The Analysis and Synthesis of Logic Translation

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Abstract

Jean-Yves Béziau (*Classical Negation can be expressed by One of its Halves*) (Béziau 1999) has given an example of a phenomenon that people consider as translation paradox. We elaborate on Béziau's case, which concerns classical negation to the half of classical negation, as well as giving some relative background to this discussion. The translation in question turns out, not to deliver the new results but instead in the interests of illustrating the development of logic translation that widely discussed in various modern applications to computer science.

Introduction

The immediate stimulus for this chapter came from some ambiguous concepts found in the analysis about the concept of the sub-logic. In some relative discussions about the conceptual analysis of translation paradox where people found the following situation paradoxical with an assumption of *stronger-weaker* distinction about the strength of logics by weakening the condition of some logical constant on purpose: given two logics, one is weaker than the other in the sense of proving everything the former proves, while at the same time the stronger logic can be translatable to the weaker one (Beziau 1999) (Humberstone 2006) (Humberstone 2005) (Mossakowski 2009) (Priest 2008).

We present two logics one is the classical logic $\mathcal{L}_{Classical}$ with the well-known semantic conditions for implication and negation, the other is $\mathcal{L}_{Classical/2}$ which is a logic with classical implication but with only the half part semantic conditions of classical negation: given any truth-assignment such that if φ is 1, then $\neg\varphi$ is 0. We demonstrate translation paradox in Béziau's case and exploit it to discuss the historical development of the translation of logics between classical logic and intuitionistic logic. Historically speaking, we see intuitionistic logic firstly appears as a *sublogic* of classical logic. By intuition, it means intuitionistic logic, which is a *sub*-logic should be *weaker* than classical logic. However, as we will see that CPL can be translated into IPL but not vice versa, in this way, it indicates that IPL is, in a sense, stronger than CPL. It seems to be that the meaning of *sublogic* and that of the *strength* of logics are not clear enough.

Many papers about the negative translation from classical to intuitionistic logic have been written, since the proposal of **Gödel-Gentzen negative translation**. Over the last seventies years, there have been various discussions on this logic translation, moreover some general concept of logic translation has been discussed that should be pertained to the area of *abstract model theory* (Mossakowski 2009). However, the focus of this chapter is on the survey and systematic the development of logic translation begun from 1930 with respect to the study of *general logic* originated from Alfred Tarski in history. Thus, we would like to relate it to the discussion between Universal Logic project and the fundamental concepts about the **deviance** of logics.

From Sub-logic to Logic Translation

Studies about logic translation could be traced back to (Kolmogorov 1925) (Glivenko 1929) (Gentzen 1933) (Gödel (1933). In this chapter, the discussion on Béziau's case of the **translation paradox** provides an easier way for people to understand how it is possible for people to consider a more general and abstract logic by the *bivaluation approach*. Moreover, it makes us realize the contents of what logic translation is caring about, especially from the abstract logical points of view. Logic translation is a relatively new realm in logical society. Not only some new logical results are generated, but the old results and concepts are also reexamined by these translation methods in recent years. For example, "proof methods" used in the "*decidability problem*", originally raised in (Rabin 1965); "Translation of superintuitionistic logics into normal extensions of $S4$" (Gabbay 2005)and "accomplishing belief revision with AGM postulates by translation" (Gabbay 1999).

Translation Paradox

Recalling the definitions shown in (Béziau 1999): an arbitrary logical structure $\mathcal{L} = \langle S, \vdash_{\mathcal{L}} \rangle$ and the bivaluation, which is a set of functions **BIV** from the set of formulas into the set $\{0, 1\}$.

The logic $\mathcal{L}_{Classical} = \langle \mathrm{S}, \rightarrow, \neg, \models_{Classical} \rangle$ with its bivaluation $\mathbf{BIV}_{Classical}$ means the classical propositional logic (CPL) with classical implication $\rightarrow$ and classical negation $\neg$. The well-known two classical semantic conditions for implication and negation are as follows:

$\beta \in \mathbf{BIV}_{Classical}$ iff given any two formulas φ, $\psi \in S$ the following two conditions hold: 1. $\beta(\varphi \rightarrow \psi) = 0$ iff $\beta(\varphi) = 1$ and $\beta(\psi) = 0$; 2. $\beta(\varphi) = 1$ iff $\beta(\neg\varphi) = 0$.

