mechanisms when agents demand more than one item of some type? Can we design strategy-proof mechanisms under other assumptions about agents’ preferences, such as LP-trees (Booth et al. 2010)? What is the computational complexity of manipulation under MTTC? What if agents’ preferences are partial orders such as CP-nets only?

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