Other classical connectives could be defined by implication and negation in $\mathcal{L}_{Classical}$ and then define the whole CPL. We consider the logic $\mathcal{L}_{Classical/2} = \langle \text{S'}, \rhd, \ominus, \models_{Classical/2}\rangle$ with its bivaluation $\mathbf{BIV}_{Classical/2}$ as follows:

a. S' is a set of formulas constructed by one binary connective $\rhd$ and one unary connective $\ominus$.

b. $\beta \in \mathbf{BIV}_{Classical/2}$ iff given any two formulas $\varphi, \psi \in S'$ the following two conditions hold:
   (1) $\beta(\varphi \rhd \psi) = 0$ iff $\beta(\varphi) = 1$ and $\beta(\psi) = 0$
   (2) $\beta(\varphi) = 1$ implies $\beta(\ominus\varphi) = 0$

c. $T \models_{Classical/2} \varphi$ iff for every $\beta \in \mathbf{BIV}_{Classical/2}$, if $\beta(a) = 1$, for every $a \in T$, then $\beta(\varphi) = 1$.

$\mathcal{L}_{Classical/2}$ is a weaker classical propositional logic (CPL) with classical implication $\rhd$ and "half classical negation" $\ominus$. Note here that "half classical negation" means we take only the following "half" condition: if $\beta(\alpha) = 1$, then $\beta(\ominus\alpha) = 0$ instead of the condition for classical negation: $\beta(\alpha) = 1$ if and only if $\beta(\ominus\alpha) = 0$.

Let us say that two formulas φ and ψ, φ is *logical equivalent* to ψ in $\mathcal{L}_{Classical/2}$ iff $\varphi \models_{Classical/2} \psi$ and $\psi \models_{Classical/2} \varphi$.

Lemma 1. *In $\mathcal{L}_{Classical/2}$, for any formula φ, ψ, ϕ,*

(1) $\varphi \rhd (\psi \rhd \phi)$ is logically equivalent to $\psi \rhd (\varphi \rhd \phi)$.

(2) $\ominus(\varphi \rhd (\psi \rhd \phi))$ is not logically equivalent to $\ominus(\psi \rhd (\varphi \rhd \phi))$.

In the following, we define some necessary concepts we will use in the discussion of Béziau's translation paradox.

Definition 2. A function $f : X \rightarrow Y$ is **injective (one-to-one)** if for any $x, y \in X$, $f(x) = f(y)$ implies $x = y$. A function $f : X \rightarrow Y$ is **surjective (onto)** if for any $y \in Y$, there is a $x \in X$ such that $y = f(x)$. A function is **bijective (one-to-one and onto)** if and only if it is both injective and surjective.

Take f as an *bijection* between the atomic formulas of $\mathcal{A}$ and the atomic formulas of $\mathcal{B}$. Given two algebras *absolute free algebras* $\langle \mathcal{A}, \rhd, \ominus\rangle$ and $\langle \mathcal{B}, \rightarrow, \neg\rangle$, there is an unique extension g of f which is an isomorphism upto g between these two *absolute free algebras*, i.e., $g(a \rhd b) = g(a) \rightarrow g(b)$ and $g(\ominus a) = \neg(g(a))$, such that $T \models_{Classical/2} \varphi$ implies $g(T) \models_{Classical} g(\varphi)$. Here g is called as a **language-isomorphism**.

Consider an atomic formula a and the bivaluation $\beta \in \mathbf{BIV}$, in $\mathcal{L}_{Classical/2}$, $\beta(a) = 0$, $\beta(\ominus a) = 0$ and $\beta(\varphi) = 0$, where $\varphi = ((\ominus a \rhd a) \rhd a)$ such that $\not\models_{Classical/2} \varphi$. By the language-isomorphism, we get a tautology $(\neg a \rightarrow a) \rightarrow a$ in $\mathcal{L}_{Classical}$, such that $\models_{Classical} g(\varphi)$. Hence, $g(T) \models_{Classical} g(\varphi)$ does not imply $T \models_{Classical/2} \varphi$.

Here, we call the logic $\mathcal{L}_{Classical/2}$ is **strictly included**, up to language-isomorphism, in the logic $\mathcal{L}_{Classical}$ in the sense of the relation $\models_{Classical/2}$ is **strictly included** in the relation $\models_{Classical}$. In this way, here is such a voice (Béziau 1999):

> "[...] it seems that we can say that the logic $\mathcal{L}_{Classical/2}$ is **strictly weaker** than the logic $\mathcal{L}_{Classical}$". One might want to interpret this fact saying that $\mathcal{L}_{Classical/2}$ is a proper **sublogic** of $\mathcal{L}_{Classical}$ [...]"

Apart from the presentations of these two logics, it is the attempt to interpret the fact that the $\mathcal{L}_{Classical/2}$ is **strictly weaker** than the logic $\mathcal{L}_{Classical}$ as the former is the proper **sub-logic** of the latter to make the paradoxical situation via the understanding of the contained-relation of deductive sense to sub-logic.

The Translation Relation

As a prelude to study Béziau's translation paradox, we set up the general idea of translation between logics, as well as discussing the historical perspective adapting to the translation between classical logic and intuitionistic logic.

Definition 3. Given two logics K_1 and K_2, consider a function Φ from $\mathcal{A}$ to $\mathcal{A}'$, $\Phi(\Sigma)$ is a translation of signature Σ from K_1 to K_2, α_Σ is a sentence translation function from the Σ-sentence to $\Phi(\Sigma)$-sentences, and γ is a model translation function from K_1-models to K_2-models, such that the $M_2 \models_2 \alpha(\varphi_1)$ if and only if $\gamma(M_2) \models_1 \varphi_1$ holds for any $\varphi_1 \in \mathcal{A}$ and any $M_2 \in K_2$-model.

One can say that the translation from $\mathcal{L}_{Classical}$ to $\mathcal{L}_{Classical/2}$ as follows:

The logic $\mathcal{L}_{Classical} = \langle \mathcal{S}, \rightarrow, \neg, \models\rangle$ and $\mathcal{L}_{Classical/2} = \langle \mathcal{S}', \rhd, \ominus, \models_{1/2}\rangle$ have sets Σ and Σ' of propositional symbols as signatures, respectively. $\mathcal{L}_{Classical}$-sentences are built from Σ with the propositional connectives $\neg$ and $\rightarrow$, and $\mathcal{L}_{Classical/2}$-sentences are built from Σ with the propositional connectives $\ominus$ and $\rhd$. Take the function Φ from Σ to Σ' as the translation of signature, and the function ρ as the translation of sentences from $\mathcal{L}_{Classical}$-sentences to $\mathcal{L}_{Classical/2}$-sentences as follows:

1. $\rho(p) = p$, for any atomic formula p
2. $\rho(p \rightarrow q) = \rho(p) \rhd \rho(q)$
3. $\rho(\neg p) = \rho(p) \rhd \ominus\rho(p)$,

such that the models translation γ along the Φ make the following holds: for any $\varphi \in \mathcal{S}$ and any $M_{1/2} \in \mathcal{L}_{Classical/2}$-model, $M_{1/2} \models_{Classical_{1/2}} \rho(\varphi)$ if and only if $\gamma(M_{1/2}) \models_{Classical} \varphi$.

We can not get a similar translation from $\mathcal{L}_{Classical/2}$ to $\mathcal{L}_{Classical}$. The sentences of $\mathcal{L}_{Classical/2}$ are the same as in $\mathcal{L}_{Classical}$ but the models are valuations of all sentences that respect the truth-table semantics of the implication $\rhd$ and the negation $\ominus$, which is only with the half of the condition in $\mathcal{L}_{Classical/2}$:

- $\varphi \rhd \psi = 1$ if and only if $\varphi = 0$ or $\psi = 1$
- $\ominus\varphi = 0$, if $\varphi = 1$.

(Béziau's Translation Paradox) Following up the previous discussions that $\models_{Classical/2}$ is strictly included (up to language-isomorphism g) in the relation $\models_{Classical}$. It might imply that $\mathcal{L}_{Classical/2}$ is strictly weaker (up to language-isomorphism g) than $\mathcal{L}_{Classical}$. Here, the immediate connection to the idea of translation is perhaps given that $\mathcal{L}_{Classical}$ is thought of as **translatable** to $\mathcal{L}_{Classical/2}$ (but not vice versa). Thus, for two logics $\mathcal{L}_{Classical}$ and $\mathcal{L}_{Classical/2}$, the situation becomes that $\mathcal{L}_{Classical/2}$ is strictly included into $\mathcal{L}_{Classical}$ that suggests the $\mathcal{L}_{Classical/2}$ is strictly weaker than $\mathcal{L}_{Classical}$, but $\mathcal{L}_{Classical}$ is translatable into $\mathcal{L}_{Classical/2}$. In other words, while $\mathcal{L}_{Classical}$ "specify a copy" of the $\mathcal{L}_{Classical/2}$, $\mathcal{L}_{Classical/2}$ should be at least strong as $\mathcal{L}_{Classical}$. The $\mathcal{L}_{Classical}$ is a "sub-logic" of the $\mathcal{L}_{Classical/2}$ in the sense of translatable.

The so-called **Béziau's translation paradox** (Humberstone 2005) is actually originated from a quite similar situation that already discussed about the development of intuitionistic logic. " [...] If an inference is intuitionistically valid, it is therefore classically valid (when $\neg$ and $\sqsupset$ are replaced with $\neg$ and $\supset$, respectively). The converse is not true, as we shall see. Hence, intuitionist logic is a sub-logic of classical logic [...] This is not true of intuitionist mathematics in general. Intuitionist mathematics endorses some mathematical principles which are not endorsed in classical mathematics; in fact, they are inconsistent classically. But because intuitionist logic is weaker than classical logic, the principles are intuitionistically consistent. For the record, it is worth noting that there is a certain way of seeing classical logic as a part of intuitionist logic too [...]" (Priest 2008)

It is this "in a certain way" that makes the phenomenon philosophically arguable, say, a weaker logic is a "sub-logic" of a stronger logic but the latter is "contained" in the former that suggests the former one is at least as strong as the latter.

The Development of Logic Translation with Abstract Logic

In order to understand this logic translation better, let us discuss it in a bit more detail. An abstract logical perspective will run between different ideas of translation, ranging from the "rough" to the "rigorous". Béziau's statement is a very specific case to provide us an approach to exam the relationship between the classical logic and intuitionistic logic in an abstract logical point of view. For example, we even do not have to know much about what the statement of Glivenko's theorem is and what principles intuitionistic logic endorses or not. We only need to own basic knowledge about some sort version of classical propositional logic.

Consequence Relations and Logic Translation

Most ideas about logic translation in literature were founded on the discussion of Tarskian logic. Some backgrounds about Tarskian logic have been mentioned in the previous chapter, we further offer some definitions to gain a deeper understanding of logic translation. First, we considered a *consequence relation* $\vdash$ to define a logical structure $\mathcal{L} = \langle \mathcal{F}, \vdash \rangle$. Second, we consider a *consequence operator* Cn to define a logical structure $\mathcal{L} = \langle \mathcal{F}, Cn \rangle$. The idea of logic translation, in this way, can be traced back to different sources by these two conceptions of logical structure.

Recalling the conception of Tarskian logic, an arbitrary logical structure is said Tarskian when it obeys the reflexivity, monotonicity, and cut. Now, we explore various ideas on translation.

Definition 4. (**Rough Translation**) A *translation* from one logic $\mathcal{L}_1$ into logic $\mathcal{L}_2$ is defined as mapping: $f : \mathcal{L}_1 \to \mathcal{L}_2$, that is to map the set of formulas in $\mathcal{L}_1$ to the set of formulas in $\mathcal{L}_2$, *such that for any formula φ, if φ is a theorem of $\mathcal{L}_1$, then $f(\varphi)$ is a theorem of $\mathcal{L}_2$.*

Definition 5. (**Revised Rough Translation (i)**) Follow **Rough Translation** with a stronger condition:

(a) For any formula φ, φ is a theorem of $\mathcal{L}_1$ *iff* $f(\varphi)$ is a theorem of $\mathcal{L}_2$.

Definition 6. (**Revised Rough Translation (ii)**) Follow **Rough Translation** with another stronger condition:

(b) For any set of formulas Γ, formula φ, if $\Gamma \vdash_{\mathcal{L}_1} \varphi$ then $f(\Gamma) \vdash_{\mathcal{L}_2} f(\varphi)$.

Definition 7. (**Conservative Translation**) Follow **Revised Rough Translation (ii)** with a stronger condition than (b):

(c) For any set of formulas Γ, formula φ, $\Gamma \vdash_{\mathcal{L}_1} \varphi$ *iff* $f(\Gamma) \vdash_{\mathcal{L}_2} f(\varphi)$.

The more noteworthy idea on translation is *schematic mapping*. Schematic mapping relies on **homomorphism** among formal languages, and involves some *diagrammatic* representation of the structures of language expressions. Here, "some" preserved diagrammatic representations are in the sense of *algebraic*, which is in tune with Tarski's paradise.

Definition 8. (Schematic Translation) Let two formal languages L_1 and L_2 with only unary $\spadesuit$ and binary connectives $\amalg$ be given. If for any atomic formulae $a_0, a_1, \ldots, a_n, \cdots \in L_1$, there are *schemata of formulae* A, $B_\spadesuit$, $C_\amalg \in L_2$ such that the mapping $r : L_1 \mapsto L_2$ satisfies the following conditions, then r is a *schematic mapping*.

(1) $r(a) = A(a)$, for every atomic formula $a \in L_1$,

(2) $r(\spadesuit\varphi) = B_\spadesuit(r(\varphi))$, for every unary connective $\spadesuit$ and formula φ of L_1,

(3) $r(\amalg(\varphi, \psi)) = C_\amalg(r(\varphi), r(\psi))$, for every binary connective $\amalg$ and formula φ, ψ of L_1.

Definition 4-7 with *schematic mapping* are *schematic translation*.

The Systematic Discussions of Logic Translation

As mentioned in the introduction, in the development of logic translation and abstract logic, three systematic discussions arise: firstly, the Prawitz and Malmnäs in th article of *A survey of some connections between classical, intuitionistic and minimal logic* (Prawitz and Malmnäs 1968), notably as the primary systematic discussion in literature, had the the definition of the term "translation" in the general level. Secondly, the Ryszard Wójcicki's discussions that adopt the abstract logical perspective fashioned in the Polish school (Wójcicki 1988). Thirdly, the Richard Epstein's discussions on the translations between propositional logics (Epstein 1990) and the translation within predicate logic (Epstein 2006). These systematic discussions and integrations relied on some given original materials about translation, such as (Kolmogorov, 1925) (Glivenko 1929) (Gödel 1933) (Gentzen 1933), and (Bloom, Brown and Suszko 1973). Moreover, we will bring forward two lines that the concepts of logic translation have been written in many different materials in literature.

History: 1968 – 1933 – 1929 – 1925 The first systematic definition of the term "translation" has been discussed in (Prawitz and Malmnäs, 1968) as follows:

(i) Consider two logical systems S_1, S_2, an interpretation[1] from S_1 to S_2 is a mapping t from formulas of S_1 to S_2 such that for any formula φ, $\vdash_{S_1} \varphi$ iff $\vdash_{S_2} t(\varphi)$.

(ii) For each set $\Gamma \cup \{\varphi\}$ of formulas in S_1, $\Gamma \vdash_{S_1} \varphi$ iff $t(\Gamma) \vdash_{S_2} t(\varphi)$ where $t(\Gamma)$ is the set of replacing all elements ψ of Γ by $t(\psi)$.

The idea in (i) is the same as the **revised rough translation (i)** (Definition 5). With regard to (i), we can say that S_1 is *intepretable* into S_2, and also that S_1 is *interpretable* into S_2 *with respect to derivability*. Note here that (ii) does not coincide with the **conservative translation** (Definition 7) that always been studied later in literature. As Mossakowski et al. stated: "Prawitz and Malmnäs also use a more permissive notion of conservative translation where the equivalence is only required for $\Gamma = \emptyset$" (Mossakowski 2009). Prawitz and Malmnäs also describe the idea of *schematically interpretable* as we mentioned in Definition 8.

These works shown by (Prawitz and Malmnäs 1968) is actually the first survey paper on other original materials in literature on translation, including (Kolmogorov 1925) (Glivenko 1929) (Gödel 1933), and (Gentzen 1933). We provide the gist of these papers without delving into the details, after which, we proceed with our main discussion. These earlier papers focused mainly on the problem of *consistency* and the relation between classical logic and intuitionistic logic. To begin with, let us examine Kolmogorov's idea on translation reflected in the following quotation:

> "The main purpose of this paper is to prove that classical mathematics is translatable into intuitionistic mathematics. For this purpose, with each formula $\Im$ of mathematics there is associated a translation $\Im^*$ in a perfectly general manner (IV, § 2)." (Kolmogorov 1925), pp. 414-415.

[1] Specially, they used the term "interpretation", whereas we used the term "translation".

Note here that we provide Hao Wang's introduction before the English translation of this article; however, in the translation, we replace the outdated terms with the modern terms.

(Kolmogorov, 1925) For $\mathcal{H}$ (Hilbert's formal system of classical propositional logic) and $\mathcal{B}$ (Brouwer's formal system of intuitionistic propositional logic), there is a translation $*$ from $\mathcal{H}$ to $\mathcal{B}$ such that for any atomic formula p there is a correspondent formula $(p)^*$ which expresses the double negation of p, denoted as $\neg\neg p$ and formulas $(\neg\varphi)^*$, $(\varphi \to \psi)^*$ are defined as $\neg\neg(\neg\varphi^*)$, $\neg\neg(\varphi^* \to \psi^*)$, respectively.

Theorem 9. *If* $\Gamma = \{\Gamma_1, \ldots, \Gamma_n\}$ *is a set of axioms in* $\mathcal{H}$ *and* $\Gamma^* = \{\Gamma_1^*, \ldots, \Gamma_n^*\}$*, then for any formula* φ*,* $\Gamma \vdash_{\mathcal{H}} \varphi$ *implies* $\Gamma^* \vdash_{\mathcal{B}} \varphi^*$.

Theorem 10. *(Glivenko's Theorem, 1929) (Glivenko's translation) An arbitrary propositional formula* φ *is a theorem of the classical propositional logic, i.e. classically provable if and only if* $\neg\neg\varphi$ *is a theorem of the intuitionistic propositional logic, i.e. intuitionistically provable.*

(Gödel, 1933) For a system of classical propositional logic $\mathcal{A}$, and a system of intuitionistic propositional logic $\mathcal{H}'$, the translation $\star$ from $\mathcal{A}$ to $\mathcal{H}'$ is defined as follows:

(1) $\varphi^\star =_{df} \varphi$

(2) $(\neg\varphi)^\star =_{df} \neg\varphi^\star$

(3) $(\varphi \land \psi)^\star =_{df} \varphi^\star \land \psi^\star$

(4) $(\varphi \lor \psi)^\star =_{df} \neg(\neg\varphi^\star \land \neg\psi^\star)$

(5) $(\varphi \to \psi)^\star =_{df} \neg(\varphi^\star \land \neg\psi^\star)$, for every atomic formula φ, ψ

Theorem 11. $\vdash_{\mathcal{A}} \varphi$ *implies* $\vdash_{\mathcal{H}'} \varphi^\star$.

(Gentzen, 1933) For a system of classical propositional logic $\mathcal{A}$, a system of intuitionistic propositional logic $\mathcal{I}$, the translation $\blacklozenge$ from $\mathcal{A}$ to $\mathcal{I}$ is defined as follows:

(1') $\varphi^\blacklozenge =_{df} \neg\neg\varphi$

(2') $(\neg\varphi)^\blacklozenge =_{df} \neg\varphi^\blacklozenge$

(3') $(\varphi \land \psi)^\blacklozenge =_{df} \varphi^\blacklozenge \land \psi^\blacklozenge$

(4') $(\varphi \lor \psi)^\blacklozenge =_{df} \neg(\neg\varphi^\blacklozenge \land \neg\psi^\blacklozenge)$

(5') $(\varphi \to \psi)^\blacklozenge =_{df} \varphi^\blacklozenge \to \psi^\blacklozenge$

Theorem 12. $\Gamma \vdash_{\mathcal{A}} \varphi$ *if and only if* $\Gamma^\blacklozenge \vdash_{\mathcal{I}} \varphi^\blacklozenge$.

Remark 13. Both Kolmogorov and Gödel's ideas on translation are special cases of Definition 5. We see that there is a translation $*$ or $\star$ that is considered between two formal systems $\mathcal{H}$ (Hilbert's formalization of propositional calculus) and $\mathcal{B}$ or $\mathcal{H}'$ (Brouwer's or Heyting's formalization of propositional calculus). Gentzen's idea on translation is a *conservative translation* (Definition 7) which indicates the importance of direction, denoted as "$\Leftrightarrow$". Moreover, Gentzen translates *implication*, denoted as "$\to$" directly rather than in terms of "$\neg$" and "$\land$" (Gödel).

History: 1988 – 1973 – 1971 – The second systematic research on translation has been discussed by Ryszard Wójcicki. Wójcicki based on the conception of consequence operators to develop his idea about logic translation. Wójcicki begun by considering *language translation*, that is, he considers "the map" between languages. Next, he considers "preserving" the consequence operators. In other words, it is a *derivability preserving* schematic translation. Apparently, his idea should be further taken as a systematic study of logic translation from an abstract logical point of view, which can be traced back to Tarski's idea of logical consequence. Before going on we should consider one different research made by Brown-Bloom-Suzuko's *abstract logic*. Brown-Bloom-Suzuko's idea is considered to be a pioneering point of view on *abstract logic*. In this way, we can moreover understand the trend of abstract logic in Polish school.

Definition 14. (Bloom-Brown-Suszko, 1971, 1973) The objects of the *category of abstract logics* are ordered pairs $\langle \mathcal{S}, Cn \rangle$ consists of an *abstract algebra* $\mathcal{S}$ and a *closure operator* C_n on $S = |\mathcal{S}|$, the carrier (universe) of $\mathcal{S}$. If S is a non-empty set, Cn is a closure operator on S then $\langle S, Cn \rangle$ is called a *closure space*.

Definition 15. Let $K_1 = \langle S_1, Cn_1 \rangle$, $K_2 = \langle S_2, Cn_2 \rangle$ be two closure spaces. A *mapping* f from S_1 to S_2 is said to be *continuous* if $\breve{f}(Z) \in Cn_1$, for all $Z \in Cn_2$. Here $\breve{f}(Z)$ is the inverse image of Z under f. The set of all continuous maps of K_1 into K_2 is denoted as $Hom(K_1, K_2)$. If both f and its inverse image $\breve{f}$ are continuous, i.e., $f \in Hom(K_1, K_2)$ and $\breve{f} \in Hom(K_1, K_2)$, then a bijective map $f : S_1 \to S_2$ is called as a *homeomorphism* between K_1 and K_2.

Note here that Brown-Bloom-Suzuko's idea on *abstract logic* is a *category-theoretical* viewpoint of logic inspired by topology. Naturally, they consider a *logical morphism* as the translation between two abstract logics from a topological point of view.

Adopting the position of abstract logic in the Polish style with respect to consequence operations, Wójcicki had a systematic study on the logic translation as follows:

(Wójcicki, 1988) Given two propositional languages S_1, S_2 with the same variables, a mapping $t : S_1 \mapsto S_2$ is a translation from S_1 to S_2 iff two conditions is satisfied:

(i) There is a formula $\varphi(p_0) \in S_2$ in one variable p_0 such that for each variable p, $t(p) = \varphi(p)$.

(ii) For each connective ρ_i of S_1 there is a formula $\varphi_i \in S_2$ such that for all terms $\alpha_1, \ldots, \alpha_k \in S_1$, k being the arity of ρ_i, we have that

$$t(\rho_i(\alpha_1, \ldots, \alpha_k)) = \varphi_i(t\alpha_1/p_1, \ldots, t\alpha_k/p_k).$$

For two propositional calculi $\mathcal{C}_1 = (S_1, C_{n_1}), \mathcal{C}_2 = (S_2, C_{n_2})$, if there is a translation t from S_1 into S_2 such that for all $X \subseteq S_1$ and all $\alpha \in S_1$,

$$\alpha \in C_{n_1}(X) \Leftrightarrow t\alpha \in C_{n_2}(t(X))$$

then $\mathcal{C}_1$ has a translation in $\mathcal{C}_2$.

History: 2006 – 1990 – The third systematic research on *translation* has been discussed by Richard Epstein. Epstein's study on translation is divided into two parts, one is to study a general idea on translation between propositional logics (Epstein, 1990) and the other is to study the translation *within* classical predicate logic (Epstein, 2006). Epstein's statements regarding *translation* can be considered at two different levels— the level of propositional logic and that of predicate logic level.

Definition 16. (Epstein, 1990) A *validity mapping* of a propositional logic $\mathcal{L}_1$ into a propositional logic $\mathcal{L}_2$ is a map t from language of $\mathcal{L}_1$ to language $\mathcal{L}_2$ such that for every φ,

$$\models_{\mathcal{L}_1} \varphi \Leftrightarrow \models_{\mathcal{L}_2} t(\varphi).$$

Definition 17. (Epstein, 1990) For any theory Γ, formula φ, if the mapping relation from t to $t(\Gamma) = \{t(\varphi) : \varphi \in \Gamma\}$, such that

$$\Gamma \models_{\mathcal{L}_1} \varphi \Leftrightarrow t(\Gamma) \models_{\mathcal{L}_2} t(\varphi)$$

then this mapping relation is a *translation* form logic $\mathcal{L}_1$ to logic $\mathcal{L}_2$, deonted as $\mathcal{L}_1 \hookrightarrow \mathcal{L}_2$.

Definition 18. (Epstein, 2006) Given two theories $\mathbf{T}, \mathbf{R}$ in first-order classical predicate logic and a mapping t from the language of $\mathbf{T}$ to the language of $\mathbf{R}$.

1. t is *validity-preserving* iff for every φ, $\models_{\mathbf{T}} \varphi$ iff $\models_{\mathbf{R}} \varphi$.
2. t is a *translation* of $\mathbf{T}$ into $\mathbf{R}$ iff for every Γ and φ, $\Gamma \models_{\mathbf{T}} \varphi$ iff $t(\Gamma) \models_{\mathbf{R}} t(\varphi)$.

Recall some basic notions: given a theory Σ and a class of models $\mathcal{S}$. $Th(\mathcal{S}) = \{\varphi \mid \text{for every } \mathcal{M} \text{ in } \mathcal{S}, \mathcal{M} \models \varphi\}$. We have the idea of **model-preserving mapping** as follows:

(Epstein, 2006) Let $\mathbf{T} = Th(\mathcal{T})$ and $\mathbf{R} = Th(\mathcal{R})$ be two theories with respect to classes of models $\mathcal{T}, \mathcal{R}$ in classical predicate logic, τ is a mapping from language of $\mathbf{T}$ to language of $\mathbf{R}$. If an onto mapping τ from $\mathcal{R}$ to $\mathcal{T}$ such that for every φ in language of $\mathbf{T}$, every model $\mathcal{M}$ in $\mathcal{R}$, the satisfaction condition: $\tau(\mathcal{M}) \models \varphi$ iff $\mathcal{M} \models \tau(\varphi)$ holds, then τ is a model-preserving mapping with respect to $\mathcal{T}$ and $\mathcal{R}$.

Theorem 19. *(Epstein, 2006) Every model-preserving mapping is a translation.*

Discussions and Outlook

So far, we have seen many different ideas about logic translation in literature. These ideas can roughly be classified as two periods:

(i) Kolmogorov-Glivenko-Gödel-Gentzen to Esptein-Wójcicki period (**KGGG-EW**): The point in this period is to focus on logical relation $\vdash$.

(ii) Bloom-Brown-Suszko to *Brazilian*-Esptein-Wójcicki period (**BBS-BEW**): The point in this period is to focus consequence relation Cn.

It is worth mentioning the difference here, since it expresses that a logic has begun to be viewed as a finitary consequence operator in period (ii). And it also expresses the trend of seeing logic *in general* or considering *abstract logic*, as Bloom-Brown-Suszko did. The research line (**BBS-*B*EW**) of abstract logic viewpoint from Bloom-Brown-Suszko to *Brazilian*-Esptein-Wójcicki, which cares about "consequence operator Cn". Logics is characterized as sets with *consequence* *operator* Cn, and translation as *continuous functions* between Cn. Although both lines are from the abstract logical point of view, **BBS-*B*EW** line is different from the **KGGG-EW**, from Kolmogorov-Glivenko-Gödel-Gentzen to Esptein-Wójcicki, since the latter one cares about the translation between classical logic and intuitionistic logic at the beginning, then coined with the **BBS-*B*EW** line. In other words, it becomes to discuss the concept of logic translation by considering the abstract logical structures instead of studying translation of individual logics directly.

By translation paradox, we can see that the concept of the strength of logics has been taken before beginning the discussion of translation of two logics. Moreover, we see that the the stronger logic could be translated into the weakened logic, but not vice versa. Similarly, historically speaking, we see intuitionistic logic first appearing as a sublogic of classical logic. In this sense, people naturally consider intuitionistic logic to be weaker than classical logic. However, it has been shown that classical logic can be part of (or translated into) intuitionistic logic in certain way, but not vice versa. It means classical logic is weaker in certain way.

As a result of this paper, the relation of "weaker than" should be clarified and distinct from the relation of sublogic. Moreover, the relation of "sublogic" should be treated in a rigorous way. We suggest that the clarification of the meaning of *sublogic* and that of the *strength* of logics should be within the framework of logic translation. Yet to clarifying the studies of logic translation will express that the difference of logics can be indicated through translation.

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